m-Functions and Floquet Exponents for Linear Differential Systems (*).

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Sunto. – *Si definisce un esponente di Floquet per certe equazioni differenziali lineari non-periodiche, la parte immaginaria del quale rappresenta una «rotazione» delle soluzioni di dette equazioni. Inoltre si discute la relazione fra l'esponente di Floquet e le funzioni m di Weyl-Kodaira, e fra la rotazione e certi problemi spettrali.*

1. – Introduction.

The Floquet exponents of a periodic linear system

$$x' = y(t)x \qquad x \in \boldsymbol{C}^k \tag{1}$$

with, say, $y(t+T) = y(t)$, are obtained by taking logarithms of the eigenvalues of the period matrix $\Phi(T)$. One obtains a set of complex numbers $w_1, \ldots, w_k$, $w_j = \beta_j + i\alpha_j$, such that the real parts β_j measure exponential growth of certain solutions of (1), and the imaginary parts measure «rotation» (in some not-too-well defined sense) of those solutions.

It is an interesting problem to define Floquet exponents when $y(t)$ is not periodic. We are going to consider this question when $y(t)$ is «stationary ergodic» (see below) and satisfies a symmetry condition, i.e., belongs to an appropriate Lie algebra g. In this paper, g will always be the Lie algebra of a Lie group $\mathcal{G}$ which preserves a non-degenerate, indefinite Hermitean form ω on $\boldsymbol{C}^k$: $\omega(x, y) = \langle x, Jy \rangle$, where $\langle\ ,\ \rangle$ is the Euclidean inner product on $\boldsymbol{C}^k$ and the non-singular matrix satisfies $J^* = -J$. For example, J might be $\begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$ with $1_n = n \times n$ identity matrix, and g might be $\mathrm{sp}\,(n, \boldsymbol{R}) = \{A : \boldsymbol{R}^{2n} \to \boldsymbol{R}^{2n} | A^*J + JA = 0\}$ = algebra of real $2n \times 2n$ infinitesimally symplectic matrices.

We will be led to study the Weyl-Kodaira m-functions [46, 32] $m_+(\lambda)$, $m_-(\lambda)$ of the family of differential equations

$$J\left(\frac{d}{dt} - y(t)\right)x = \lambda\gamma(t)x \qquad x \in \boldsymbol{C}^k\,,\ \lambda \in \boldsymbol{C}\,, \tag{2$_\lambda$}$$

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where $\gamma^*(t) = \gamma(t) \geqq 0$ [3, Chpt. 9]. Using the m-functions, we will define a function $w = w(\lambda)$ (Im $\lambda \geqq 0$) which has properties related to those of the usual Floquet exponents. This function w in turn can be used to study the spectral problem $(2)_\lambda$.

Some observations are in order.

(i) We obtain one (not k) Floquet exponents w for equation (1). Our methods indicate how one might define others; however, there is as yet no general technique for doing so.

(ii) The appearance of the parameter λ is not an accident. The significance and utility of w only become apparent when λ is introduced. In general, it is a good idea to study (1) from this point of view: embed it in a one (or more)-parameter family $(2)_\lambda$, and consider quantities related to this family.

(iii) In the body of the paper, we will let $g = u(p, q)$ $(p \leqq q)$, the Lie algebra of the Lie group $U(p, q)$ of matrices preserving the skew-form $\omega_0(x_1, x_2) = \langle x_1, J_0 x_2\rangle$ with $J_0 = i\begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix}$. Here 1_p resp. 1_q is the $p \times p$ resp. $q \times q$ identity matrix. Explicitly, $u(p, q) = \{A: \mathbf{C}^k \to \mathbf{C}^k | k = p + q, A^* J_0 + J_0 A = 0\}$. As is well-known, any spectral problem $(2)_\lambda$ may be transformed into one with $J = J_0$ by a constant charge of variables $x = Bz$ (the proof is repeated below). This holds in particular if $J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$ and $y(t)$ is infinitesimally symplectic.

(iv) Finally, we will find it very convenient that $u(p, q)$ (or rather the semisimple algebra su $(p, q) = \{A \in u(p, q)$: trace $A = 0\}$ is the Lie algebra of the isometry group of a bounded (Cartan) symmetric domain D. In fact, the m-functions $m_\pm$ take values in such a domain. However, the presence of D is not crucial, and it will be clear that one can define analogues of the m-functions in more general circumstances.

Before discussing our results in more detail, it seems appropriate to outline previous work on m-functions and Floquet exponents, and to put the present paper in perspective.

First a quick review of the long history of the Weyl-Kodaira functions; we apologize for its sketchy and superficial nature. H. Weyl introduced his m-functions for the Sturm-Liouville operator

$$(3) \qquad (p\varphi')' + \varphi = \lambda\varphi \qquad p, q \text{ real}, \ \lambda \in \mathbf{C}$$

in 1909 [46]; his paper retains a fresh and original quality to this day. TITCHMARSH [45] mode a systematic application of the m-functions and their function theory to the spectral problem (3). KODAIRA [32] defined quantities closely related to the m-functions for higher-dimensional symmetric differential operators; he adopted a geometric point of view. Later authors, including ATKINSON [3], EVERITT and EVERITT-KUMAR [17, 18] and HINTON-SHAW [23, 24], refined and extended Kodaira's work, using analytical methods. They used the m-functions and the closely related « characteristic function » to study self-adjoint boundary value problems corresponding to $(2)_\lambda$.

In this paper (§ 3), we construct ab initio the m-functions for $(2)_\lambda$ when $y(t)$ is stationary ergodic. We have tried to combine the geometric insights of Kodaira with the analytical convenience aimed at by later authors. To this end, we rely heavily on the theory of exponential dichotomy (COPPEL [9], SACKER-SELL [39, 40], SELGRADE [42]). We will show that a stationary ergodic $y(t)$ is in the limit-point case at $t = \pm\infty$, and will identify the quantities $m_\pm$ as elements of a bounded symmetric domain. The domain « collapses » to the m-function [32, 3, 18].

Floquet exponents in the sense of this paper have only been considered in the last few years. After anticipatory papers by PASTUR [37] and THOULESS [44], JOHNSON-MOSER [29] introduced and studied the function $w(\lambda)$ for the almost periodic Schrödinger equation

$$\left(\frac{-d^2}{dt^2} + q(t)\right)\varphi = \lambda\varphi \qquad q \text{ real}, \ \lambda \in \boldsymbol{C}\,. \tag{4}$$

In fact the present paper grew out of an attempt to understand the « complex rotation » considered in [29]. Avron-Simon [4] considered the Floquet exponent for the difference analogue of (4):

$$x_{m+1} + x_{m-1} + V(m)x_m = \lambda x_m\,. \tag{5}$$

GIACHETTI-JOHNSON [20] treated $w(\lambda)$ for the AKNS operator [1]:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left[\frac{d}{dt} - y(t)\right]x = \lambda x\,, \qquad y(t) \text{ real}\,,\ \operatorname{tr} y(t) \equiv 0\,;$$

in [20], they also considered the non-self-adjoint problem when $y(t) \in \mathrm{sl}\,(2, \boldsymbol{C})$. KOTANI [33] showed that $w(\lambda)$ determines the absolutely continuous spectrum of the Schrödinger operator $-d^2/dt^2 + q(t)$. Moser [34] used w in his book relating the finite-band Schrödinger potentials $q(t)$ to the classical Neumann problem. DE CONCINI and JOHNSON [12] used it in characterizing the finite-band AKNS potentials $y(t)$.

Finally, CRAIG-SIMON [11] studied the symplectic difference equation obtained by letting $V(m)$ in (5) be an $n\times n$ symmetric matrix. They consider a quantity completely analogous to the $w(\lambda)$ of the present paper. The contributions of the present paper might be summarized as follows: (i) a more general framework; (ii) a detailed study of the relation between w and the m-functions (§ 4); and (iii) a geometric approach to the study of w, which complements the analytic style of [11]. In particular we clarify the notion of rotation in higher dimensions (§ 2), and relate it to the density of states (§ 5) and, for symplectic (Hamiltonian) systems, to the Arnold-Maslov index ([2]; see § 2). We also prove a « gap-labelling » theorem [5, 27].

It is time to describe $w(\lambda)$ and the m-functions more precisely. Suppose for the moment that $J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$ and that $g = \mathrm{sp}\,(n, \boldsymbol{R})$. Thus $(*)_\lambda$ is a Hamiltonian spectral problem.

If λ is real, the complex number $w(\lambda) = \beta(\lambda) + i\alpha(\lambda)$ is defined as follows. Let $\Phi(t) = \Phi_\lambda(t)$ be the fundamental matrix solution of $(2)_\lambda$ such that $\Phi(0) = I$. Letting Λ^n denote the n-th wedge product [19], we define

$$\beta(\lambda) = \lim \frac{1}{t} \ln |\Lambda^n \Phi(t)| \,.$$

Thus $\beta(\lambda)$ is a Lyapounov exponent. As for $\alpha(\lambda)$, let $\mathfrak{L}$ be the set of Lagrange subspaces of $\boldsymbol{R}^{2n}$; thus $l \in \mathfrak{L} \Leftrightarrow l \subset \boldsymbol{R}^{2n}$ is an n-dimensional subspace such that $\langle x_1, Jx_2 \rangle = 0$ for all $x_1, x_2 \in l$. Fix $l_0 \in L$, say $l_0 = [e_1, \ldots, e_n]$, the subspace spanned by the first n unit vectors. Let C be the Maslov cycle: $C = \{l \in \mathfrak{L}: l \cap l_0 \geqq 1\}$. Then $C \subset \mathfrak{L}$ has codimension one. Now if $\bar{l} \in \mathfrak{L}$, then so is $\Phi(t)\bar{l}$, since $\Phi(t)$ is symplectic. Consider the number $n(t)$ of oriented intersections of the curve $s \to \Phi(s)\bar{l}$ with C for $0 \leqq s \leqq t$. Then

$$\alpha(\lambda) = \lim_{t \to \infty} \frac{n(t)}{t} \,.$$

Thus $\alpha(\lambda)$ is a rotation number. It is clearly related to the Arnold-Maslov index (BOTT [6], ARNOLD [2], DUISTERMAAT [13]).

An obvious problem with these definitions is that, in general, the limits need not exist. It is at this point that we use the fact that y is stationary ergodic, i.e., is a typical path of a stationary ergodic process. We use the Birkhoff ergodic theorem [35] to show that the limits exist for almost all y.

A remarkable and useful property of $w(\lambda)$ is that it admits a holomorphic extension (also called $w(\lambda)$) into the upper half-plane $\mathrm{Im}\,\lambda > 0$. We will see that this extension is intimately related to the Weyl-Kodaira functions $m_\pm(\lambda)$, which we now describe. Let M_n^s be the set of symmetric, $n \times n$ complex matrices. Let $H_s = \{m \in M_n^s: \mathrm{Im}\, m > 0\}$; thus H_s is the Siegel upper half-plane [43], and is one of the Cartan bounded symmetric domains. Observe that M_n^r parametrizes an open dense subset U of the set L^c of *complex* Lagrange planes in $\boldsymbol{C}^{2n}$, In fact, $l \in U \Leftrightarrow l$ has a basis of column vectors of the form $\begin{pmatrix} 1_n \\ m \end{pmatrix}$, where $m \in M_n^s$.

Relying heavily on results of SACKER-SELL ([39, 40]; see also SELGRADE [42]), we will show that, if $\mathrm{Im}\,\lambda \neq 0$, then equation (2) has *exponential dichotomy* (*ED* for short). This means that $\boldsymbol{C}^{2n} = V^s + V^u$, where solutions of $(2)_\lambda$ with initial conditions in $V^s(V^u)$ tend to zero esponentially as $t \to \infty$ $(t \to -\infty)$. We will see that V^s lies in U if $\operatorname{Im} \lambda > 0$; let $\begin{pmatrix} 1_n \\ m_+(\lambda) \end{pmatrix}$ be its representation. It turns out that $m_+(\lambda) \in H_s$; the map $\lambda \to m_+(\lambda)$ is one of the Weyl-Kodaira functions.

The connection of $w(\lambda)$ with m_+ is the following. Write $\lambda J^{-1}\gamma + y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; then $m \in M_n^s$ satisfies the Riccati equation

$$m' = -mbm + dm - ma + c\,. \tag{6}$$

Linearize this equation around $\hat{m}_+(t) = \Phi(t)\cdot m_+(\lambda) =$ solution of (6) with initial condition $m_+(\lambda)$:

$$(\delta m)' = f_+(\hat{m}_+(t))\,\delta m\,. \tag{7}$$

Then

$$w(\lambda) = \lim_{t\to\infty} \frac{1}{2n}\cdot\frac{1}{t}\int_0^t \operatorname{tr} f_+(\hat{m}_+(s))\,ds\,, \qquad \operatorname{tr} = \text{trace}\,.$$

Thus by Liouvilles formula, $w(\lambda)$ is the average of the logarithm of the determinant of the fundamental matrix solution of (7).

There is a similar formula relating $w(\lambda)$ and $m_-(\lambda)$. The starting point is the observation that V^u has a parametrization $\begin{pmatrix} m_-(\lambda) \\ 1_n \end{pmatrix}$ where $m_-(\lambda) \in S$.

We finish this introduction by discussing terminology and some basic results. First let $\mathrm{gl}\,(k, \boldsymbol{C})$ be the Lie algebra of all $k\times k$ complex matrices. Let $g \subset \mathrm{gl}\,(k, \boldsymbol{C})$ be a *real* Lie subalgebra, and let $\xi_g = \left\{y\colon \boldsymbol{R} \to g \,|\, \sup_t \int_t^{t+1} |y(s)|\,ds < \infty\right\}$, where $|\cdot|$ is the Euclidean norm on g. We give ξ_g the distribution topology: $y_n \to y$ in ξ_g iff $\int_{\infty-}^{\infty} y_n\varphi\,ds \to \int_{-\infty}^{\infty} y\varphi\,ds$ for all $\varphi \in C_c^\infty(\boldsymbol{R}) =$ set of C^∞ real functions on $\boldsymbol{R}$ with compact support. Let $\tau\colon \xi_g \times \boldsymbol{R} \to \xi_g$ be the *translation flow* defined by $\tau(y, t)(s) = y(t + s)$. We usually write $\tau_t(y)$ for $\tau(y, t)$. For any bounded subset $B \subset \xi_g$ (i.e., there exists $K > 0$ such that $\sup_t \int_t^{t+1} |y(s)|\,ds \leqq K$ for all $y \in B$), the restriction $\tau\colon B\times\boldsymbol{R} \to B$ is jointly continuous.

Next let $Y \subset \xi_g$ be a bounded translation-invariant subset (i.e., $\tau_t(Y) \subset Y$ for all $t \in \boldsymbol{R}$). Suppose further that

$$\lim_{\varepsilon\to 0} \sup_t \int_t^{t+\varepsilon} |y(s)|\,ds = 0 \quad \text{uniformly in } y \in Y\,. \tag{8}$$

This condition holds if, for instance, $\text{ess}_t \sup |y(t)| \leqq K < \infty$ for all $y \in Y$. Then Y is compact metric in the distribution topology. Finally, let μ be an *ergodic measure* on Y [35] such that $\mu(W) > 0$ for each open $W \subset Y$. (Recall that a Radon probability measure on Y is ergodic if (i) $\mu(\tau_t(B)) = \mu(B)$ for each Borel set $B \subset Y$; i.e., μ is invariant; (ii) $\mu(\tau_t(B) \Delta B) = 0$ $(t \in \boldsymbol{R})$ implies either $\mu(B) = 0$ or $\mu(B) = 1$).

1.1 DEFINITION. – A triple (Y, τ, μ) as just described is (in this paper) a *stationary ergodic process*.

We will need two lemmas, the first of which is a simple consequence of ergodicity of μ and the Birkhoff ergodic theorem [35].

