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# $N$-person Nim and $N$-person Moore's Games

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*Abstract*: We present one way of defining $n$-person perfect information games so that there is a reasonable outcome for every game. In particular, the theory of Nim and Moore's games is generalized to $n$-person games.

## 1. Background

This article is on the generalization of the theory of 2-person perfect information games to $n$-person games. In a 2-person perfect information game two players alternate moves until one of them is unable to move at his turn and that player loses. There is no chance move or any probability involved. The goal of the game is to win or at least not to lose. Among the games of this type are checkers, tic-tac-toe, Nim, Hackenbush, etc.

During the last few years, the theory of 2-person perfect information games has been promoted to an advanced level. Naturally it is of interest to generalize as much as possible of the theory to $n$-person games. Yet there has not been any general result of this kind. One reason is as follows. In 2-person perfect information games, one can always talk about what the outcome of the game *should* be, when each player plays it right, i.e., when each player adopts an optimal strategy. But when there are more than two players, it may not make sense to talk about the same thing. For instance, it may so happen that one of the players can help any of the others to win, but anyhow, he himself has to lose (this situation actually occurs in Chinese checkers). So the outcome of the game depends on how the group coalitions are formed among the players. This makes it too complicated to develop a general theory. In order to circumvent this difficulty one has to be very careful in defining the rules for ending the game and for deciding winners and losers. In section 2, we shall present one set of such rules under which it makes sense to talk about what the outcome of an $n$-person perfect information game should be.

Moreover, with these rules we shall generalize *Bouton*'s [1902] Nim Theory, to $n$-person games. Our results will also encompass a slightly wider class of games, called the Moore's games [*Conway*, p. 181].

One reason that Nim theory is of particular interest to be generalized to $n$-person games is because of its simple mathematical structure. In fact, Nim heaps are the only *impartial* games with monotonically decreasing Sprague-Grundy numbers [*Sprague;*

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*Grundy*]. Secondly, the solution of Nim involves the expression of numbers in the basis of *two* and the addition of numbers in the scale of *two* without carry. This leads to the philosophical question of what is the intrinsic property of Nim that is connected to the number *two*. The only obvious answer seems to be the number of players, although there is not enough evidence to support this answer. As it has turned out, the solution of $n$-person Nim depends on the addition of numbers in the scale of $n$, but still it involves the binary expressions of numbers. This accidentally coincides with the solution of Moore's game $\text{Nim}_{n-1}$. Thus the phenomenon that the binary expression of numbers is essential to the solution of Nim does not originate from the number of players in Nim.

## 2. Introduction to Combinatorial Games

Nim is played with a number of heaps of counters, and the move is to remove some counters from any one heap. Two players alternate moves until the counters are all gone; whoever removes the last counter wins the game. Every Nim game is either a mover-winning position or a mover-losing position. To determine the outcome of a Nim game, we first express the number of counters in each heap in the basis of two and then add these numbers in the scale of two without carry. The resulting binary number is called the *Nim-sum* of these numbers. If this Nim-sum is not 0, then the first player to move can force a win. Otherwise, the second mover can win.

The rules of the Moore's game $\text{Nim}_k$ are the same as those of Nim except that, at each move, the player is allowed to remove counters from any number of heaps up to $k$. Ordinary Nim is the particular case $\text{Nim}_1$. The strategy for Nim can be easily generalized to $\text{Nom}_k$. Again, we write the numbers of counters in the binary notation. But then add these numbers without carry, and in the scale of $k + 1$. The game is a mover-losing position if and only if the resulting "number" is zero.

Two-person perfect information games ranging from simple ones like Nim and $\text{Nim}_k$ up to complex ones like chess and checkers are all called *combinatorial games*. In general, a combinatorial game may be defined as a vertex in a finite edge-bicolored directed graph. Edges of one color represent legal moves for one player, the other color for the opponent. At each turn a player must move from one vertex to another along the direction of an edge of his color. When a player is unable to move at his turn, that player loses. The sum (disjunctive compound) of two games is defined to be the compound game in which a player may move in either component game. For instance, the sum of a 2-heap Nim game with a 3-heap Nim game is simply the Nim game consisting of all the 5 heaps.

