Optimal Enumerations and Optimal Gödel Numberings

by

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1. Introduction. A Gödel-numbering $\varphi = (\varphi_i | i \in N)$ of all partial recursive (p.r.) functions $f\colon N \to N$ is considered to be optimal if any other recursive sequence $(g_i | i \in N)$ of p.r. functions can be effectively translated into φ by a linearly bounded translation. This implies that all p.r. functions have rather low Gödel numbers relative to φ when these numbers are compared with the lowest Gödel numbers relative to any other Gödel numbering. The lowest Gödel number of f relative to an optimal Gödel numbering φ is a distinguished measure of size of p.r. functions in the sense of Blum's axiomatic approach [1]. Lowest Gödel numbers have been independently considered by A. R. Meyer [7].

We prove that for any two optimal Gödel numberings φ and $\bar{\varphi}$ there exists a recursive permutation t such that t and t^{-1} are linearly bounded and $\varphi_i = \bar{\varphi}_{t(i)}$ for all i. It is the author's experience that this result cannot be obtained merely by careful attention to the details of the standard proof of Rogers recursive isomorphism theorem for Gödel numberings [3] in spite of the fact that optimal Gödel numberings admit a linearly bounded S_1^1-function. The original proof of Theorem 4 was based on an improved version of the recursion theorem. However, the referee pointed out that an improved version of the injective translation theorem (Theorem 2) together with a combinatorial construction (Lemma 2) enable us to obtain this result as a corollary of an improved version of Roger's Isomorphism Theorem.

An analogous isomorphism theorem holds for optimal enumerations of recursively enumerable sets. Optimal enumerations are a specification of optimal (universal resp.) algorithms according to Kolmogorov [2] and Solomonoff [5]. Next we establish a relationship between the relative frequency of Gödel numbers of a p.r. function f and the minimal program complexity of f. Functions with low minimal program complexity have a high relative frequency of programs. This seems to be somewhat surprising since one might expect that optimal Gödel numberings avoid many repetitions. But this fact corresponds with theorems of Friedberg [6]. Hereby we know that any one-one enumeration of all p.r. functions must not be effective. Optimal Gödel numberings and optimal

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enumerations are strongly related to one another. It can be shown that the shortest programs of p.r. functions relative to an optimal Gödel numbering are asymptotically random.

2. Optimal Gödel Numberings. $\mathbf{P}_k(\mathbf{R}_k)$ denotes the set of all p.r. (rec.) functions $h: N^k \to N$. Let $g \in \mathbf{P}_{k+1}$ then for every $i \in N$ we denote $g_i = \lambda x[g(i, x)]$. We shall use the concept of (acceptable) Gödel numbering in the sense of Rogers [3].

Definition 1. $\varphi^k \in \mathbf{P}_{k+1}$ is a Gödel numbering (G.N.) of $\mathbf{P}_k$ if and only if $\forall g \in \mathbf{P}_{k+1}: \exists t \in \mathbf{R}_1: \forall i \in N: g_i = \varphi^k_{t(i)}$. Obviously φ^k is G.N. if and only if it satisfies the S-m-n theorem and the universal Turing-machine theorem. In the following let φ^k be a fixed G.N. of $\mathbf{P}_k$. We define the *program complexity* $K_g(h)$ of $h \in \mathbf{P}_1$ relative to $g \in \mathbf{P}_2$ (we use the convention min $\varnothing = \infty$):

$$K_g(h) = \min\{i | g_i = h\}.$$

Let $\|i\|$ be the length of the binary notation of the integer i. We wish to ensure that the size of the functional $\|K_g\|$ essentially does not depend on g. Therefore we introduce the concept of optimal G.N's. A function $g: N \to N$ is linearly bounded (l.b.) if and only if $\lim\sup_n g(n)/n < \infty$. This means that there exists $c \in N$ such that $\|g(i)\| \leq \|i\| + c$ for all $i > 0$.