1.2 LEMMA. – For μ - a.a.y, $\{\tau_t(y) \colon (t > 0)\}$ and $\{\tau_t(y) \colon t < 0\}$ are dense in Y.

The second lemma produces an « evaluation function » e: $Y \to g \colon y \to y(0)$. Since ξ_g consists of equivalence classes of functions, it is not clear how e should be defined. Nevertheless,

1.3 LEMMA. – There exists $e \in L^1(Y, g, \mu)$ such that, for μ - a.a. $y \in Y$:

(i) the function $t \to e(\tau_t(y))$ is defined and equals $y(t)$ for a.a. $t \in \boldsymbol{R}$;

(ii) $\displaystyle \frac{1}{t}\int_0^t y(s)\,ds = \frac{1}{t}\int_0^t e(\tau_s(y))\,ds \to \int_Y e(y)\,d\mu(y)$ as $t \to \pm\infty$.

PROOF. – Though the proof is standard, we give the details. Note first that (ii) follows from (i) and the Birkhoff ergodic theorem, so it suffices to prove (i).

Define $f_n \colon \boldsymbol{R} \times Y \to \boldsymbol{R} \colon (t, y) \to n\int_t^{t+1/n} y(s)\,ds$. Then f_n is continuous. Using Fubini's theorem, we see that $f(t, y) \overset{\text{def}}{=} \lim_{n\to\infty} f_n(t, y)$ exists for $m \times y$ - a.a. (t, y) (m = Lebesgue measure on $\boldsymbol{R}$). Thus we can find $t_0 \in \boldsymbol{R}$ such that $f(-t_0, y)$ is defined μ - a.e. and is μ-measurable. Since τ is continuous and μ is invariant, the function

$$e(y) = f(-t_0, \tau_{t_0}(y)) = \lim_{n\to\infty} n\int_0^{1/n} y(s)\,ds$$

is defined μ - a.e. and is μ-measurable. Clearly $e(\tau_t(y)) = \lim_{n\to\infty} n\int_t^{t+1/n} y(s)\,ds = y(t)$ for m - a.a.t.

To prove that $|e| \in L^1(Y, \mu)$, note that for μ - a.a. y,

$$\frac{1}{t}\int_0^t |e(\tau_s(y))|\,ds \leqq \frac{1}{t}\int_0^t ds \lim_{n\to\infty} n\int_s^{s+1/n} |y(u)|\,du = \frac{1}{t}\int_0^t |y(s)|\,ds \leqq K < \infty,$$

independent of $y \in Y$. Let $B_n = \{y \in Y: n \leqq |e(y)| < n+1\}$ $(n \geqq 0)$. Then $|e| \in L^1(Y, \mu)$ iff $\sum_{n=1}^{\infty} n\mu(B_n) < \infty$. Given $\varepsilon > 0$ and an integer $N > 1$, choose T so large that

$$t \geqq T \Rightarrow \left|\frac{1}{t}\int_0^t \chi_n(\tau_s(y))\,ds - \mu(B_n)\right| < \frac{\varepsilon}{nN}, \qquad n = 1, 2, \ldots, N.$$

Here χ_n is the characteristic function of B_n. Such a $T = T(y)$ can be found for μ - a.a. $y \in H$, by the Birkhoff theorem. Then for $t \geqq T$:

$$K \geqq \frac{1}{t}\int_0^t |e_0(\tau_s(y))|\,ds \geqq \sum_{n=1}^{N} n\cdot\frac{1}{t}\int_0^t \chi_n(\tau_s(y))\,ds \geqq \sum_{n=1}^{N} n\mu(B_n) - \varepsilon\,.$$

This completes the proof.

1.4 NOTATION. – We will write $\int_Y \operatorname{tr} y\,d\mu(y) = \int_Y \operatorname{tr} e(y)\,d\mu(y)$.

Now let $\langle\,,\,\rangle$ be the Euclidean inner product on C^k, and let J be a non-singular $k \times k$ matrix such that $J^* = -J$.

1.5 LEMMA. – There is a non-singular matrix B such that $B^*JB = J_0 = \begin{pmatrix} -i1_p & 0 \\ 0 & i1_q \end{pmatrix}$ where $K = p + q$.

PROOF. – First diagonalize J by means of a unitary matrix u_1, then permute the basis elements of $\boldsymbol{C}^k$ with an appropriate u_2, finally choose an appropriate diagonal matrix d with positive diagonal entries and let $B = u_1 u_2 d$.

Next let $g_J = \{A \in \mathrm{gl}(K, \boldsymbol{C}): A^*J = -JA\}$. Then g_J is a real Lie subalgebra of gl $(K, \boldsymbol{C})$. There is a $1-1$ correspondence between elements of g_J and Hermitean matrices A_1: namely A_1 is Hermitean iff $J^{-1}A_1 \in g_J$.

Let Y be a stationary ergodic process with values in g_J. Consider the following family of ordinary differential equations:

$$(2)_{y,\lambda} \qquad x' = [\lambda J^{-1}\gamma_y(t) + y(t)]x \qquad x \in \boldsymbol{C}^k, \quad \lambda \in \boldsymbol{C}, \quad y \in Y.$$

We make the following

1.6 ASSUMPTIONS. – (i) iJ has at least one positive and one negative eigenvalue;

(ii) there is a continuous function γ: $Y \to \mathrm{gl}(K, \boldsymbol{C})$ such that $\gamma^*(y) = \gamma(y)$ and $\gamma(y) \geqq 0$ $(y \in Y)$, and $\gamma_y(t) = \gamma(\tau_t(y))$ $(y \in Y,\ t \in \boldsymbol{R})$;

(ii) given $y \in Y$ and $\lambda \in \boldsymbol{C}$ with $\operatorname{Im}\lambda \neq 0$, there exists a constant $C = C(y, \lambda)$ such that, if $x(t)$ is a non-zero solution of $(2)_{y,\lambda}$, then

$$\int_{-\infty}^{\infty} \langle x(t), x(t)\rangle\,dt \leqq C\int_{-\infty}^{\infty} \langle \gamma_y(t)\cdot x(t), x(t)\rangle\,dt\,.$$

Note that the last condition strengthens somewhat the one imposed by Atkinson [3, Chapt. 9].

Most any spectral problem defined by an ordinary differential operator can be put in the form $(2)_{y,\lambda}$. For example, let $L\varphi = -\varphi'' + q(t)\varphi = \lambda\varphi$ where $\varphi \in \boldsymbol{C}^n$ and $q(t)$ is real $n\times n$ and symmetric. Letting $\gamma_g(t) = \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}$, $y(t) = \begin{pmatrix} 0 & 1_n \\ q(t) & 0 \end{pmatrix}$, $J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$, we see that $L\varphi = \lambda\varphi$ is equivalent to $(2)_{y,\lambda}$ with $x = \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}$. For another example, let $J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$, $y(t) = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ with $\operatorname{Re} a = 0$, $\gamma_g(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$: We obtain the (two-dimensional) AKNS spectral problem [1].

Now make the change of variables $x = Bz$, where B is as in Lemma 1.5. We obtain

$$z' = [\lambda J_0^{-1}(B^*\gamma_g(t)B) + B^{-1}y(t)B]z\,.$$

Furthermore, replacing t by $-t$ if necessary, we can assume that $p \leqq q$. With these remarks in mind, we make the

1.7 CONVENTION. – Unless otherwise specified, we assume that $J = J_0 = \begin{pmatrix} -i1_p & 0 \\ 0 & i1_q \end{pmatrix}$ with $p \leqq q$ in equations $(2)_{y,\lambda}$. Hence, unless otherwise specified,

$$g = g_{J_0} = \{A \in \mathrm{gl}\,(K, \boldsymbol{C})\colon A^*J_0 + J_0A = 0\} = u(p, q).$$

1.8 REMARK. – Observe that $\mathrm{sp}\,(n, \boldsymbol{R})$ can be embedded in $\mathrm{su}\,(n, n) \subset u(n, n)$ via the map $A \to u_1Au_1^{-1}$, where $u_1 = \begin{pmatrix} i1_n & i1_n \\ -1_n & 1_n \end{pmatrix}$. See, e.g. [43, p. 124].

1.9 TERMINOLOGY. – We collect here some standard terms from topological dynamics. Let X be a space. A *flow* on X is a continuous map $\tau\colon X\times\boldsymbol{R} \to X\colon (x, t) \to \tau_t(x)$ such that: (i) $\tau_0(x) = x$; (ii) $\tau_t\circ\tau_s = \tau_{t+s}$ $(x \in X,\ t, s \in \boldsymbol{R})$. If $x \in X$, then the *orbit* through x is $\{\tau_t(x)\colon t \in \boldsymbol{R}\}$. The *ω-limit set* $\omega(x) = \Big\{\bar{x} = \lim_{n\to\infty} \tau_{t_n}(x)$ for a sequence $t_n \to \infty\Big\}$. The *α-limit set* $\alpha(x)$ is defined similarly, except $t_n \to \infty$ becomes $t_n \to -\infty$. Both $\omega(x)$ and $\alpha(x)$ are *invariant*, i.e., $\tau_t(\omega(x)) \subset \omega(x)$, $\tau_t(\alpha(x)) \subset \alpha(x)$ for all $t \in \boldsymbol{R}$. If X is compact, then X is *minimal* if every orbit is dense in X. Let $\mathcal{G}$ be a topological group. A *continuous* map $\Phi\colon X\times\boldsymbol{R} \to \mathcal{G}$ is a *cocycle* if

(i) $\Phi(x, 0) = idy$;

(ii) $\Phi(x, t + s) = \Phi(\tau_t(x), s)\cdot\Phi(x, t)$ $(x \in X;\ t, s \in \boldsymbol{R})$.

See ELLIS [16].

We end this introduction by recalling the definition of exponential dichotomy [9, 39]. Fix $\lambda \in \boldsymbol{C}$. Let $\Phi_y(t)$ be the fundamental matrix solution of $(2)_{y,\lambda}$ such that $\Phi_y(0) = I$. It is easy to see that $\Phi\colon Y \times \boldsymbol{R} \to U(p, q) = \{B \in GL(n, \boldsymbol{C})\colon \langle Bx_1, J_0 Bx_2\rangle = \langle x_1, J_0 x_2\rangle$ for all $x_1, x_2 \in \boldsymbol{C}^k\}$ is a cocycle in the sense of 1.9. Also the map $\hat{\tau}\colon Y \times \boldsymbol{C}^k \times \boldsymbol{R} \to Y \times \boldsymbol{C}^k\colon (y, x, t) \to (\tau_t(y), \Phi_y(t)x)$ defines a flow on $Y \times \boldsymbol{C}^k$.

1.10 DEFINITION. – Fix $\lambda \in \boldsymbol{C}$. We say that equations $(2)_{y,\lambda}$ have *exponential dichotomy* (ED) if there are continuous vector subbundles $V^s, V^u \subset Y \times \boldsymbol{C}^k$ such that:

(i) $V^s \oplus V^u = Y \times \boldsymbol{C}^k$;

(ii) V^s, V^u are invariant (with respect to $\hat{\tau}$);

(iii) there are constants $K > 0$, $\alpha > 0$ such that, if $(y, x_0) \in V^s$, then $|\Phi_y(t)x_0| \leqq Ke^{-\alpha t}|x_0|$ $(t > 0)$, and if $(y, x_0) \in V^u$, then $|\Phi_y(t)x_0| \leqq Ke^{\alpha t}|x_0|$ $(t < 0)$. Here $|\cdot|$ is the Euclidean norm on $\boldsymbol{C}^k$.

2. – $w(\lambda)$ for real λ.

In this section we define the Floquet exponent $w(\lambda)$ when λ is real. We write $w(\lambda) = \beta(\lambda) + i\alpha(\lambda)$, and consider β and α separately. Following 1.7, we let $J = J_0 = \begin{pmatrix} -i1_p & 0 \\ 0 & i1_q \end{pmatrix}$, $g = u(p, q)$, and we suppose $0 < p \leqq q < p + q = k$.

2.1 DEFINITION. – Let $\Phi_y(t)$ be the fundamental matrix solution of $(2)_{y,\lambda}$ with $\Phi_y(0) = I$. Define

$$\beta = \beta(\lambda) = \lim_{t\to\infty} \frac{1}{t} \ln |\Lambda^q \Phi_y(t)|\,.$$

It is not immediately clear that $\beta(\lambda)$ is well-defined; however, by the theorem of Oseledec [36]:

2.2 THEOREM. – For each $\lambda \in \boldsymbol{R}$, the limit in 2.1 exists and is independent of y for μ - a.a. $y \in Y$.

2.3 REMARK. – We can also write $\beta(\lambda) = \lim_{t\to\infty} (1/t) \ln |\Lambda^p \tilde{\Phi}_y(t)|$. The reason is as follows. Let $\beta_1 \geqq \ldots \geqq \beta_k$ be the Lyapounov numbers of $(2)_{y,\lambda}$, counted with multiplicities [36]. Then $\beta(\lambda) = \sum_{i=1}^{q} \beta_i$ for μ - a.a. y. Now $\lim_{t\to\infty} (1/t) \ln |\Lambda^p \Phi_y(t)|$ equals $-\sum_{i=q+1}^{k} \beta_i$ However, $\tilde{\Phi}_y(t) \in U(p, q) \Rightarrow |\det \Phi_y(t)| = 1$. Using Liouvilles formula and the regularity [7, 36] of $(2)_{y,\lambda}$ for μ - a.a. y, we see that $\sum_{i=1}^{q} \beta_i = -\sum_{i=q+1}^{k} \beta_i$.

We turn to the rotation number α. Though one can give a purely geometric definition of this quantity, it is convenient to choose another starting point and then derive its geometric properties.

We introduce the space M_{pq} of $q\times p$ complex matrices m. This space parametrizes an open dense subset of the manifold $\mathfrak{S}_p$ of complex p-dimensional subspaces of C^k. In fact, M_{pq} parametrizes those $l\in\mathfrak{S}_p$ which have a basis of the form $\begin{pmatrix} e_1\\ m_1\end{pmatrix}, \dots, \begin{pmatrix} e_p\\ m_p\end{pmatrix}$, where $\{e_1, \dots, e_p\}$ is the standard basis in C^p and $m_1, \dots, m_p \in C^q$. If such a basis for l exists, then $m = (m_1, \dots, m_p)$ is the corresponding element of M_{pq} The components of m are the « Plücker coordinates » of l.

In M_{pq} consider the set $D = \{m\in M_{pq}\colon 1_p - m^t m > 0$, i.e., is positive definite$\}$. This set is an analogue of the unit disc, and reduces to it if $p = q = 1$. Its boundary ∂D consists of points m for which $1_p - m^t m$ is positive semi-definite. The set D is a Cartan symmetric domain [21]:

Let $U(p, q)$ be the (real) Lie group of complex $k\times k$ matrices preserving the form $\omega_0(x_1, x_2) = \langle x_1, J_0 x_2\rangle$; thus $U(p, q)$ has Lie algebra $u(p, q)$. Note that $U(p, q)$ acts on M_{qp} in the following way: if $\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in U(p, q)$, then the action is $m\to$ $\to (C + Dm)(A + Bm)^{-1}$. This action is induced by the linear action of $U(p, q)$ on $\mathfrak{S}_p$:

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix}\begin{pmatrix} 1_p\\ m\end{pmatrix} = \begin{pmatrix} A + Bm\\ C + Dm\end{pmatrix} \to \begin{pmatrix} 1_p\\ (C + Dm)(A + Bm)^{-1}\end{pmatrix}.$$

Observe that $U(p, q)$ preserves D and ∂D [21]. In particular, if $m\in\bar{D}$, then $(A + Bm)^{-1}$ exists.

Next we introduce a decomposition (Iwasawa decomposition) of $u(p, q)$. Define Lie subalgebras $t_0, a_0, n_0 \in u(p, q)$ as follows:

$$t_0 = \left\{\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix} \middle| a\in u(p)\,,\ d\in u(q)\ \text{(thus } a^* = -a,\ d^* = -d)\right\},$$

$$a_0 = \left\{H^{-1}\sigma H \middle| \sigma = \begin{pmatrix} \begin{pmatrix} t_1 & & 0\\ & \ddots & \\ 0 & & t_p\end{pmatrix} & 0 & 0\\ 0 & 0_{q-p} & 0\\ 0 & 0 & \begin{pmatrix} t_1 & & 0\\ & \ddots & \\ 0 & & t_p\end{pmatrix}\end{pmatrix}\right\};\ t_1, \dots, t_p \in \Theta\,,$$

$$n_0 = \left\{H^{-1}\sigma H \middle| \sigma = \begin{pmatrix} d_1 & \alpha & \beta\\ 0 & 0 & \gamma\\ 0 & 0 & -d_1^*\end{pmatrix}\right\},$$

where d_1 is $p\times p$ upper triangular with zero diagonal, and α, β, γ are arbitrary complex matrices of appropriate sizes. Here the matrix H is defined by

$$H=\begin{pmatrix} 1_p & 0 & 1_p \\ 0 & 1_{q-p} & 0 \\ -1_p & 0 & 1_p \end{pmatrix}. \tag{9}$$

The algebras t_0, a_0, n_0 are compact, abelian, and nilpotent respectively.