Both Nim and $\text{Nim}_k$ belong to a special class of combinatorial games, namely the *short impartial* games. A game is said to be *short* if any chain of moves (not necessarily alternating) must terminate; and a game is called *impartial* if, at any stage, the possible moves are independent of whose turn it is to move. The outcome of a short impartial game is either mover-winning or mover-losing and is completely determined by its

*Sprague/Grundy* number (for abbreviation, the S/G number), which is defined recursively as the smallest non-negative integer that is not the S/G number of any of the positions into which the game can be moved. Thus the game is mover-losing if and only if its S/G number is 0. For example the S/G number of a Nim game is simply the Nim-sum of the heap sizes. The importance of the S/G number lies in its Nim-additivity, i.e., the S/G number of the sum of two games is the Nim-sum of the S/G numbers of the summand games. This enables us to describe the outcome and the optimum strategies for the sum of short impartial games in terms of information about the summand games.

## 3. Defining the Rules for $N$-person Games

It is natural to define an $n$-person game as a vertex in a finite directed graph such that the edges are in $n$ colors, one color for each player. The players rotate turns moving from one vertex to another along directed edges. We need to define the rules for ending the game and for deciding the winners or losers and meanwhile, avoid the possibility of group coalition among players.

We shall call the players $P_1, P_2, \ldots, P_n$ according to the initial order of turns. As before, a game is ended when any player is unable to move at his turn; especially for impartial games this is the only reasonable definition for ending. Naturally we define the loser to be the player unable to move. If that player is $P_m$, say, we assign a different rank to each player, ranging from bottom to top in the order of $P_m$, $P_{m+1}, \ldots, P_n, P_1, \ldots, P_{m-1}$. In particular, the last player able to move is the top winner. Under these rules the rank of any one player automatically determines the ranks for all. For this reason, it makes sense to say what the outcome of the game should be when each player adopts an optimal strategy toward his own highest possible rank. Of course the optimality of a move is based on the assumption that the consequent moves by all the players shall be optimal.

*Remark*: In fact, if $\sigma$ is any permutation on $\{1, 2, \ldots, n\}$, we may also rank the players from bottom to top according to the order $P_{\sigma(m)}, P_{\sigma(m+1)}, \ldots,$ $\ldots, P_{\sigma(m-1)}$; and there will still be a *reasonable* outcome for every game. So we have $n!$ choices for the rule just as in the case of 2-person games the rule can be last-mover-win or last-mover-lose. Here we have chosen $\sigma$ to be the identity mapping not only because it seems to be the natural choice, but also because it allows simple solutions to $n$-person Nim and $n$-person Moore's games. In fact if there were any permutation $\sigma$ that could lead to a generalization of the *Sprague/Grundy* function, we might find it an even more natural choice than the identity mapping.

## 4. $N$-person Nim

In $n$-person Nim the move is defined in the same way as in the ordinary Nim, and the rules for deciding winners and losers are as in the last section. If the $m$-th player to move should be the biggest loser, the game will be called an $(m-1)$- position. Since $n$-person Nim, as well as regular Nim, is impartial, this number $m$ is independent of whose turn it is to move. The same terminology applies to all the $n$-person short impartial games.

Thus for a player to achieve his highest possible rank, he should always try to move into a 0-position if possible. If no 0-position is available, he should try to move into a 1-position, and so on. We conclude this optimal strategy in the following.

*Theorem 1*: An $n$-person short impartial game is a 0-position if the only possible moves are all into $(n-1)$-positions; and it is a $q$-position, $n > q > 0$, if $q$ is the smallest number such that there is a move into a $(q-1)$-position.

*Corollary 1*: An $n$-person short impartial game is a $q$-position, $q > 0$, if $q$ is the smallest number such that the game can be moved into a 0-position within $q$ moves; and it is a 0-position if the game can not be moved into another 0-position within less than $n$ moves.

The strategy described above is by far an efficient algorithm. In fact, a direct application would require a computational complexity which grows exponentially with respect to the number of moves. A Nim type game is always by definition a compound game with components of trivial structures. For this kind of games, it is most desirable if their outcomes and optimum strategies can be described in terms of information about the component games. This is the goal for the remainder of this article. The next theorem provides an efficient criterion for determining whether an $n$-person Nim game is a 0-position. Because of Corollary 1, we need only to apply this theorem to all positions within $n-2$ moves from the initial position in order to determine the excat outcome of the game.