There is a corresponding decomposition $U(p, q) = K_0A_0N_0$; K_0, A_0, N_0 are the Lie subgroups of $U(p, q)$ corresponding to t_0, a_0, n_0 respectively. (In the case at hand, $K_0 = \exp t_0$, $A_0 = \exp a_0$, $N_0 = \exp n_0$). That is, each $v \in U(p, q)$ decomposes uniquely in the form $v = uan$ $(u \in K_0,\ a \in A_0,\ n \in N_0)$, and the decomposition defines a C^∞ diffeomorphism of $K_0A_0N_0$ onto $U(p, q)$. Let $S_0 = A_0N_0$. Then S_0 is a closed subgroup of $U(p, q)$, and each $v \in U(p, q)$ decomposes uniquely in the form $v = us$ $(u \in K_0,\ s \in S_0)$. This decomposition (which is the one we will see later) defines a C^∞ diffeomorphism of K_0S_0 onto $U(p, q)$. See [21, Chpt. 6].

The decomposition $U(p, q) = K_0A_0N_0$ is the Iwasawa decomposition of $U(p, q)$ [25, 21]. It is the analogue for $U(p, q)$ of the Gram-Schmidt decomposition of $GL(n, \mathbf{R})$, used in [30] to prove the Oseledec theorem.

2.4 REMARKS. – (i) Observe that the point $m^* = \begin{pmatrix} 0_{q-p,p} \\ 1_p \end{pmatrix}$ is preserved by each $s \in S_0$: $sm_* = m_*$. This is easily seen by noting that $H^{-1}\begin{pmatrix} 1_p \\ m^* \end{pmatrix} = \begin{pmatrix} 1_p \\ 0 \end{pmatrix}$ and using the description of a_0 and n_0.

(ii) The action of K_0 on M_{qp} is *linear*: if $u = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \in K_0$, then $u\cdot\begin{pmatrix} 1_p \\ m \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 m \end{pmatrix}$, hence $u\cdot m = u_2 m u_1^{-1}$. This is a special case of a general fact about Lie algebras. Set $p_0 = \left\{\begin{pmatrix} 0 & \overline{m}^t \\ m & 0 \end{pmatrix}\middle|\, m \in M_{qp}\right\} \subset u(p, q)$. Then $u(p, q) = t_0 + p_0$ is a Cartan decomposition [21]. The map $m \to u\cdot m$ coincides with the adjoint map Ad_u: $p_0 \to p_0$: $p \to upu^{-1}$. See [21, Chpt. VIII].

With these preliminaries out of the way, we can define α. The idea is that α should be the average «rotation» due to the action of $\Phi_y(t)$ on M_{qp}. We expect that, if $\Phi_y(t) = u_t T_t$ with $u_t \in K_0$, $T_t \in S_0$, then α should depend only on u_t.

For $t \in \mathbf{R}$ and $m_0 \in M_{pq}$, let $d_{m_0}\Phi_y(t)$ be the Frechet derivative at m_0 of the map $m \to \Phi_y(t)m$. Then (for small t) $d_{m_0}\Phi_y(t)$ is a non-singular linear map of M_{qp} to itself.

2.5 DEFINITION. – Let $m_0 \in \overline{D}$. Define

$$\alpha = \alpha(\lambda) = \lim_{t\to\infty} \frac{1}{p+q}\,\frac{1}{t}\,\mathrm{Im}\ln\det d_{m_0}\Phi_y(t)\,,$$

where we take any continuous branch of the logarithm. (Note that, since $\Phi_y(t)$ preserves $\bar{D}$, $d_{m_0}\Phi_y(t)$ is defined for all $t \in \boldsymbol{R}$).

We must show that α is well-defined and depends only on u_t.

To begin, let $u_0 \in K_0$. We factor $\Phi_y(t)u_0 = u_y(t)T(y, u_0, t)$, where $u_y(t) \in K_0$ and $T(y, u_0, t) \in S_0$. We further write $u_y(t) \equiv u_y(t, u_0) = u(y, u_0, t)u_0$: Using uniqueness in the Iwasawa decomposition, it is easily shown that: (i) the map $(y, u_0, t) \to u_y(t)$ defines a flow on $Y \times K_0$; (ii) the maps u: $Y \times K_0 \times \boldsymbol{R} \to K_0$ and T: $Y \times K_0 \times \boldsymbol{R} \to S_0$ are cocycles with respect to this flow (see 1.9 for definitions). In fact,

$$u_y(t+s)T(y, u_0, t+s) = \Phi_y(t+s)u_0 =$$
$$= \Phi_{\tau_t(y)}(s)u_y(t)T(y, u_0, t) = u_{\tau_t(y)}(s)T\big((\tau_y(y), u_y(t), s\big)T(y, u_0, t),$$

and statements (i) and (ii) follow.

Let $u_0 \in K_0$, and write $\Phi_y(t)u_0 = u_t T_t$ with $T_t = T(y, u_0, t)$. Then $d_{m_0}\Phi_y(t)u_0 = u_t d_{m_0} T_t$, where we use 2.5 (ii). We show now that $d_{m_0}T_t$ does not contribute to the rotation number.

2.6 PROPOSITION. – Let $m_0 \in \bar{D}$. Then Im ln det $d_{m_0}T_t$ is uniformly bounded, where ln is any continuous branch of the logarithm.

This proposition is a corollary of a stronger one.

2.7 PROPOSITION, – There is a continuous map σ: $S_0 \times D \to \boldsymbol{C}$ such that:

(i) $\exp \sigma(T, m_0) = \det d_{m_0}T$;

(ii) $|\text{Im}\, \sigma(T, m)| < \pi p(p+q)$ $(T \in S_0,\ m \in D)$;

(iii) $m \to \sigma(T, m)$: $D \to \boldsymbol{C}$ is holomorphic $(T \in S_0)$.

One derives 2.6 from 2.7 by a limiting argument, letting $m_n \to m_0 \in \bar{D}$ for $m_n \in D$.

PROOF OF 2.7. – Begin with the linear map H defined in (8). It induces a map η: $D \to M_{qp}$. Explicitly, write $m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$ where m_1 is $(q-p) \times p$ and m_2 is $p \times p$; then

$$\eta\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} m_1(1_p + m_2)^{-1} \\ (-1_p + m_2)(1_p + m_2)^{-1} \end{pmatrix}$$

Let $m_0 = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in D$. Then the derivative $d_{m_0}\eta$ is given by

$$d_{m_0}\eta\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 1_{q-p} & -m_1(1_p + m_2)^{-1} \\ 0 & 2(1_p + m_2)^{-1} \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}(1_p + m_2)^{-1}.$$

The determinant is then easy to compute, and we find $\det d_{m_0}\eta = 2^p[\det(1_p + m_2)^{-1}]^{p+q}$; here $(1_p + m_2)^{-1}$ is viewed as an operator on $\mathbf{C}^p$.

Next recall that $1_p - \bar{m}_0^t m_0 > 0$, hence the eigenvalues of $m_2\colon \mathbf{C}^p \to \mathbf{C}^p$ all lie in the unit disc, hence all the eigenvalues of $(1 + m_2)^{-1}$ lie in the right half-plane.

The domain D is simply connected, as is every Hermitean symmetric domain [21, VIII. 4.6]. The map $m \to \det d_m\eta\colon D \to \mathbf{C}$ is holomorphic and non-zero. Choose $m_0 = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in D$ such that all the eigenvalues $\lambda_1, \ldots, \lambda_p$ of $(1 + m_2)^{-1}$ are distinct. Then $\lambda_1, \ldots, \lambda_p$ remain distinct in a neighborhood of m_0. Using analytic continuation (see [43, pp. 23-24]), define a holomorphic function $\sigma_1\colon D \to \mathbf{C}$ such that: (i) $e^{\sigma_1(m)} = \det d_m\eta$ $(m \in D)$; (ii) in a neighborhood of m_0, $\sigma_1(m) = (p + q)\sum_{i=1}^{p} \ln \lambda_i$, where $-\pi/2 < \arg \lambda_i < \pi/2$ $(1 \leqslant i \leqslant p)$.

We claim that $|\operatorname{Im} \sigma_1(m)| < \pi/2p(p + q)$ for all $m \in D$. To see this, let $\tilde{c}\colon [0, 1] \to D$ be a curve joining m_0 and m. Then $\tilde{c}$ is homotopic to a real analytic curve c joining m_0 and m. We can assume that $c(0) = m_0$, $c(1) = m$, and that c is defined and analytic on a complex neighborhood B of $[0, 1] \subset \mathbf{C}$. Write $c(s) = \begin{pmatrix} c_1(s) \\ c_2(s) \end{pmatrix}$ $(s \in B)$. By [31, Chpt. 2], the eigenvalues $\lambda_1(c_2(s)), \ldots, \lambda_p(c_2(s))$ are branches of algebraic functions, hence all λ_i are distinct and holomorphic in s except at isolated points in B. Perturbing c slightly, we can assume that $\lambda_1(c_2(s)), \ldots, \lambda_p(c_2(s))$ are distinct for $0 \leqq s < 1$. Hence $\sigma_1(c(s)) = (p + q)\sum_{i=1}^{p} \ln \lambda_i(c_2(s))$ for $0 \leqq s < 1$, and by continuity of the eigenvalues [31], this equation holds also for $s = 1$. We conclude that $|\operatorname{Im} \sigma_1(m)| \leqq (p + q)\sum_{i=1}^{p} |\arg \lambda_i(c_2(1))| < (\pi/2)(p + q)p$, as desired.

Let us write $\sigma_1(m) = \ln \det d_m\eta$ $(m \in D)$. Letting $E = \eta(D) \subset N_{qp}$, it is clear that we can define a holomorphic branch of $|\operatorname{Im} \ln \det d_n\eta^{-1}| < (\pi/2)p(p + q)$ for all $n \in E$.

Now let $T \in S_0$. Let $t\colon D \to D$ be the map induced by T. Let $\hat{t} = j_0 t j_0^{-1}$, where j_0 is induced on D by J_0 (thus $j_0(m) = -m$). The mapping $\eta\hat{t}\eta^{-1}\colon E \to E$ has the property that $\det d_n(\eta\hat{t}\eta^{-1})$ is real and positive for all $n \in E$. This is true because $HJ_0TJ_0^{-1}H^{-1}$ is a matrix of the form

$$\begin{pmatrix} d_1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \gamma & d_1^{*-1} \end{pmatrix}$$

where d_1 is lower triangular with positive real diagonal; hence $\eta\hat{t}\eta^{-1}$ has the form $n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \to \begin{pmatrix} \alpha d_1^{-1} \\ \beta d_1^{-1} \end{pmatrix} + \begin{pmatrix} n_1 d_1^{-1} \\ \gamma n_1 d_1^{-1} + d_1^{*-1} n_2 d_1^{-1} \end{pmatrix}$. Thus $d_n\hat{t}$ has positive real determinant.

Let $m_0 \in D$, $n_0 = \eta(m_0) \in E$. We have

$$\det d_{m_0} t = \det d_{-m_0} \hat{t} = \det d_{\eta\hat{t}(-m_0)} \eta^{-1} \cdot \det d_{\eta(-m_0)} (\eta \hat{t} \eta^{-1}) \cdot \det d_{-m_0} \eta \, .$$

Making use of the branches of ln defined earlier, we see that

$$\sigma(T, m_0) = \ln \det d_{\eta\hat{t}(-m_0)} \eta^{-1} + \ln \det d_{\eta(-m_0)} (\eta \hat{t} \eta^{-1}) + \ln \det d_{-m_0} \eta$$

is a function with properties (i)-(iii) of 2.7.

Proposition 2.6 shows that the limit in 2.5, if it exists, depends only on $u_t = u(y, id, t)$. The existence of the limit is guaranteed for μ - a.a. y by

2.8 THEOREM. – Consider the cocycle $u(y, u_0, t)$. There is a set $Y_1 \subset Y$ of full μ-measure such that, if $y \in Y_1$ and $u_0 \in K_0$, then $(p + q) i\alpha(\lambda) = \lim_{t\to\infty} (1/t) \ln \cdot \det u(y, u_0, t)$ exists and is independent of $(y, u_0) \in Y_1 \times K_0$.

PROOF. – We basically just repeat the argument in [29, § 4]. First let $\hat{\mu}$ be an invariant measure on $Y \times K_0$ which projects to μ under the map π: $Y \times K_0 \to Y$: $(y, u_0) \to y$. Using [35], it is easily seen that such a measure exists. Next let

$$ig(y, u_0) = \frac{d}{dt} \ln \det u(y, u_0, t) \bigg|_{t=0} .$$

Using smoothness of the Iwasawa decomposition, and arguing as in the proof of Lemma 1.4, one shows that $g \in L^1(Y \times K_0, \hat{\mu})$. Using the Birkhoff theorem, there is a set $B \subset Y \times K_0$ of full $\hat{\mu}$-measure such that, if $(y, u_0) \in B$, then

$$(10) \qquad \hat{\alpha}(y, u_0) \stackrel{\text{def}}{=} \lim_{t\to\infty} \frac{1}{t} \ln \det u(y, u_0, t) = \lim_{t\to\infty} \frac{1}{t} \int_0^t g\big(\tau_s(y), u(y, u_0, s) u_0\big) \, ds$$

exists.

Let $Y_1 = \pi(B)$, so that Y_1 has μ-measure 1. Let $y \in Y_1$, and suppose $(y, u_0) \in B$. Fix $m \in \bar{D}$, and let $u_1 \in K_0$. Using 2.4 (ii) and 2.6, we have

$$\lim_{t\to\infty} \frac{-i}{t} \ln \det u(y, u_0, t) = \lim_{t\to\infty} \frac{1}{t} \operatorname{Im} \ln \det d_m \Phi_y(t) u_0 =$$
$$= \lim_{t\to\infty} \frac{1}{t} \operatorname{Im} [\ln \det d_m \Phi_y(t) u_0 + \ln \det u_0^{-1} u_1] = \lim_{t\to\infty} \frac{1}{t} \operatorname{Im} \ln \det d_m \Phi_y(t) u_1 =$$
$$= \lim_{t\to\infty} \frac{-i}{t} \ln \det u(y, u_1, t) \, .$$

Hence $\hat{\alpha}(y, u_0) = \hat{\alpha}(y)$ exists and is independent of u_0 for all $y \in Y_1$. Clearly $\hat{\alpha}(y)$ is invariant: $\hat{\alpha}(\tau_t(y)) = \hat{\alpha}(y)$ for all $y \in Y_1$. By the Birkhoff theorem, $\hat{\alpha}$ is independent of y for μ - a.a.y. Shrinking Y_1 by a set of measure zero, we obtain 2.8.

2.9 REMARK. – Combining 2.4 (ii), 2.6, and 2.8 shows that there is a set $Y_1 \subset Y$ with $\mu(Y_1) = 1$ such that, if $m_0 \in \bar{D}$, then

$$\alpha(\lambda) = \lim_{t\to\infty} \frac{1}{p+q} \cdot \frac{1}{t} \operatorname{Im} \ln \det d_{m_0} \Phi_y(t) u_0 = \lim_{t\to\infty} \frac{-i}{p+q} \frac{1}{t} \ln \det u(y, u_0, t)$$

exists and is independent of m_0 and of $(y, u_0) \in Y_1 \times K_0$.

2.10 PROPOSITION. – $\lambda \to \alpha(\lambda)$ is continuous ($\lambda \in \boldsymbol{R}$).

PROOF. – If the function of Lemma 1.3 were continuous, we could apply the simple ergodic-theoretic argument of [29]. In the present situation, another argument is necessary.

Let $\lambda_n \to \lambda_0 \in \boldsymbol{R}$. We use the index $n = 0, 1, 2, \ldots$ to refer to cocycles, etc. having to do with equations $(2)_{y,\lambda_n}$.

First note that, by the form of equations $(2)_{y,\lambda}$ the continuity of the cocycles $\Phi_y^n(t)$, and smoothness of the Iwasawa decomposition, $u_n(y, \tilde{u}, t) \xrightarrow[n\to\infty]{} u_0(y, \tilde{u}, t)$ uniformly on compact subsets of $Y \times K_0 \times \boldsymbol{R}$.

Next we observe that, for $n = 0, 1, 2, \ldots$ and $\tilde{u}_0, \tilde{u}_1 \in K_0$, $|\ln \det u_n(y, \tilde{u}_0, t) - \ln \det u_n(y, \tilde{u}_1, t)| < 2\pi(p+q)$ uniformly in n and in $t \in \boldsymbol{R}$. Here of course we always choose that continuous branch of the logarithm such that $\ln 1 = 0$. To prove this assertion, let $\{v(s) : 0 \leqq s \leqq 1\}$ be a path joining $\tilde{u}_0$ and $\tilde{u}_1$ such that $\sup_s |\ln \det v(s) - \ln \det v(0)| < 2\pi(p+q)$. Such a path can always be found. Let $m_* \in M_{qp}$ be as in 2.4 (i). Then

$$\ln \det u_n(y, v(s), t) = i \operatorname{Im} \ln \det d_{m_*} \Phi_y^n(t) v(s) =$$
$$= i \operatorname{Im} \ln \det d_{m_*} \Phi_y^n(t) u_1 + \ln \det v(s) u_1^{-1} = \ln \det u_n(y, u_1, t) + \ln \det v(s) u_1^{-1},$$

and the assertion follows. We have used the fact that, if $T \in S_0$, then $\det d_{m_*} T$ is real.

Now choose $y \in Y$ such that the limit in (10) exists for all n and all $u_0 \in K_0$. Let $\varepsilon > 0$ be small, and choose $T > 0$ so that $2\pi(p+q)/T < \varepsilon$. Then choose N so large that $n \geqq N$, $(y, \tilde{u}) \in Y \times K_0 \Rightarrow |\ln \det u_n(y, \tilde{u}, t) - \ln \det u_0(y, \tilde{u}, t)| < \varepsilon$ for all $|t| \leqq T$. For $r = 0, 1, \ldots, R-1$, let $y_r = \tau_{rn}(y)$, $u_r^T = u_n(y, id, rT)$ $(n = 0, 1, \ldots)$. Then using the cocycle identity (1.9):

$$\frac{1}{RT} |\ln \det u_0(y, id, RT) - \ln \det u_n(y, id, RT)| \leqq$$
$$\leqq \frac{1}{RT} \sum_{r=0}^{R-1} |\ln \det u_0(y_r, u_r^0, T) - \ln \det u_0(y_r, u_r^n, T)| +$$
$$+ \frac{1}{RT} \sum_{r=0}^{R-1} |\ln \det u_0(y_r, u_r^n, T) - \ln \det u_n(y_r, u_r^n, T)| < \varepsilon + \varepsilon = 2\varepsilon .$$

This completes the proof of 2.10.

2.11 REMARKS. – (i) If μ is the only invariant measure on Y, then the limit in (10) is defined for *all* $(y, u_0) \in Y \times K_0$, is everywhere constant, and is uniform in (y, u_0, t) [29].

(ii) The argument used in proving 2.10 can clearly be applied to much more general perturbations of the coefficient matrix in $(2)_{y,\lambda}$. One needs only the continuity in the perturbation of $u(y, u_0, t)$ used above.

(iii) The proof of 2.10 is very similar to a proof of Ruelle [38]. The rotation number α discussed here is presumably equal to that of Ruelle.

Let us now discuss the geometric significance of α. We consider only the case $g = \mathrm{sp}\,(n, \boldsymbol{R})$, i.e., $J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$. One can interpret α in a similar way if $g = u(p, q)$ by using self-adjoint boundary conditions [3], but we do not do so here (the basic idea is in BOTT [6]).

First of all, recall (1.8) that $u_1^{-1} \cdot \mathrm{sp}\,(n, \boldsymbol{R}) \cdot u_1 \subset \mathrm{su}\,(n, n) \subset u(n, n)$, where $u_1 = \begin{pmatrix} i1_n & i1_n \\ -1_n & 1_n \end{pmatrix}$. Then the rotation number α can be defined just as above. Translating back to $\mathrm{sp}\,(n, \boldsymbol{R})$ via $A \to u_1 A u_1^{-1}$, we obtain the following statement.

2.12 PROPOSITION. – Let $K \subset \mathrm{Sp}\,(n, \boldsymbol{R})$ = symplectic group be the maximal compact subgroup defined as follows: $K = \left\{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \middle| A, B \text{ are real } n \times n,\ A^* = -A,\ B^* = B\right\}$. Let $u\colon Y \times K \times \boldsymbol{R} \to K$ be the cocycle induced by equation $(2)_{y,\lambda}$ (where now $y(t) \in \mathrm{sp}\,(n, \boldsymbol{R})$).

Then for μ - a.a. y,

$$-i \lim_{t \to \infty} \frac{1}{t} \ln \widetilde{\det}\, u(y, u_0, t) = \alpha(\lambda) = \alpha \tag{11}$$

exists and is independent of $u_0 \in K$.

We must explain the notation $\widetilde{\det}$. Recall that K is isomorphic to the unitary group $U(n)$ via the map $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \to A + iB$. Let $\widetilde{\det}$ be the usual determinant of an $n \times n$ complex matrix. The relation between $\widetilde{\det}$ and the determinant det of the induced map on M_{nn} is simply $\det = (\widetilde{\det})^{2n}$. Hence there is no factor $1/(p + q) = 1/2n$ in (11).

Now let $\mathfrak{L}$ be the set of Lagrange subspaces $l \subset \boldsymbol{R}^{2n}$ defined in § 1. As in § 1, let $l_0 = [e_1, \dots, e_n] \in \mathfrak{L}$ be the plane spanned by the unit vectors $e_1, \dots, e_n$. Let $C = \{l \in \mathfrak{L}\colon \dim l \cap l_0 \geqq 1\}$, the Maslov cycle. Then one can use C to define a generator of the first cohomology group $H^1(\mathfrak{L}, Z) = Z$ as follows. Let $c\colon [0, 1] \to \mathfrak{L}$ be a closed curve; then $h(c)$ = number of oriented intersections of c with C. See Duistermaat [13].

Next recall (Arnold [2]) that h can be expressed in another way. Let $O(n) = \{u \in K: ul_0 = l_0\}$; then $O(n)$ is isomorphic to the real rotation group of dimension n, and $\mathfrak{L} = K/O(n)$. The map det: $K \to \boldsymbol{C}$ induces a function $\det^2$: $\mathfrak{L} \to C$ via $\det^2(l) = (\det u)^2$ where $ul_0 = l$. Arnold shows that $h(c)$ equals the winding number of the map $\det^2 c: [0, 1] \to \boldsymbol{C}$.

The relation of α to h is now easily described. The complement of C in $\mathfrak{L}$ is simply connected [2]. Choose $l \in \mathfrak{L}$, and consider the curve $\tilde{c}: t \to \Phi_y(t)l$ $(0 \leq t \leq T)$. If l and/or $\Phi_g(T)l \in C$, we perturb $\tilde{c}$ slightly so as to make the intersection transversal. Then we deform $\tilde{c}$ to a closed curve c by sliding the endpoint $\Phi_g(t)l$ to l through $\mathfrak{L}\setminus C$. Let $n(T) = h(c)$. Using 2.8, the limit

$$\frac{\alpha}{\pi} = \lim_{t\to\infty} \frac{n(T)}{T}$$

is independent of the construction and exists for all $l \in \mathfrak{L}$, for μ - a.a.$y \in Y$. Thus α/π measures average rotation in the sense of « average number of oriented intersections with the Maslov cycle ».

2.13 REMARKS. – (i) Consider a difference equation $x_{n+1} = V(n)x_n$ where $V(n) \in U(p, q)$ or $\mathrm{Sp}(n, \boldsymbol{R})$. One can define a rotation number for such an equation by first suspending it [16] and then applying the methods discussed above. See [27].

(ii) There are other Lie algebras g for which one can define a rotation number analogous to the one discussed above. This is true in particular if g is the Lie algebra of the isometry group $\mathcal{G}$ of a bounded symmetric domain. In addition to $\mathrm{su}(p, q)$ and $\mathrm{sp}(n, \boldsymbol{R})$, these algebras are $g = SO^*(2\mathrm{n})$, $\mathrm{so}(2, q)$ $(q \geq 2)$, eIII, and eVII [21]. The basic reason is that a maximal compact subgroup $K \subset \mathcal{G}$ has center isomorphic to the circle group T.

3. – The m-functions.

In this section we define and study the Weyl-Kodaira m-functions for the equations

$$(2)_{y,\lambda} \qquad \frac{dx}{dt} = [\lambda J_0^{-1}\gamma_y(t) + y(t)]x \qquad x \in \boldsymbol{C}^k,\ k = p + q\,,$$

where $J_0 = \begin{pmatrix} -i1_p & 0 \\ 0 & i1_q \end{pmatrix}$, $0 < p \leq q < k$, and $y(t) \in u(p, q)$. We impose the conditions (ii), (iii) of 1.6. Thus $\gamma_y(t) = \gamma(\tau_t(y))$ is symmetric and positive semi-definite. Moreover, given $y \in Y$, $\lambda \in \boldsymbol{C}$, and a solution $x(t)$ of $(2)_{y,\lambda}$, there is a constant C such that

$$\int_{-\infty}^{\infty} \langle x(s), x(s)\rangle\, ds \leq C \int_{-\infty}^{\infty} \langle \gamma_y(s)\cdot x(s), x(s)\rangle\, ds\,.$$

This condition is practically always satisfied.

In § 3, the ergodic measure μ plays no role. Hence we assume only that Y is a bounded translation invariant subset of ξ_g which satisfies (8).

A basic result is

3.1 THEOREM. – Suppose $\operatorname{Im}\lambda \neq 0$. Then equations $(2)_{y,\lambda}$ have exponential dichotomy (ED). Moreover the stable and unstable bundles $V^s(\lambda)$, $V^u(\lambda)$ (see 1.10) satisfy $\dim V^s(\lambda) = p$, $\dim V^u(\lambda) = q$ if $\operatorname{Im}\lambda > 0$; $\dim V^s(\lambda) = q$, $\dim V(\lambda) = p$ if $\operatorname{Im}\lambda < 0$.

PROOF. – We first assume that the base space Y is chain recurrent (e.g., [8]; we do not use the definition directly, hence do not repeat it). In this case, equations $(2)_{y,\lambda}$ have ED iff no equation $(2)_{y,\lambda}$ admits a nonzero bounded solution [39, 42].

Suppose that $x_0(t)$ is a non-zero bounded solution of some equation $(2)_{y,\lambda}$. We use *Green's identity*: writing $L_g = J_0(d/dt - y(t)) - \lambda\gamma_g$, we have

$$\int_a^b [\langle f, L_y g\rangle - \langle L_y f, g\rangle]\, dt = \langle f, J_0 g\rangle\Big|_{t=a}^{t=b} + 2i \operatorname{Im}\lambda \int_a^b \langle \gamma_y f, g\rangle\, dt \tag{12}$$

where $a < b \in \boldsymbol{R}$ and $f, g\colon \boldsymbol{R} \to \boldsymbol{C}^k$ are absolutely continuous with integrable derivatives. Letting $f = g = x_0$, we find that the left-hand side is zero, and that the first term on the right is uniformly bounded in a, b. So if $\operatorname{Im}\lambda \neq 0$, the condition 1.6 (iii) implies that $\int_{-\infty}^{\infty} \langle x_0(s), x_0(s)\rangle\, ds < \infty$. Hence there are sequences $a_n \to -\infty$, $b_n \to \infty$ such that $\lim_{n\to\infty} x_0(a_n) = 0 = \lim_{n\to\infty} x_0(b_n)$. Using (12) again, this implies that $\int_{-\infty}^{\infty} \langle \gamma_y x_0, x_0\rangle\, ds = 0$, which by 1.6 (iii) implies that $x_0(t) \equiv 0$.

We have arrived at a contradiction. Thus if Y is chain recurrent and $\operatorname{Im}\lambda \neq 0$ then equations $(2)_{y,\lambda}$ have ED.

To find the dimensions of the bundles $V^s(\lambda)$, $V^u(\lambda)$, we use a « principle of infection » based on the perturbation theorem of Sacker and Sell [40]. Consider the two-parameter family of differential systems

$$\frac{dx}{dt} = [(1-\varepsilon)\lambda J_0^{-1} + \varepsilon\lambda J_0^{-1}\gamma_y(t) + y(t)]x\,, \tag*{$(13)_{\lambda,\varepsilon}$}$$

where $0 \leqq \varepsilon \leqq 1$. If $\varepsilon = 1$, we obtain equations $(2)_{y,\lambda}$. Write $\lambda = |\lambda| e^{i\theta}$ and suppose, e.g. $\operatorname{Im}\lambda > 0$, i.e., $0 < \theta < \pi$. Make the change of variables $s = |\lambda| t$, and write $\tilde{x}(s) = x\big((1/|\lambda|)s\big)$. Then we have

$$\frac{d\tilde{x}}{ds} = \left[(1-\varepsilon)\, e^{i\theta} J_0^{-1} + \varepsilon e^{i\theta} J_0^{-1} \gamma_y\left(\frac{s}{|\lambda|}\right) + \frac{1}{|\lambda|}\, y\left(\frac{s}{|\lambda|}\right)\right] \tilde{x}(s)\,. \tag*{$(14)_{\lambda,\varepsilon}$}$$

Fix $\theta \in (0, \pi)$. Let ξ_g be the space of § 1 with $g = u(p, q)$, and let N be a neighborhood of the constant function $e^{i\theta} J_0^{-1}$. Using property (8) in § 1, we see that, for small ε and large $|\lambda|$, the coefficient of $(14)_{\lambda,\varepsilon}$ lies in N. The constant system $d\tilde{x}/ds = e^{i\theta}J_0^{-1}\tilde{x}$ clearly has ED, and the stable resp. unstable bundles have dimensions p resp. q. Now the Sacker-Sell result [40] implies that, for small ε and large $|\lambda|$, equations $(14)_{\lambda,\varepsilon}$ have ED as well, and the dimensions of the bundles remain p and q. Returning to the original variable t, we see that equations $(13)_{\lambda,\varepsilon}$ have ED.

Now the first part of the proof shows that equations $(13)_{\lambda,\varepsilon}$ have ED for all $0 \leqq \varepsilon \leqq 1$ and $\operatorname{Im} \lambda > 0$. Since the bundles $V^{s,u}(\lambda, \varepsilon)$ vary continuously (COPPEL [9], SACKER-SELL [40]), we see by a connectedness argument that $\dim V^s(\lambda) = p$, $\dim V(\lambda) = q$ if $\operatorname{Im} \lambda > 0$.

If $\operatorname{Im} \lambda < 0$, similar arguments show that $\dim V^s(\lambda) = q$, $\dim V^u(\beta) = p$. This completes the proof of 3.1 if Y is chain recurrent.

To prove 3.1 in full generality, we use another theorem of Sacker and Sell [40]. Let $Y_1 \subset Y$ be any minimal subset. Then Y_1 is chain-recurrent, so if $\operatorname{Im} \lambda > 0$, then $\dim V^s(\lambda) = p$ and $\dim V^u(\lambda) = q$ over Y_1. By [40], equations $(2)_{y,\lambda}$ have ED over all of Y, and the dimensions of $V^{s,u}(\lambda)$ are p, q if $\operatorname{Im} \lambda > 0$. One argues analogously if $\operatorname{Im} \lambda < 0$. This completes the proof of 3.1.