*Theorem 2*: Consider the $n$-person Nim game of $h$ heaps of sizes $c_1, c_2, \ldots, c_h$, respecitively. Express the $c$'s in binary notation and add them together without carry, and in the scale of $n$. Then the resulting $n$-ary number is 0 if and only if the game is a 0-position.

*Proof*: Let the game under consideration be denoted as $g$. Let the resulting $n$-ary number be $\Delta(g)$. If $\Delta(g) \neq 0$, let $\delta(g)$ be the leftmost nonzero digit of $\Delta(g)$. While if $\Delta(g) = 0$, define $\delta(g) = 0$. We need only to prove the following two statements.

A) If $\delta(g) = 0$, then $\delta(f) \neq 0$ for any position $f$ reachable within $n-1$ moves from $g$.

B) If $\delta(g) \neq 0$, then there exists a position $f$ reachable within $n-1$ moves from $g$ such that $\delta(f) = 0$.

To prove A), we consider two general positions $j$ and $k$ such that $j$ can be moved into $k$. If $\delta(j) \neq 0$, then clearly $\delta(k) \geqslant \delta(j) - 1$. While if $\delta(j) = 0$, then $\delta(k) = n-1$. These observations together with the induction prove A). The statement B) is a direct consequence of the following lemma.

*Lemma 1*: Let $n$ be a positive integer and $c_1, c_2, \ldots, c_h$ be non-negative integers. Take a sufficiently large number $t$ and express $c_i$ in the binary notations as $c_{i1}\, c_{i2} \ldots c_{it}$ for all $i$. Let $s$ be the smallest index such that $\sum_{i=1}^{h} c_{is}$ is not divisible by $n$. Assume that $c_{is}\, c_{i,s+1} \ldots c_{it} = 1\ 1 \ldots 1$ is true if and only if $i \leqslant k$; here $k$ is a non-negative integer not exceeding $h$. Then there exists non-negative integers $d_1, d_2, \ldots, d_h$ satisfying the following conditions.

1. $d_i \leqslant c_i$ for all $i = 1, \ldots, h$.
2. If $d_{i1}\, d_{i2} \ldots d_{it}$ is the binary expression of $d_i$, then $\sum_{i=1}^{h} d_{iu}$ is divisible by $n$ for $u = 1, \ldots, t$.
3. There exists a permutation $\pi$ on $\{1, \ldots, h\}$ which fixes $\{1, \ldots, k\}$ such that $d_{\pi(i)} = c_{\pi(i)}$ for all $i \geqslant n$.

*Proof*: The proof is by induction on $t - s$. Let $\delta$ be the remainder of $\sum_{i=1}^{h} c_{is}$ divided by $n$. Reordering $c_{k+1}, \ldots, c_h$ if necessary, we may assume that $c_{is} = 1$ for $i \leqslant \delta$. Define $\bar{c}_i$ to be the number with the binary expression $c_{i1}\, c_{i2} \ldots c_{i,s-1}\ 0\ \overbrace{1\ 1 \ldots 1}^{t-s}$ for $i \leqslant \delta$. Let $\bar{c}_i = c_i$ for $i > \delta$. Thus $\bar{c}_i \leqslant c_i$ for all $i$. Also let $\bar{c}_{i1}\, \bar{c}_{i2} \ldots \bar{c}_{it}$ be the binary expression of $\bar{c}_i$ for all $i$. If $\sum_{i=1}^{h} \bar{c}_{in}$ is divisible by $n$ for $u = 1, \ldots, t$, then the lemma can be proved by taking $d_i = \bar{c}_i$. So we assume this is not the case and let $\bar{s}$ be the smallest index such that $\sum_{i=1}^{h} \bar{c}_{is}$ is not divisible by $n$. Clearly $\bar{s} > s$. Also we know that $\bar{c}_{i\bar{s}}\bar{c}_{i,\bar{s}+1} \ldots \bar{c}_{it} = 1\ 1 \ldots 1$ for $i \leqslant k$. Reordering $\bar{c}_{k+1}, \ldots, \bar{c}_h$ if necessary, we may assume the existence of $\bar{k}$ such that $\bar{k} > k$ and that $\bar{c}_{i\bar{s}}\bar{c}_{i,\bar{s}+1} \ldots \bar{c}_{it} = 1\ 1 \ldots 1$ if and only if $i \leqslant \bar{k}$. The inequalities $\bar{c}_i \leqslant c_i$ for all $i$ and $\bar{k} \leqslant k$, together with the induction hypothesis on $t - s$ imply the existence of the numbers $d_1, d_2, \ldots, d_h$ as desired.