Now we consider the location of the bundles V^s, V^u. We will show that, if $\operatorname{Im} \lambda > 0$, then $V_y^s(\lambda) \underset{\text{def}}{=} V^s(\lambda) \cap (\{y\} \times C^k)$ has a basis of column vectors $\begin{pmatrix} 1_p \\ m_+ \end{pmatrix}$ with $m_+ \in D \subset M_{qp}$. Similarly, letting M_{pq} be the set of $p \times q$ complex matrices, and letting $D' = \{m \in M_{pq}: 1_q - m^t m > 0\}$, the fiber $V_y^u(\lambda) = V^u(\lambda) \cap (\{y\{ \times C^k)$ has a basis of column vectors of the form $\begin{pmatrix} m_- \\ 1_q \end{pmatrix}$ with $m_- \in D'$. These relations define the Weyl-Kodaira functions $m_\pm = m_\pm(y, \lambda)$ if $\operatorname{Im} \lambda > 0$. If $\operatorname{Im} \lambda < 0$, we will find that (with analogous notation) $m_+(y, \lambda) \in D'$ and $m_-(y, \lambda) \in D$.

To begin, recall that, if $A \in U(p, q)$, then $A(D) \subset D$. Since $A: \mathfrak{S}_p \to \mathfrak{S}_p$ is a diffeomorphism, and since D defines a subset of $\mathfrak{S}_p$, we see that $A: \bar{D} \to \bar{D}$ is a homeomorphism. (Recall $\mathfrak{S}_p =$ set of complex p-planes in C^k).

As always, let $\Phi_y(t)$ be the fundamental matrix solution of $(2)_{y,\lambda}$ with $\Phi_y(0) = I$. Fix λ with $\operatorname{Im} \lambda > 0$, and let $t < 0$. Then $\Phi_y(t) \notin U(p, q)$. Nevertheless it induces a diffeomorphism φ_t of $\mathfrak{S}_p$ onto itself. We claim that φ_t maps $\bar{D}$ strictly into D.

Intuitively, this is easy to see. Consider the Riccati equation satisfied by $m \in M_{qp}$: writing $\lambda J_0^{-1}\gamma_y + y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$m' = -mbm + dm - ma + c\,. \tag{15}$$

Write $\lambda = \lambda_1 + i\lambda_2$. If $m_0 \in \partial D$ and if $\lambda_2 = 0$, then the tangent vector m' at m_0 points «parallel to ∂D», since $\Phi_y(t)$ preserves ∂D if $\operatorname{Im} \lambda = 0$. If $\lambda_2 > 0$, then $-m'$ has an extra component which points into D. Since «the stable bundle attracts solutions as $t \to -\infty$», we must have $V_y^s(\lambda) \in D (y \in Y)$.

A formal proof, though somewhat tedious, is not hard. We must sidestep the problem that ∂D is not a manifold (it is a stratified manifold). Consider the equations

$$(16)_{\lambda,\varepsilon} \qquad \frac{dx}{dt} = [\varepsilon\lambda J_0^{-1} + \lambda J_0^{-1}\gamma_y(t) + y(t)]x \qquad \varepsilon > 0\,.$$

Fix $t < 0$, $y \in Y$, $\lambda = \lambda_1 + i\lambda_2$ with $\lambda_2 > 0$, and let $\eta_\varepsilon\colon \mathfrak{S}_p \to \mathfrak{S}_p$ be the diffeomorphism induced by the fundamental matrix solution $\Phi_y^\varepsilon(t)$ of $(16)_{\lambda,\varepsilon}$.

3.2 LEMMA. – $\eta_\varepsilon(\bar{D}) \subset D$.

PROOF. – Let $m_0 \in \partial D$. The matrix $1_p - \bar{m}_0^t m_0$ is Hermitean and positive semidefinite. Let $m(s)$ be the solution of the Riccati equation (15) corresponding to $(16)_{\lambda,\varepsilon}$ which satisfies $m(0) = m_0$.

Let $z_0 \in \boldsymbol{C}^p$ be a vector of norm 1 such that $\langle m_0 z_0, m_0 z_0\rangle = 1$. Suppose for contradiction that there is a sequence $0 > s_n \uparrow 0$ such that $\langle m(s_n)z_n, m(s_n)z_n\rangle \geqq 1$. Choosing a subsequence and replacing z_0 if necessary, we can assume that $z_n \to z_0$.

Assume for the time being that $y(t)$ is continuous. Let

$$\varphi(s) = \langle (1_p - \overline{m(s)}^t\, m(s))\, z_0, z_0\rangle\,.$$

Computing the derivative at $s = 0$, we find

$$\varphi'(0) = -\,2\lambda_2\varepsilon\langle m_0 z_0, m_0 z_0\rangle + h(z_0) = -\,2\lambda_2\varepsilon + h(z_0)\,,$$

where $h(z_0)$ is the contribution to $\varphi'(0)$ from the terms other than $\varepsilon\lambda J_0^{-1}$ in $(16)_{\lambda,\varepsilon}$.

We claim that $h(z_0) \leqq 0$. To see this, write $h(z_0) = h_1(z_0) + h_2(z_0)$, where $h_1(z_0)$ is the contribution to $\varphi'(0)$ from $\lambda_1 J_0^{-1}\gamma_y + y$, and $h_2(z_0)$ is that from $\lambda_2 J_0^{-1}\gamma_y$. Then $h_1(z_0) \leqq 0$ because if $\lambda \in \boldsymbol{R}$, then the fundamental matrix solution of $(16)_{\lambda,\varepsilon}$ preserves $\bar{D}$. Also $h_2(z_0) \leqq 0$ (this is most easily seen by diagonalizing $\gamma_y(0)$).

We conclude that $\varphi'(0) \leqq -\,2\lambda_2\varepsilon$. Replacing z_0 by z_n, and calling the resulting curves $\varphi_n(s)$, we get $\varphi_n'(0) \leqq -\,\lambda_2\varepsilon$ for large n. This implies that, if n is large, then $\varphi_n(s_n) > 0$, a contradiction. Thus $m(s) \in D$ for small $s < 0$. An elementary argument which we omit shows that $\eta_\varepsilon(\bar{D}) \subset D$ for all $\varepsilon > 0$. This proves 3.2 if y is continuous.

To remove the continuity assumption, approximate $y(t)$ by continuous functions $y_n(t)$ in such a way that the fundamental matrix solutions $\Phi_n(t;\lambda)$ converge to $\Phi_y(t;\lambda)$ in $U(p,q)$, uniformly for (t,λ) in compact subsets of $\boldsymbol{R}\times\boldsymbol{C}$. Here $\Phi_n(t,\lambda)$, $\Phi_y(t,\lambda)$ have the obvious meaning. Let H_0 be a domain in $\boldsymbol{C}$ whose closure is compact in $H^+ = \{\lambda \in \boldsymbol{C}\colon \operatorname{Im}\lambda > 0\}$. Then there exist $\sigma > 0$ and a domain $D_0 \subset M_{qp}$ such that $\bar{D} \subset D_0$ and such that the induced map $m \to \Phi_y(t;\lambda)m$ maps D_0 entirely into M_{qp} for all $-\,\sigma \leqq t \leqq 0$ and all $\lambda \in csH_0$. For sufficiently large n, the same holds for $\Phi_n(t,\lambda)$. Moreover $\Phi_n \to \Phi_y$ uniformly on compact subsets of D_0, and this convergence is itself uniform on H_0.

Now let $m_0 \in \bar{D}$. Then $\Phi_n(t;\lambda)m_0 \in D$ for all $t < 0$ and all $\lambda \in H^+$. Hence $\Phi_y(t;\lambda)m_0 \in \bar{D}(-\sigma \leq t \leq 0, \lambda \in H_0)$. Suppose for contradiction that there exists $\lambda_0 \in H_0$ such that $g(\lambda_0) \subset \partial D$, where $g(\lambda) \overset{\text{def}}{=} \Phi_y(t;\lambda)m_0$ for fixed $t \in [-\sigma,)$. Let $z_0 \in C^p$ be a vector of norm 1 such that $\langle g(\lambda_0)z_0, g(\lambda_0)z_0\rangle = 1$. Consider the holomorphic function $\lambda \to \langle g(\lambda)z_0, g(\lambda_0)\lambda_0\rangle$. The real part of the logarithm of this function has no interior maximum in H_0, hence there exists $\lambda \in H_0$ such that $\ln|\langle g(\lambda)z_0, g(\lambda_0)z_0\rangle| > 0$. Hence for large n, $|\langle \Phi_n(t;\lambda)m_0z_0, g(\lambda_0)z_0\rangle| > 1$. Since $g(\lambda_0)z_0$ has norm 1, the norm of $\Phi_n(t;\lambda)m_0z_0$ must be > 1, a contradiction. Hence $g(\lambda) \in D$ for all $\lambda \in H_0$, and hence $\Phi_y(t,\lambda)m_0 \in D$ for small negative t. This implies 3.2 in complete generality.

Now let $V^{s,u}(\lambda, \varepsilon)$ be the stable and unstable bundles for equations $(16)_{\lambda,\varepsilon}$ ($\text{Im}\,\lambda > 0$, $\varepsilon \geqslant 0$). If $\varepsilon = 0$ we regain the bundles $V^{s,u}(\lambda)$ defined by equations $(2)_{y,\lambda}$.

3.3 LEMMA. – Let $\varepsilon > 0$. The p-plane $V_y^s(\lambda,\varepsilon) = V^s(\lambda,\varepsilon) \cap (\{y\} \times C^k)$ has a basis of column vectors of the form $\begin{pmatrix} 1_p \\ m_+ \end{pmatrix}$ where $m_+ \in D$ $(y \in Y)$. We say (with slight imprecision) that $V_y^s(\lambda,\varepsilon) \in D$.

PROOF. – Fix $\bar{y} \in Y$, and let $l_u = V_{\bar{y}}^u(\beta,\varepsilon)$. Then l_u is a q-plane in C^k. Let $l \subset C^k$ be a p-plane such that $l \cap l_u = \{0\}$. Then any non-zero solution of $(16)_{\bar{y},\lambda,\varepsilon}$ with initial condition in l grows exponentially as $t \to -\infty$; moreover $\Phi_{\bar{y}}(t;\lambda,\varepsilon)\cdot l$ approaches $\{V_y^s(\lambda,\varepsilon) : y \in Y\} \subset Y \times \mathfrak{S}_p$ as $t \to -\infty$. These statements follows easily from the definition of ED (1.10).

Now choose $l \in D$ such that $l \cap l_u = \{0\}$. Simple dimensional considerations show that this can be done. From 3.2 and the preceding paragraph, we see that $V_y^s(\lambda,\varepsilon) \in D$ for all points y in the α-limit set of $\bar{y}$ (1.9).

Next let Y_ω be the *ω-limit set* of $\bar{y}$ (1.9). Since Y_ω is invariant, we can find $y_1 \in Y_\omega$ which is in the α-limit set of some other point $y_2 \in Y_\omega$. By the argument just given, $V_{y_2}^s(\lambda,\varepsilon) \in D$. Now, $y \to V_y^s$ is continuous, hence there exists a positive t such that $V_{\tau_t(\bar{y})}^s(\lambda,\varepsilon) \in D$. Since $Y \times D$ is negatively invariant (3.2), we see that $V_y^s(\lambda,\varepsilon) \in D$, as desired. The proof of 3.3 is complete.

We now remove the assumption $\varepsilon > 0$. Fix $y \in Y$, and write $m_+(\lambda,\varepsilon)$, $m_+(\lambda)$ for the parameters corresponding to $V_y^s(\lambda,\varepsilon)$, $V_y^s(\lambda)$. Since the bundles $V^{s,u}$ vary continuously in (λ,ε) [9], we have $m_+(\lambda,\varepsilon) \to m_+(\lambda)$ as $\varepsilon \to 0^+$. Hence $m_+(\lambda) \in \bar{D}$ ($\text{Im}\,\lambda > 0$). Since $\lambda \to m_+(\lambda)$ is holomorphic [26], we can apply the argument in the last part of the proof of 3.2 to conclude that $m_+(\lambda) \in D$.

All of the above arguments apply with trivial modifications to $V^s(\lambda)$, $V^u(\lambda)$ for all $\text{Im}\,\lambda \neq 0$. Summarizing:

3.4 THEOREM. – Let $D \subset M_{qp}$, $D' \subset M_{pq}$ be as defined above. If $\text{Im}\,\lambda > 0$, then $V_y^s(\lambda)$ has a basis $\begin{pmatrix} 1_p \\ m^+(y,\lambda) \end{pmatrix}$ where $m_+(y,\lambda) \in D$. Also $V_y^u(\lambda)$ has a basis $\begin{pmatrix} m_-(y,\lambda) \\ 1_q \end{pmatrix}$

with $m_+(y, \lambda) \in D'$. If $\operatorname{Im} \lambda < 0$, then $V_y^s(\lambda)$ has a basis $\begin{pmatrix} m_+(y, \lambda) \\ q \end{pmatrix}$ with $m_+(y, \lambda) \in D'$, and $V_y^u(\lambda)$ has a basis $\begin{pmatrix} 1_p \\ m_-(y, \lambda) \end{pmatrix}$ with $m_-(y, \lambda) \in D$.

We have the following

3.5. COROLLARY. – $\lim_{t\to\infty} \Phi_{\tau_t(y)}(-t) m_0 = m_+(y, \lambda)$ uniformly in $y \in Y$, $m_0 \in \bar{D}$, and in $\lambda \in C$ for any compact $C \subset H^+$. Also $\lim_{t\to\infty} \Phi_{\tau_t(y)}(-t) m_0 = m_-(y, \lambda)$ uniformly in $y \in Y$, $m_0 \in \bar{D}'$, and in $\lambda \in C$. There are analogous results for $\operatorname{Im} \lambda < 0$.

It is this « collapsing in » of $\bar{D}$, $\bar{D}'$ that is characteristic of limit-point systems $(2)_{y,\lambda}$.

PROOF. – Consider only the first statement. In view of the ED for $\operatorname{Im} \lambda > 0$, it suffices to show that, if $l \in \mathfrak{S}_p$ has basis $\begin{pmatrix} 1_p \\ m_0 \end{pmatrix}$ with $m_0 \in \bar{D}$, then $l \cap V_y^u(\lambda) = \{0\}$ (see the proof of 3.3). However this follows from $m_-(y, \lambda) \in D'$: if there were a non-zero vector in $l \cap V_y(\lambda)$, then $m_0 \cdot m_-(y, \lambda)$ would have 1 as an eigenvalue, which is impossible.

For the Lie algebra $\operatorname{sp}(n, \boldsymbol{R})$, more can be said about the location of the m-functions. As usual, embed $\operatorname{sp}(n, \boldsymbol{R})$ in $\operatorname{su}(n, n)$ via $A \to u_1^{-1} A u_1$, where $u_1 = \begin{pmatrix} i1_n & i1_n \\ -1_n & 1_n \end{pmatrix}$. By [43, p. 125], the fundamental matrix solution $\Phi_y(t)$ of $(2)_{y,\lambda}$ preserves the Siegel unit disc $D_s = \{m \in M_{nn}: m^t = m\} \cap D$ if λ is real. It is then easy to see that $m_+(y, \lambda) \in D_s$ if $\operatorname{Im} \lambda \neq 0$. In fact, one can use the trick already used in proving 3.3 and 3.5. Namely, if, say, we want to show that $m_+(y, \lambda) \in D_s$, we look for $m \in D_s$ such that the corresponding n-plane $\begin{pmatrix} 1_n \\ m \end{pmatrix}$ intersects $V_y(\lambda)$ in $\{0\}$. This is true for any n-plane $l = u_1 l_0$ where $l_0 \subset \boldsymbol{R}^{2n}$ is a real Lagrange plane: it follows from Green's identity (12) that solutions $x(t)$ of $(2)_{\lambda,y}$ with $0 = x(0) \in u_1 l_0$ are unbounded both as $t \to \infty$ and as $t \to -\infty$.

Translating back to $\operatorname{sp}(n, \boldsymbol{R})$ via $B \to u_1 B u_1^{-1}$, and recalling [43] that $u_1 \cdot D_s$ is the *Siegel upper half-space* $H_s = \{m \in M_{nn}: m^t = m, \operatorname{Im} m > 0\}$, we see that $m_\pm(y, \lambda) \in H_s$ $(y \in Y, \operatorname{Im} \lambda \neq 0)$.

The Lie algebra $so^*(2n) = \left\{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| a, b\ n \times n \text{ complex}, a^t = -a, b^* = b\right\}$ admits a similar discussion. Replacing $so^*(2n)$ by $u_1^{-1}\, so^*(2n) u_1$, we find that $\Phi_y(t)$ preserves $D \cap \{m \in M_{nn}: m^t = -m\} = D_a$. Hence $m_\pm \in u_1 \cdot D_a$. See [21, p. 527].

For the Lie algebras $g = so(2, q)$, eIII, eVII, one can find « m-functions » in the corresponding symmetric domain by introducing a certain operator J_0 [21, Corollary 7.13], and viewing $\Phi_y(t)$ as an element of the adjoint group of g.

4. – $w(\lambda)$ for complex λ.

We return to the quantity $w(\lambda) = \beta(\lambda) + i\alpha(\lambda)$ defined for real λ in § 2. We will show that there is a function (also called w) holomorphic in the upper half-plane H^+ such that $\lim w(\lambda + i\varepsilon) = \beta(\lambda) + i\alpha(\lambda)$.

The definition of w is motivated by that of the rotation number α in § 2. We can interpret what was done there as follows. Consider the Riccati equation for $m \in M_{qp}$: with $\lambda J_0^{-1}\gamma_y + y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$m' = -mbm + dm - ma + c\,. \tag{15}$$

Linearize it, obtaining

$$(\delta m)' = f_+(m)\,\delta m\,. \tag{17}$$

Then

$$\alpha = \lim_{t\to\infty} \frac{1}{p+q}\,\frac{1}{t}\,\mathrm{Im}\int_0^t \mathrm{tr}\, f_+(m(s))\,ds\,, \qquad \mathrm{tr} = \mathrm{trace}\,,$$

where $m(s)$ is a solution of (15) with $m(0) \in \bar{D}$. We can write α in this way because of Liouville's formula and the fact that $d_{m_0}\Phi_y(t)$ is the fundamental matrix solution of (17).

We are led to the following

4.1. DEFINITION. – Let $\lambda \in H^+$, and let $m_+(y, \lambda)$ be the m-function defined by $V^s(\lambda)$. For $m \in M_{qp}$, let $f_+(m)$ be the linear opeator on M_{qp} obtained by linearizing (15); explicitly

$$f_+(m)\cdot r = -mbr - rbm + dr - ra\,. \tag{18}$$

We do not indicate the dependence of f_+ on $y \in Y$. Define

$$w(\lambda) = \frac{1}{p+q}\int_Y \mathrm{tr}\, f_+(m_+(y, \lambda))\,d\mu(y)\,. \tag{19}$$

By the Birkhoff ergodic theorem, for μ - a.a.y:

$$w(\lambda) = \frac{1}{p+q}\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathrm{tr}\, f_+\big(m_+(\tau_s(y), \lambda)\big)\,ds\,. \tag{20}$$

We see that $\mathrm{Re}\, w(\lambda)$ measures the average rate of change of volume determined by the motion of vectors tangent to $m_+(y, \lambda)$, and that $\mathrm{Im}\, w(\lambda)$ measures average rotation «around» $m_+(y, \lambda)$.

Since the bundles $V^s(\lambda)$ vary holomorphically in λ [26], we see without difficulty that w is holomorphic in H^+.

We now derive two other formulas for $w(\lambda)$, which will also be used in § 5. Define

$$(21)\qquad \begin{cases} i\alpha_0 = \int_Y \operatorname{tr} J_0^{-1}\gamma(y)\,d\mu(y) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \operatorname{tr} J_0^{-1}\gamma_y(s)\,ds & \mu\text{ - a.e.}\,, \\ iy_0 = \int_Y \operatorname{tr} y\,d\mu(y) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \operatorname{tr} y(s)\,ds & \mu\text{ - a.e.}\,. \end{cases}$$

For the notation in these formulas, see 1.4 and 1.6 (ii). Here tr means the usual trace of a $k\times k$ matrix. Then $\alpha_0, y_0 \in \boldsymbol{R}$.

Consider the mapping $\eta_1\colon r\to dr - ra$ of M_{qp} to itself. Then $\operatorname{tr}\eta_1 = p\operatorname{tr} d - q\operatorname{tr} a$. In addition, $\int(\operatorname{tr} a + \operatorname{tr} d)\,d\mu(y) = i(\lambda\alpha_0 + y_0)$. Hence $\left(\text{noting that } \eta_1 \text{ depends on } y \text{ through the coefficient matrix } \begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)$:

$$\int_Y \operatorname{tr}\eta_1(y)\,d\mu(y) = -(p+q)\int_Y \operatorname{tr} a(y)\,d\mu(y) + ip(\lambda\alpha_0 + y_0)\,.$$

Similarly, let $\eta_2(r) = -mbr - rbm$ for $m\in M_{qp}$. Then setting $m = m_+(y,\lambda)$:

$$\int_Y \operatorname{tr}\eta_2(y)\,d\mu(y) = -(p+q)\int_Y \operatorname{tr} b(y)m_+(y,\lambda)\,d\mu(y)\,.$$

Combining these two formulas, we get

$$(22)\qquad w(\lambda) = -\int_Y \operatorname{tr}(a + bm_+)\,d\mu(y) + \frac{ip}{p+q}(\lambda\alpha_0 + y_0)\,.$$

The quantity $-\int_Y \operatorname{tr}(a+bm_+)\,d\mu(y)$ may be interpreted as follows. For fixed $y\in Y$ and $\lambda\in H^+$, let

$$(23)\qquad N(t) = \begin{pmatrix} 1_p & m_-(\tau_t(y),\lambda) \\ m_+(\tau_t(y),\lambda) & 1_q \end{pmatrix}$$

and make the change of variable $x = N(t)z$ in $(2)_{y,\lambda}$. Then

$$\frac{dz}{dt} = \begin{pmatrix} a + bm_+ & 0 \\ 0 & cm_- + d\end{pmatrix} z\,.$$

Let $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ where $z_1\in C^p$ and $z_2\in C^q$. Let $Z_1(t)$ be the fundamental matrix solution of $z_1' = (a + bm_+)z_1$ satisfying $Z_1(0) = 1_p$. Then by Liouville's formula and the Birkhoff theorem one has for μ - a.a. y:

$$\lim_{t\to\infty} \frac{1}{t} \ln \det Z_1(t) = \int_Y \operatorname{tr}(a + bm_+)\, d\mu(y) . \tag{24}$$

Thus $w(\lambda)$ measures the (exponential) growth and rotation of the matrix $Z_1(t)$, which, it should be noted, is induced by p linearly independent solutions of $(2)_{y,\lambda}$ with initial conditions in $V_y^s(\lambda)$. In fact, a basis for solutions of $(2)_{y,\lambda}$ initiating in $V_y(\lambda)$ is given by $\begin{pmatrix} Z_1(t) \\ m_+(\tau_t(y), \lambda)\, Z_1(t) \end{pmatrix}$.

The quantity $\int_Y (cm_- + d)\, d\mu(y)$ can be treated similarly. In fact, a basis for solutions with initial conditions in $V_y^u(\lambda)$ is given by $\begin{pmatrix} m_-(\tau_t(y), \lambda)\, Z_2(t) \\ Z_2(t) \end{pmatrix}$, where $Z_2' = (\mathrm{cm}_- + d) Z_2$ and $Z_2(0) = 1_q$. We have for μ - a.a. y:

$$\lim_{t\to\infty} \frac{1}{t} \ln \det Z_2(t) = \int_Y \operatorname{tr}(\mathrm{cm}_- + d)\, d\mu(y) . \tag{25}$$

To get a formula for $w(\lambda)$, note that det $N(t)$ is bounded above and bounded away from zero. This follows from a computation similar to and easier than one which will be carried out in § 5, hence we omit details here. Hence for μ - a.a.y:

$$i(\lambda\alpha_0 + y_0) = \int_Y \operatorname{tr} a) + \operatorname{tr} d)\, d\mu(y) = \lim_{t\to\infty} \frac{1}{t} \ln \det \Phi_y(t) =$$
$$= \lim_{t\to\infty} \frac{1}{t} \ln \det Z_1(t) Z_2(t) = \int_Y \operatorname{tr}(a + bm_+)\, d\mu(y) + \int_Y \operatorname{tr}(cm_- + d)\, d\mu(y) .$$

We emphasize that, in the last two integrals, tr means the (usual) trace of a $p \times p$ matrix resp. a $q \times q$ matrix. Combining (22) with the preceding equation yields

$$w(\lambda) = \int_Y \operatorname{tr}(cm_- + d)\, d\mu(y) - \frac{iq}{p+q} (\lambda\alpha_0 + y_0) . \tag{26}$$

Now we can analyze the boundary behavior of $w(\lambda)$. We temporarily write $\hat{w}(\lambda) = \hat{\beta}(\lambda) + i\hat{\alpha}(\lambda)$ for the quantity introduced in § 2 (i.e., $\lambda \in \boldsymbol{R}$).

We consider first the real part β of w. It follows from (26) that $\beta(\lambda) = \lim_{t\to\infty} (1/t) \cdot \ln |\Lambda^q \Phi_y(t)|$ for μ - a.a.y. In fact, the Oseledec theory [36] tells us that, for μ - a.a.y, $\lim_{t\to\infty} (1/t) \ln |\Lambda^q \Phi_y(t)|$ is the sum $\sum_{i=1}^{q} \beta_i$ of the q largest Lyapounov exponents of $(2)_{y,\lambda}$. By 3.1, equation $(2)_{y,\lambda}$ has q positive and p negative Lyapounov exponents, and moreover the positive exponents are all defined by solutions with initial conditions in $V_y^u(\lambda)$. By (25) and [36] we have for μ - a.a.y: $\lim_{t\to\infty} (1/t) \ln |\Lambda^q \Phi_y(t)| = \sum_{i=1}^{q} \beta_i = \beta(\lambda)$.

Next we borrow an idea from Herman [22] and Craig-Simon [10] and note that the function $\sigma(\lambda) = \varliminf_{t\to\infty} (1/t) \ln |\Lambda^q \Phi_y(t; \lambda)|$ is subharmonic in the entire complex plane. We have $\beta(\lambda) = \sigma(\lambda)$ $(\lambda \in H^+)$, and $\hat{\beta}(\lambda) = \sigma(\lambda)$ $(\lambda \in \boldsymbol{R})$. The function $\sigma(\lambda)$ has the following properties:

(i) $\varlimsup_{\lambda\to\lambda_0} \sigma(\lambda) \leqq \sigma(\lambda_0)$;

(ii) $\sigma(\lambda_0) = \lim_{r\to 0^+} \frac{1}{\pi} \int\limits_{|\varrho|\leqq 1} \sigma(\lambda_0 + r\varrho)\, dA$ for all $\lambda_0 \in \boldsymbol{C}$.

Fix $\lambda_0 \in \boldsymbol{C}$, and let $\tilde{\sigma}(\varrho) = \varlimsup_{r\to 0^+} \sigma(\lambda_0 + r\varrho) \leqq \sigma(\lambda_0)$ for $|\varrho| \leqq 1$. Then

$$\sigma(\lambda_0) \leqq \frac{1}{\pi} \int\limits_{|\varrho|\leqq 1} \tilde{\sigma}(\varrho)\, dA \leqq \sigma(\lambda_0)$$

we have used Fubini's theorem and the uniform boundedness of σ on compact subsets of $\boldsymbol{C}$. We conclude that $\tilde{\sigma}(\varrho) = \sigma(\lambda_0)$ for almost all ϱ, $|\varrho| \leqq 1$.

The last remark is applied as follows. Since β is positive and harmonic on H^+, it has non-tangential boundary values $\lim_{r\to 0^+} \beta(\lambda_0 + r\varrho)$ $(\varrho \in H^+)$ for a.a. $\lambda_0 \in \boldsymbol{R}$. From the preceding paragraph, we get

$$\lim_{r\to 0^+} \beta(\lambda_0 + r\varrho) = \hat{\beta}(\lambda_0) \qquad (\varrho \in H^+)\,, \tag{27}$$

for a.a.$\lambda_0 \in \boldsymbol{R}$. This is the convergence result we wanted.

Let us turn to $\operatorname{Im} w(\lambda) = \alpha(\lambda)$, and show that $\hat{\alpha}(\lambda_0) = \lim_{r\to 0^+} \alpha(\lambda_0 + r\varrho)$ for *all* $\lambda_0 \in \boldsymbol{R}$ and all $\varrho \in H^+$. In fact α is continuous on cls H^+.

First of all, for a.a. $\lambda_0 \in \boldsymbol{R}$, the limit $\hat{m}(y) = \lim_{r\to 0^+} m_+(y, \lambda + r\varrho) \in \bar{D}$ exists for μ - a.a. y and is μ-measurable (and independent of $\varrho \in H^+$). This follows from Fubini's theorem and a standard result on boundary behavior of bounded holomorphic functions [15].

Let $\lambda_0 \in \boldsymbol{R}$ be such that $\hat{m}(y)$ is well-defined for μ - a.a.y. Consider the μ-integrable functions $g_\lambda(y) = \operatorname{tr} f_+(y, m_+(y, \lambda))$ $(\lambda \in H^+)$. If $\lambda \to \lambda_0$ non-tangentially $(\lambda \in H^+)$, then $g_\lambda(y) \to \hat{g}(y) = \operatorname{tr} f_+(y, \hat{m}(y))$ for μ - a.a. y. Moreover we can apply Lebesgues dominated convergence theorem (see (18) and 1.3): we get

$$\lim_{r\to 0^+} w(\lambda_0 + r\varrho) = \lim_{r\to 0^+} \int\limits_Y g_{\lambda_0 + r\varrho}(y)\, d\mu(y) = \int\limits_Y \hat{g}(y)\, d\mu(y)$$

for all $\varrho \in H^+$. By the Birkhoff theorem and Liouvilles formula, we have for μ - a.a. y:

$$\int\limits_Y \hat{g}(y)\, d\mu(y) = \lim_{t\to\infty} \frac{1}{t} \ln \det d_{\hat{m}(y)} \Phi_y(t)$$

and since $\hat{m}(y) \in \bar{D}$ we have by 2.9:

$$\operatorname{Im}\int_Y \hat{g}(y)\, d\mu(y) = (p+q)\hat{\alpha}(\lambda_0)\,.$$

Since $\hat{\alpha}$ is continuous on $\boldsymbol{R}$ (2.10), we see that in fact $\alpha = \operatorname{Im} w$ is continuous on cls H^+ with boundary value $\hat{\alpha}$ [15].

Summing up:

4.2. THEOREM. – Let $\hat{w}(\lambda) = \hat{\beta}(\lambda) + i\hat{\alpha}(\lambda)$ be the quantity defined in § 2. The function $w(\lambda)$ is holomorphic on H^+ with boundary value $\hat{w}$: that is, $\operatorname{Re} w(\lambda) \to \hat{\beta}(\lambda_0)$ non-tangentially for a.a. $\lambda_0 \in \boldsymbol{R}$, and $\operatorname{Im} w(\lambda) \to \hat{\alpha}(\lambda_0)$ continuously for $\lambda_0 \in \boldsymbol{R}$.

4.3 REMARK. – It is perhaps worth noting that one can prove (27) without appealing to subharmonicity, by introducing the (Iwasawa) decomposition $\mathfrak{S}L(K, \boldsymbol{C}) = KS$, where $K_0 \subset K = U(p, q)$ and $S_0 \subset S$. Let $\operatorname{Im} \lambda > 0$, and let $u_0 \in K$ be such that $u_0 m_* = m_+(y, \lambda)$ where $m_* = \begin{pmatrix} 0 \\ 1_p \end{pmatrix} \in \partial D$. Writing $\Phi_y(t) u_0 = u_y(t) T(y, u_0, t)$, one finds that the individual Lyapounov exponents $0 > \beta_{q+1} \geq \ldots \geq \beta_{q+p}$ are obtained by averaging certain elements of the matrix function $(y, u_0) \to (d/dt) T(y, u_0, t)|_{t=0}$.