The above lemma provides an algorithm for finding a 0-position within $n - 1$ moves from any given nonzero position. The following is an illustrative example.

*Example*: Consider the 5-person Nim game of nine heaps of sizes 55, 31, 60, 20, 9, 55, 53, 53, and 4, respectively. We are to reduce the sizes of four or fewer heaps in order to arrive at a 0-position. First we express these nine numbers in binary notation and compute their $n$-ary sum without carry. The computation is done digit by digit, and from left to right until a nonzero digit, say $\delta$, in the sum is arrived at. Place a vertical line to the left of this digit (see the following table). Consider the nine binary numbers formed by those digits to the right of the vertical line. Select $\delta$ largest among them. Each selected number is then replaced by $01 \ldots 11^{+}$. Here $1^{+}$ means 1 as far as the addition is concerned, but should be regarded as infinitesimally greater when compared to 1.

By iterating the above process, we eventually arrive at the 0-position with heap sizes 53, 3, 34, 20, 2, 51, 53, 53, and 4.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 10111 | | 11 | 0111 | | 110 | 111 | | 11011 | 1 | | 110111 |
| | 11111 | → | 0 | $1111^+$ | → | 00 | $111^+$ | → | 0001 | $1^+$ | | $00011^+$ |
| 1 | 11100 | → | 10 | $1111^+$ | → | 100 | $111^+$ | → | 10001 | $1^+$ | → | 100010 |
| | 10111 | | 1 | 0100 | | 10 | 100 | | 1010 | 0 | | 10100 |
| | 1001 | | | 1001 | → | 0 | $111^+$ | → | 001 | $1^+$ | → | 0010 |
| 1 | 10111 | | 11 | 0111 | | 110 | 111 | → | 11001 | $1^+$ | | $110011^+$ |
| 1 | 10101 | | 11 | 0101 | | 110 | 101 | | 11010 | 1 | | 110101 |
| 1 | 10101 | | 11 | 0101 | | 110 | 101 | | 11010 | 1 | | 110101 |
| | 100 | | | 100 | | | 100 | | 10 | 0 | | 100 |
| 0 | 2 . . . . | | 00 | 3 . . . | | 000 | 4 . . | | 00000 | 2 | | 000000 |

Using the same notations as in the proof of Theorem 2, we have

*Corollary 2*: If $\delta(g) = 0$, then $g$ is a 0-position. If $\delta(g) = n - 1$, then $g$ is an $(n - 1)$-position. If $1 \leqslant \delta(g) \leqslant n - 2$, then $g$ is a $q$-position for some $q \geqslant \delta(g)$

## 5. *N*-person $\text{Nim}_k$

In this section we generalize the results in the last section to $\text{Nim}_k$. The rules for $n$-person $\text{Nim}_k$ are defined in the obvious way as an analogue of $n$-person Nim. The following theorem parallels Theorm 2.

*Theorem 3*: Consider the $n$-person Moore's game $\text{Nim}_k$ of $h$ heaps of size $c_1$, $c_2, \ldots, c_k$, respectively. Express the $c$'s in the binary notation and add them together without carry, and in the scale of $nk - k + 1$. Then the resulting number is 0 if and only if the game is a 0-position.

*Proof*: We shall use the same notations as in the proof of Theorem 2. Consider two general positions $j$ and $k$ of $n$-person $\text{Nim}_k$ such that $j$ can be moved into $k$. If $\delta(j) \neq 0$, then $\delta(k) \geqslant \delta(j) - k$. While if $\delta(j) = 0$, then $\delta(k) \geqslant (n - 2)k + 1$. With these two observations, the proof can proceed in exactly the same way as in the proof of Theorem 2.

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