One gets (27) by a limiting argument, using the measurable section $y \to \hat{m}(y)$ discussed above. One must show that $u_y^{\infty}(t) \in K$ does not contribute to exponential growth of solutions; one does so by using a metric on $\mathfrak{S}_p$ with respect to which each $u \in K$ acts isometrically [21, Chapt. VIII]. See [30] for similar ideas and for various techniques needed to rigorize this discussion.

5. – $w(\lambda)$ and spectral theory.

Our final project is to apply $w(\lambda)$ to the spectral theory of

$$(2)_{y,\lambda} \qquad \frac{dx}{dt} = \left(\lambda J_0^{-1} \gamma_y(t) + y(t)\right) x\,.$$

We will use the following basic formula. Fix $\lambda \in H^+$, and write $\varrho_y(t) = \lambda J_0^{-1}\gamma_y(t) + y(t)$. Consider variations of the form $\delta\varrho_y(t) = \delta r(\tau_t(y))$, where $\delta r\colon Y \to u(p, q)$ is continuous. Then

$$(28) \qquad -\delta w = \int_Y \operatorname{tr}\left(Q_y - \frac{1}{2} I\right) \delta r(y)\, d\mu(y) + \frac{i}{2}\frac{q-p}{q+p}\int_Y \operatorname{tr} \delta r(y)\, d\mu(y)\,,$$

where $Q_y\colon \boldsymbol{C}^k \to \boldsymbol{C}^k$ is the projection with range $V_y^s(\lambda)$ and kernel $V_y^u(\lambda)$. Here tr = trace of a $k \times k$ matrix. Of course the first term in (28) is the interesting one. See [29] for a special case of (28).

It is worthwhile to state explicity the precise meaning of (28). Let $\mathcal{R} = \{r\colon Y \to u(p, q) | r$ is continuous$\}$ with the uniform norm $|\cdot|_\infty$ For fixed $\lambda \in H^+$, $w = w(\lambda)$ defines a mapping from $\mathcal{R}$ into $\boldsymbol{C}$ via $r \to w(\lambda J_0^{-1} + y + r_y)$, where $r_y(t) = r(\tau_t(y))$. We write $r \to w(r)$ for this mapping (see two paragraphs below for even more precision in its definition). Then (28) is to be interpreted as saying that w is Frechet differentiable at $r = 0$, and $\delta w \overset{\text{def}}{=} (d_{r=0} w)(\delta r) = -\int_Y \operatorname{tr}(Q_y - \frac{1}{2}I)\,\delta r(y)\,d\mu(y)$.

The proof of (28) does not depend on the spectral theory of $(2)_{y,\lambda}$. Therefore we first prove (28), then use it to obtain spectral information.

During the proof of (28), we fix $\lambda \in H^+$ and drop it from the notation.

We begin with the promised comment on the definition of $w(r)$. We take the point of view that Y is a fixed compact metric space with flow $\{\tau_t\colon t \in \boldsymbol{R}\}$ and ergodic measure μ such that Supp $\mu = Y$. Writing $y(t) = e(\tau_t(y))$ where $e \in L^1(Y, q, \mu)$ (see 1.3), we have differential equations

$$(29)_{y,r} \qquad \frac{dx}{dt} = [\lambda J_0^{-1}\gamma_y(t) + e(\tau_t(y)) + r(\tau_t(y))]x$$

for $y \in Y$ and $r \in \mathcal{R}$ (λ is omitted from the subscript). Clearly $(29)_{y,0}$ coincides with $(2)_{y,\lambda}$. It is equally clear that we can carry out all steps of § 1-4 for equations $(29)_{y,r}$, obtaining m-functions $m_\pm(y, r)$ and a Floquet exponent $w(r)$.

We need a result on perturbation of the bundles V^s, V^u due to Coppel ([9]; also [28]). Write $V^s(r)$, $V^u(r)$ for the bundles defined by equations (29) (recall $\lambda \in H^+$ is fixed). Let $Q_y(r)$ be the projection with range $V_y^s(r)$ and kernel $V_y^u(r)$. Then $Q_y(0)$ corresponds to equation $(2)_{y,\lambda}$ ($y \in Y$).

5.1 THEOREM. – (i) There is an open set $B \subset \mathcal{R}$ containing $r = 0$ and a constant C such that, if $r \in B$, then

$$\sup_y |Q_y(r) - Q_y(0)| \leq C|r|_\infty ,$$

where $|\cdot$ is the Euclidean norm on linear self-maps of $\boldsymbol{C}^k$.

(ii) The constants K, α of 1.10 can be chosen independent of $r \in B$.

With Theorem 5.1 at hand, it is easily seen that $r \to Q_y(r)\colon \mathcal{R} \to$ the Banach space of continuous maps $Y \to \mathrm{gl}\,(k, \boldsymbol{C})$ with the sup norm is Frechet differentiable at $r = 0$. We outline the argument. Let $C_\pm = \{f\colon Y \to M_{qp}$ (plus sign) or M_{qp} (minus sign)$|f$ continuous$\}$ with the sup norms. It is sufficient to show that the functions $\varphi_\pm\colon \mathcal{R} \to \mathcal{C}_\pm\colon \varphi_\pm(r)(y) = m_\pm(y, r)$ are Frechet differentiable at $r = 0$.

Fix $y \in Y$, and write $m_0(t) = m_+(\tau_t(y), 0)$. Then $m_0(t)$ satisfies $\left(\text{with } \lambda J_0^{-1}\gamma_y + y = \right.$ $\left. = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$:

$$(15) \qquad m' = -mbm + dn - ma + c\,.$$

Write $\delta r_y = \begin{pmatrix} \delta a & \delta b \\ \delta c & \delta d \end{pmatrix}$ where $\delta r \in \mathfrak{R}$, and consider the equation

$$(30) \qquad (\delta m)' = f_+(m_0(t))\,\delta m = -\,m_0\,\delta b m_0 + \delta d m_0 - m_0\,\delta a + \delta c \equiv q(t)\,.$$

Let $\Psi(t)$ be the fundamental matrix of the homogeneous equation $(\delta m)' = f_+(m_0(t))\,\delta m$ such that $\Psi(0) = I$. Since the homogeneous equation is uniformly stable as $t \to -\infty$ (this follows from (22) and (24)), equation (30) has a unique bounded solution $\delta m(t) = \int_{-\infty}^{t} \Psi(t)\Psi^{-1}(s) q(s)\,ds$. One can show that $\delta m(t) = \delta \hat{m}(\tau_t(y))$ where $\delta\hat{m} \in \mathfrak{C}_+$. Using 5.1 (i) and the fact that $\sup_t |q(t)| = O(|\delta r|_\infty)$, one can show that $m_+(\tau_t(y), r) - m_t(\tau_t(y), 0) = \delta m(t) + o(|\delta r|_\infty)$. The mapping $\delta r \to \delta m$ is bounded (this again uses 5.1 (ii)), and hence is the Frechet derivative of φ_+ at $r = 0$.

One can similarly show that φ_- is Frechet differentiable at $r = 0$. In fact a little more work shows that $\varphi_\pm$ are C^1 functions on $B \subset \mathfrak{R}$.

Let us now turn to the proof of (28). For $r \in \mathfrak{R}$ and $y \in Y$, let

$$N_r(t) = \begin{pmatrix} 1_p & m_-(\tau_t(y), r) \\ m_+(\tau_t(y), r) & 1_q \end{pmatrix}$$

Let Q_* be the constant projection with matrix $\begin{pmatrix} 1_p & 0 \\ 0 & 0 \end{pmatrix}$: The change of variables $x = N_r(t)z$ brings $(29)_{y,r}$ to diagonal form:

$$(31) \qquad \frac{dz}{dt} = \begin{pmatrix} a + bm_+ & 0 \\ 0 & cm_- + d \end{pmatrix} z \equiv \sigma_r(t) z\,.$$

With an eye to (22) and (26), consider

$$w_1(r) = \tfrac{1}{2}\int_Y [\operatorname{tr}(a + bm_+) - \operatorname{tr}(cm_- + d)]\,d\mu(y)\,.$$

For μ - a.a. y we have:

$$w_1(r) = \lim_{t\to\infty} \frac{1}{2t}\int_0^t [\operatorname{tr}(a + bm_+) - \operatorname{tr}(cm_- + d_-)]\,ds = \lim_{t\to\infty} \frac{1}{t}\int_0^t \operatorname{tr}\left(Q_* - \frac{1}{2} I\right)\sigma_r(s)\,ds\,.$$

Now, by the change of variable formula

$$\sigma_r(t) = N_r^{-1}(\varrho_y(t) + r_y(t))\,N_r - N_r^{-1} N_r'\,,$$

where $\varrho_y(t) = \lambda J_0^{-1}\gamma_y(t) + y(t)$. Hence for μ - a.a. y:

$$(32) \qquad w_1(r) - w_1(0) = I_1 - I_2\,,$$

where

$$I_1 = \lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr} \left\{ \left(Q_* - \frac{1}{2} I \right) (N_r^{-1} (\varrho_y + r_y) N_r - N_0^{-1} \varrho_y N_0) \right\} ds ,$$

$$I_2 = \lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr} \left\{ \left(Q_* - \frac{1}{2} I \right) (N_r^{-1} N_r' - N_0^{-1} N_0') \right\} ds .$$

Let us now show that $\int_0^t \{\operatorname{tr}(2Q_* - I) N_0^{-1} N_0'\}\, ds$ is uniformly bounded. The same argument will show that $\int_0^t \operatorname{tr}\{(2Q_* - I) N_r^{-1} N_r'\}\, ds$ is bounded, and it will follow that $I_2 = 0$.

To begin, consider the $p \times p$ matrix function $H(t) = 1_p - m_-(t) m_+(t)$, where $m_\pm(t) = m_\pm(\tau_t(y), 0)$. Using [31, Chpt. 2], one can show that $\ln \det H(t)$ is uniformly bounded (the eigenvalues of $H(t)$ lie in the right half-plane). Therefore, using Liouvilles formula,

$$\int_0^t \operatorname{tr}\{(-m_-' m_+ - m_- m_+')(1_p - m_- m_+)^{-1}\}\, ds \quad \text{is bounded} . \tag{33}$$

Similarly,

$$\int_0^t \operatorname{tr}\{(-m_+' m_- - m_+ m_-')(1_q - m_+ m_-)^{-1}\}\, ds \quad \text{is bounded} . \tag{34}$$

Now,

$$N_0^{-1} = \begin{pmatrix} 1_p & -m_- \\ -m_+ & 1_q \end{pmatrix} \begin{pmatrix} (1_p - m_- m_+)^{-1} & 0 \\ 0 & (1_q - m_+ m_-)^{-1} \end{pmatrix} \quad \text{and} \quad 2Q_* - I = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix}.$$

Thus

$$\operatorname{tr}\{(2Q_* - I) N_0^{-1} N_0'\} = \operatorname{tr}(N_0' N_0^{-1} (2Q_* - I)\} =$$

$$= \operatorname{tr}\left\{ \begin{pmatrix} 0 & m_-' \\ m_+' & 0 \end{pmatrix} N_0^{-1} \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix} \right\} = - m_-' m_+ (1_p - m_- m_+)^{-1} + m_+' m_- (1_q - m_+ m_-)^{-1} .$$

In the next-to-last term, we can replace $\begin{pmatrix} 0 & m_-' \\ m_+' & 0 \end{pmatrix} N_0^{-1}$ by

$$\begin{pmatrix} 1_p & -m_- \\ -m_+' & 1_q \end{pmatrix} \begin{pmatrix} 0 & m_-' \\ m_+' & 0 \end{pmatrix} \begin{pmatrix} (1_p - m_- m_+)^{-1} & 0 \\ 0 & (1_q - m_+ m_-)^{-1} \end{pmatrix} .$$

Doing so yields $\operatorname{tr}\{(2Q_* - I) N_0^{-1} N_0'\} = - m_- m_+' (1_p - m_- m_+)^{-1} + m_+ m_-' (1_q - m_+ m_-)^{-1}$.

Now add the two expressions for tr $\{(2Q_* - I)N_0^{-1}N_0'\}$ and use (33) and (34). We conclude that $I_2 = 0$, as desired.

Turning now to I_1, we can write $I_1 = I_3 + I_4$, where

$$I_3 = \lim_{t\to\infty}\frac{1}{t}\int_0^t \operatorname{tr}\left\{\left(Q_* - \frac{1}{2}I\right)[N_r^{-1}(\varrho_y + r_y)N_r - N_0^{-1}(\varrho_y + r_y)N_0\right\}ds\,,$$

$$I_4 = \lim_{t\to\infty}\frac{1}{t}\int_0^t \operatorname{tr}\left\{\left(Q_* - \frac{1}{2}I\right)N_0^{-1}r_yN_0\right\}ds\,.$$

We clain that $I_3 = o(|r|_\infty)$ if $r \in B$. This follows easily from Frechet differentiability of $\varphi_\pm$. For,

$$N_r^{-1}(\varrho_y + r_y)N_r - N_0^{-1}(\varrho_y + r_y)N_0 = N_0^{-1}(\varrho_y + r_y)(N_r - N_0) -$$
$$- N_0^{-1}(N_r - N_0)N_0^{-1}(\varrho_y + r_y)N_0 + o(|N_r - N_0|_\infty)\,.$$

Multiplying by $Q_* - \frac{1}{2}I$, taking the trace, and permuting factors, we get $I_3 = o(|N_r - N_0|_\infty) = o(|r|_\infty)$.

We conclude that, if $r \in B$, then for μ - a.a.y,

$$w_1(r) - w_1(0) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \operatorname{tr}\left\{\left(Q_* - \frac{1}{2}I\right)N_0^{-1}r_yN_0\right\}ds + o(|r|_\infty) =$$
$$= \lim_{t\to\infty}\frac{1}{t}\int_0^t \operatorname{tr}\left\{\left(Q_y - \frac{1}{2}I\right)r_y\right\}ds + o(|r|_\infty) = \int \operatorname{tr}\left\{\left(Q_y - \frac{1}{2}I\right)r(t)\right\}d\mu(y) + o(|r|_\infty)\,.$$

We have used the fact that $Q_{\tau_s(y)} = N_0^{-1}(t)Q_*N_0(t)$ $(y \in Y,\, t \in \boldsymbol{R})$ and the Birkhoff ergodic theorem. This completes the proof of (28).

5.2 REMARK. – If one leaves out the factor $2Q_* - I$ in the computation showing that $I_2 = 0$, one obtains that $\ln\det N_0(t)$ is uniformly bounded. This fact was used in proving (26).

Let us show how to apply (28) to spectral problems. We consider two examples.

5.3 EXAMPLE. – Let $J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$, $\gamma_y(t) = \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}$, $y(t) = \begin{pmatrix} 0 & 1_n \\ q(t) & 0 \end{pmatrix}$ where $q(t)$ is $n\times n$ real and symmetric. Then $(2)_{y,\lambda}$ is equivalent to the operator equation

$$L_y\varphi = \left(-\frac{d^2}{dt^2} + q(t)\right)\varphi = \lambda\varphi\,, \qquad \varphi \in \boldsymbol{C}^n\,,$$

where $x = \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}$. We define L_y to be the closure of the operator $-d^2/dt^2 + q(t)$ with domain $C_c^\infty(R, \boldsymbol{C}^n)$ and range $L^2(\boldsymbol{R}, \boldsymbol{C}^n)$; then L is self-adjoint ([14]; by 3.1, no boundary conditions at $\pm\infty$ are needed).

Applying (28) with $\delta r = \delta\lambda J^{-1}\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}$ and using analyticity of w, we have

$$(35)\qquad -\frac{dw}{d\lambda} = \int_Y \operatorname{tr}\left\{\left(Q_y - \frac{1}{2}I\right)J^{-1}\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}\right\} d\mu(y)\,.$$

Define

$$(36)\qquad \mathfrak{S}_y(t, s; \lambda) = \begin{cases} \Phi_y(t) Q_y \Phi_y^{-1}(s) & t \geqq s \\ -\Phi_y(t)(I - Q_y)\Phi_y^{-1}(s) & t < s\,, \end{cases}$$

and note that

$$(37)\qquad Q_y - \tfrac{1}{2}I = \tfrac{1}{2}\left[\lim_{s\to 0^+} \mathfrak{S}_y(0, s; \lambda) + \lim_{s\to 0^-} \mathfrak{S}_y(0, s; \lambda)\right].$$

We clearly have

$$\begin{pmatrix} (L_y - \lambda)^{-1} f(t) \\ 0 \end{pmatrix} = \int_{-\infty}^{\infty} \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} \mathfrak{S}_y(t, s; \lambda) J^{-1} \begin{pmatrix} f(s) \\ 0 \end{pmatrix} ds \qquad (\operatorname{Im}\lambda > 0,\ f \in L^2(\boldsymbol{R}, \boldsymbol{C}^n))\,.$$

Write

$$\begin{pmatrix} \hat{\mathfrak{S}}_y & * \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} \mathfrak{S}_y J^{-1}\,.$$

Recall that, if $\Delta\hat{\mathfrak{S}}_y(\lambda) = \frac{1}{2}\left[\lim_{s\to 0^+} \hat{\mathfrak{S}}_y(0, s, \lambda) + \lim_{s\to 0^-} \hat{\mathfrak{S}}_y(0, s, \lambda)\right]$, then

$$(38)\qquad \frac{\operatorname{Im} \Delta\hat{\mathfrak{S}}_y(\lambda)}{\operatorname{Im}\lambda} = \int_{-\infty}^{\infty} \frac{dP_y(t)}{|t - \lambda|^2} \qquad (\operatorname{Im}\lambda > 0)\,,$$

where P_y is the spectral matrix of L_y (thus $P_y(\cdot)$ is symmetric, $P_y(t) - P_y(s) \geqq 0$ if $t \geqq s$, and the increase points of P_y determine the spectrum of L_y).

By (35), (37), and (38), we have

$$(39)\qquad -\frac{dw}{d\lambda} = \int_Y \operatorname{tr} \Delta\hat{\mathfrak{S}}_y(\lambda)\, d\mu(y) \qquad (\operatorname{Im}\lambda > 0)\,.$$

Arguing as in [20] or [29], we find that, if $f \in C_c^\infty(\boldsymbol{R})$, then

$$(40)\qquad -\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\, d\alpha(t) = \int_Y \left(\operatorname{tr}\int_{-\infty}^{\infty} f(t)\, dP_y(t)\right) d\mu(y)\,,$$

or less prosaically

$$(41)\qquad -\frac{1}{\pi}\, d\alpha = \int_Y (\operatorname{tr}\, dP_y)\, d\mu(y)\,.$$

Using ergodicity of μ, one shows in a well-known way [4, 37] that the spectrum of L_y is independent of y for μ - a.a.y. Hence $-\alpha$ is non-decreasing and increases exactly on the spectrum of L_y for μ - a.a. y.

5.4 EXAMPLE. – We consider the AKNS operator ([1], also ZAKHAROV-SHABAT [47]):

$$L_y x = J_0\left(\frac{d}{dt} - y(t)\right)x = \lambda x \qquad x \in \boldsymbol{C}^{2n}, \qquad J_0 = \begin{pmatrix} -i1_n & 0 \\ 0 & i1_n \end{pmatrix}$$

where $y(t) \in U(n, n)$. The closure of $J_0(d/dt - y(t))$ on $C_c^\infty(\boldsymbol{R}, \boldsymbol{C}^{2n})$ is a self-adjoint operator on $L^2(\boldsymbol{R}, \boldsymbol{C}^{2n})$ (note that this is not true if $p \neq q$, because then by 3.1 the deficiency indices [14] p and q of $J_0(d/dt - y(t))$ are not equal).

Let $\mathfrak{S}_y(t, s; \lambda)$ be as in (36). Define

$$\hat{\mathfrak{S}}_y = \mathfrak{S}_y J_0^{-1}, \qquad \Delta\hat{\mathfrak{S}}_y(\lambda) = \frac{1}{2}\left[\lim_{s\to 0^+} \hat{\mathfrak{S}}_y(0, s, \lambda) + \lim_{s\to 0^-} \hat{\mathfrak{S}}_y(0, s, \lambda)\right] = \left(Q_y - \frac{1}{2}I\right)J_0^{-1}.$$

Then

$$\frac{\operatorname{Im} \Delta\hat{\mathfrak{S}}_y(\lambda)}{\operatorname{Im} \lambda} = \int_{-\infty}^{\infty} \frac{dP_y(t)}{|t - \lambda|^2}$$

where $P_y(\cdot)$ is the $2n \times 2n$ spectral matrix of L_y, and

$$-\frac{dw}{d\lambda} = \int_Y \operatorname{tr} \Delta\hat{\mathfrak{S}}_y(\lambda)\, d\mu(y) \qquad (\operatorname{Im} \lambda > 0),$$

$$-\frac{1}{\pi}\, d\alpha = \int_Y (\operatorname{tr} dP_y)\, d\mu(y).$$

As before, $-\alpha$ is non-decreasing, and increases exactly on the spectrum of L_y for μ - a.a.y.

5.5 REMARK. – We can put these examples in a more general framework, as follows. Consider the general equations $(2)_{y,\lambda}$. Following Atkinson [3, Chpt. 9],1 et $a < 0 < b$, and introduce self-adjoint boundary conditions N, M at a, b respectively. Thus N, M are $(p + q)\times(p + q)$ matrices satisfying $N^* J_0 N = M^* J_0 M$, and $Mx = Nx = 0 \Rightarrow x = 0$. Consider $(2)_{y,\lambda}$ on $[a, b]$ with the boundary conditions $x(a) = Nv$, $x(b) = Mv$ for $v \in \boldsymbol{C}^k$. One obtains a spectral matrix $P_y^{NM}(t)$ and a « characteristic function » [3] $F_y^{NM}(\lambda)$ such that

$$\frac{\operatorname{Im} F_y^{NM}(\lambda)}{\operatorname{Im} \lambda} = \int_{-\infty}^{\infty} \frac{dP_y^{NM}(t)}{|t - \lambda|^2} \qquad (y \in Y;\ \operatorname{Im} \lambda > 0).$$

Now let $a \to -\infty$, $b \to \infty$. Using 3.1 and (37), one can show that $F_y^{NM}(\lambda) \to$ $\to (Q_y(\lambda) - \frac{1}{2}I)J_0^{-1} \overset{\text{def}}{=} F_y(\lambda)$ uniformly on compact subsets of H^+, independent of N, M. It follows that dP_y^{NM} converges weakly to a matrix-valued measure dP_y, which is also independent of N and M.

Using (28), we obtain

$$-\frac{dw}{d\lambda} = \int_Y \operatorname{tr}\{F_y(\lambda)\cdot\gamma(y)\}\,d\mu(y) + \frac{i}{2}\frac{q-p}{q+p}\int_Y \operatorname{tr} J_0^{-1}\gamma(y)\,d\mu(y)\,,$$

$$-\frac{1}{\pi}\,d\alpha = \int_Y \operatorname{tr}\{dP_y\cdot\gamma(y)\}\,d\mu(y) + \frac{1}{2}\frac{q-p}{q+p}\int_Y \operatorname{tr} J_0^{-1}\gamma(y)\,d\mu(y)\,.$$

We finish the paper with a discussion of «gap-labelling» for equations $(2)_{y,\lambda}$. To avoid obscuring the simple ideas involved with technical complications, we assume equations $(2)_{y,\lambda}$ take the form either of Example 5.3 or that of Example 5.4. Thus $(2)_{y,\lambda}$ is equivalent to $L_y\varphi = \lambda\varphi$ resp. $L_y x = \lambda x$ where L_y is as in 5.3 resp. 5.4.

We need a preliminary result which is of independent interest.

5.6 THEOREM. – Let $Y \subset \xi_g$ be a bounded translation invariant subset which satisfies (8) of § 1 (hence is compact metric). Let $\bar{y} \in Y$ have dense orbit, and let $L_{\bar{y}}$ be the corresponding operator. Then λ_0 is in the resolvent of $L_{\bar{y}}$ iff equations $(2)_{y,\lambda_0}$ have ED.

The ergodic measure μ plays a role neither in the statement nor in the proof of 5.6.

PROOF. – The proof generalizes that given in [27] in the case $k = p + q = 2$. The «if» part of the theorem is easy: one uses the function $\mathfrak{S}_y(t, s; \lambda_0)$ of (36) and the Riesz-Thorin interpolation theorem [48].

To prove the «only if» statement, we first show that no equation $(2)_{y,\lambda}$ admits a non-trivial bounded solution. For if $x_0(t)$ is a bounded solution of $(1)_{y,\lambda}$, then it can be used to construct a sequence $\{x_s\}_{s=1}^{\infty} \subset L^2(\boldsymbol{R}, \boldsymbol{C}^n \text{ or } \boldsymbol{C}^{2n})$ such that

$$\|L_y x_s - \lambda_0 x_s\|_2 < \frac{\|x_s\|_2}{s}\ (s = 1, 2, \ldots)\,.$$

Hence λ_0 is in the spectrum of L_y [14]. Now, there is an interval $(\lambda_0 - \delta, \lambda_0 + \delta)$ in the resolvent of $L_{\bar{y}}$. Since $L_{\bar{y}}$ and $L_{\tau_t(\bar{y})}$ have the same spectrum for all $t \in \boldsymbol{R}$ (they are conjugate under translation by t), it follows that $P_{\tau_t(\bar{y})}$ is constant on $(\lambda_0 - \delta,$ $\lambda_0 + \delta)$ for all $t \in \boldsymbol{R}$. Next, the family of spectral measures $\{\operatorname{tr} dP_y | y \in Y\}$ is weakly continuous in y, i.e., $\int_{\infty}^{\infty} f(t) \operatorname{tr} dP_{y_j}(t) \to \int_{\infty}^{\infty} f(t) \operatorname{tr} dP_y(t)$ if $y_j \to y$, for all $f \in C_c^{\infty}(\boldsymbol{R})$. This follows from joint continuity of the characteristic function $F_y(\lambda) = (Q_y(\lambda) - \frac{1}{2}I)J_0^{-1}$. It follows from these statements and density of $\{\tau_t(\bar{y}) : t \in \boldsymbol{R}\}$ that $\operatorname{tr} P_y(t) = \text{const.}$ on $(\lambda_0 - \delta, \lambda_0 + \delta)$. This is a contradiction. Hence no equation $(1)_{y,\lambda_0}$ has a bounded solution.

Next let $Y_1 \subset Y$ be a minimal set. Then [39, 42] equations $(2)_{y,\lambda_0}$ have ED over Y_1. Recall that the bundles $V^s(\lambda)$, $V^u(\lambda)$ are continuous in λ [9]. Hence there is a disc D centered at λ_0 such that equations $(2)_{y,\lambda}$ have ED and $V^s(\lambda)$, $V^u(\lambda)$ have constant dimension. By 3.1, these dimensions are both equal to n (in Example 5.3 and Example 5.4). Since this is true for any minimal $Y_1 \subset Y$, the Sacker-Sell result [40] implies that equations $(2)_{y,\lambda_0}$ have ED.

5.7 REMARK. – The last paragraph of the proof shows that, if $y(t) \in U(p, q)$ and $\lambda \in \boldsymbol{R}$, then equations $(2)_{y,\lambda}$ can have ED only if $p = q$.

Now we prove gap labelling for the operators L_y. The ergodic measure μ plays a crucial role in this result.

5.8 THEOREM. – There is a countable set $A_0 \subset \boldsymbol{R}$, depending only on the topological space Y, such that, if $(\lambda_1, \lambda_2) \subset \boldsymbol{R}$ is in the resolvent of L_y for μ - a.a. y, then $\alpha(\lambda) \in A_0$ for all $\lambda \in (\lambda_1, \lambda_2)$.

PROOF. – This result is proved for $k = p + q = 2$ in [27].

We first define A_0. Following Schwarzmann [41], let $H(Y, T)$ be the set of homotopy classes of continuous maps φ from Y to the unit circle $T \subset \boldsymbol{C}$. Each such class $[\varphi]$ contains a map φ such that $y \to (d/dt)\varphi(\tau_t(y))|_{t=0} = \varphi'(y)$ is continuous. The map h:

$$[\varphi] \to \int_Y \frac{\varphi'(y)}{\varphi(y)}\, d\mu(y) = \lim_{t\to\infty} \frac{1}{t} \arg \varphi(\tau_t(y)) \quad \mu\text{ - a.e.}$$

defineds a homomorphism from $H(Y, T)$ to the additive reals (the group structure on $H(Y, T)$ is defined by multiplication). In fact h induces a homomorphism from $H^1(Y; Z)$ = group of *real* Cech 1-cocycles taking integer values on *integer* Cech cycles into $\boldsymbol{R}$. Let $A_0 = \{\frac{1}{2}h([\varphi]) | [\varphi] \in H(Y, T)\}$.

Next let $\lambda_0 \in (\lambda_1, \lambda_2)$, and let $V^s(\lambda_0)$ be the corresponding stable bundle. Let $m_* = 1_n \in M_{nn}$, and let $N_1 = \text{cls}\{g \cdot m_*: g \in U(n, n)\}$. By 2.4 (i), $N_1 = \{g \cdot m_*: g \in K_0\}$, where $K_0 = \left\{\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \middle| u_1, u_2 \in U(n)\right\}$. (We assume from now on that Example 5.3 has been conjugated into $su(n, n)$ via the usual matrix $\begin{pmatrix} i1_n & i1_n \\ -1_n & 1_n \end{pmatrix}$.) Observe that N_1 is homeomorphic to K_0/Δ, where

$$\Delta = \left\{\begin{pmatrix} u_1 & 0 \\ 0 & u_1 \end{pmatrix} \middle| u_1 \in U(n)\right\}.$$

Let $m_0 \in M_{nn}$. It is easily seen that there is an $n_1 \in N_1$ such that the planes $l_0, l_1 \in \mathfrak{S}$ parametrized by m_0, n_1 satisfy $l_0 \cap l_1 = \{0\}$. In fact, a non-zero element of $l_0 \cap l_1$ is defined by a vector $v \in \boldsymbol{C}^n$ such that $(n_1 - m_0)v = 0$, so one need only choose $n_1 \in U(n)$ for which no $v \neq 0$ with this property exists.

Let $y \in Y$. We show that, if $m_y \in \bar{D}$ parametrizes $V_y^s(\lambda_0)$, then $m_y \in N_1$. To see this, let $\bar{y} \in Y$ be as in Lemma 1.2. Choose $n_1 \in N_1$ such that the corresponding plane l_1 satisfies $l_1 \cap V_{\bar{y}}^u(\lambda_0) = \{0\}$. Since the set $Y \times N_1$ is invariant under the flow $(y, m) \to (\tau_t(y), \Phi_y(t)m)$, and since $\Phi_y(t_n)n_1 \to m_y$ if $t_n \to -\infty$ and $t_{t_n}(\bar{y}) \to y$ (1.10), we must have $m_y \in N_1$.

Now let $u = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & u_2 u_1^{-1} \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ 0 & u_1 \end{pmatrix} \in K_0$. Since $\det(m \to u_1 m u_1^{-1}) = 1$, we have $\det u = \det u_2 \det u_1^{-1}$. Let $\widetilde{\det}\, u_i$ $(i = 1, 2)$ be the usual determinant of the $n \times n$ matrix u_i. Then $\det u_i = (\widetilde{\det}\, u_i)^n$. We see that $(\det u)^{1/n}$ factors through $\Delta = \left\{\begin{pmatrix} u_1 & 0 \\ 0 & u_1 \end{pmatrix} \middle| u_1 \in U(n)\right\}$ and hence defines a continuous map $\varphi: N_1 \to T$.

Using 2.9, we see that $2\alpha(\lambda_0) = h(\varphi \circ m_+)$ where h is the Schwarzschild homomorphism. Hence $\alpha(\lambda_0) \in A_0$, as desired. We have written m_+ for the map $y \to m_+(y, \lambda_0)$: $Y \to N_1$. This completes the proof of Theorem 5.8.

5.8 REMARK. – As has been emphasized by BELLISSARD, LIMA, and TESTARD [5], gap labelling is closely related to properties of the trace on a certain crossed-product C^*-algebra.

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