Definition 2. $\varphi \in \mathbf{P}_2$ is an *optimal* G.N. of $\mathbf{P}_1$ if and only if $\forall f \in \mathbf{P}_2: \exists t \in \mathbf{R}_1$: t is l.b. $\wedge f_i = \varphi_{t(i)}$ for all i.

COROLLARY 1. *Let φ be an optimal G.N. of $\mathbf{P}_1$ and $\bar{\varphi} \in \mathbf{P}_2$. Then there exists $c \in N$: $\|K_\varphi(h)\| \leq \|K_{\bar{\varphi}}(h)\| + c$ for all $h \in \mathbf{P}_1$.*

Optimal Gödel numberings are well known in terms of Turing machines. Any universal Turing machine which is able to simulate the Turing-machine table of any other Turing machine can be considered to be an optimal G.N. We give an easy formal proof of the existence of optimal G.N.'s.

THEOREM 1. *There exists an optimal G.N. φ of $\mathbf{P}_1$.*

Proof. Consider the following pairing function $g \in \mathbf{R}_2: g(n, i) = 2^n + i2^{n+1} - 1$. g is a bijective function such that g_n is l.b. for all n. Then we define $\varphi \in \mathbf{P}_2$ as follows:

$$\varphi(g(n, i), x) = \varphi^2(n, i, x).$$

This implies that φ is an optimal G.N. of $\mathbf{P}_1$.

In the following we shall use that optimal G.N.'s admit a linearly bounded S^1_1-function.

LEMMA 1. *A G.N. φ of $\mathbf{P}_1$ is optimal if and only if it admits a linearly bounded S^1_1-function, i.e., a function $S^1_1 \in \mathbf{R}_1$ such that $\varphi^2_e(i, x) = \varphi_{S^1_1(e, i)}(x)$ for all $e, i, x \in N$ and $\lambda i[S^1_1(e, i)]$ is linearly bounded for all e.*

Proof. Suppose φ is a linearly bounded S^1_1-function. Then for every $f \in \mathbf{P}_2$ we can find an integer e_0 such that $f = \varphi^2_{e_0}$. Hence $\varphi_{S^1_1(e_0, i)}(x) = \varphi^2_{e_0}(i, x) = f(i, x)$. Clearly $t = \lambda i[S^1_1(e_0, i)]$ is the desired t. Suppose now that φ is optimal and let $\bar{\varphi}$ be the optimal G.N. in the proof of Theorem 1. Then by Definition 2 there exists a l.b. $t \in \mathbf{R}^1_1$ such that $\varphi_{tg(n, i)}(x) = \bar{\varphi}_{g(n, i)}(x) = \varphi^2(i, n, x)$, where

$g(n, i) = 2^n + i2^{n+1} - 1$. Hence φ admits the S_1^1-function tg, where $\lambda i[tg(n, i)]$ is l.b. for all i.

As a prelude to our linear isomorphism theorem for optimal Gödel numberings we give a new proof of the injective translation theorem.

The statements and proofs of Theorem 2, Theorem 3 and Lemma 4 are all due to the referee. He pointed out this way to prove Corollary 2, Corollary 3 and Theorem 4.

THEOREM 2. *Let $S_1^1 \in \mathbf{R}_2$ be an S_1^1-function for the G.N. φ of $\mathbf{P}_1$. Then for every $g \in \mathbf{P}_2$ we can effectively find an integer e_0 such that $g_i = \varphi_{S_1^1(e_0, i)}$ for all i and $\lambda i[S_1^1(e_0, i)]$ is injective.*

Proof. Via a standard application of the recursion theorem, we can effectively find an integer e_0 such that

$$\varphi_{e_0}^2(i, x) = \begin{cases} \text{undefined if } \exists j < i\colon [S_1^1(e_0, j) = S_1^1(e_0, i)] \\ 0 \quad \text{if } \forall j < i\colon [S_1^1(e_0, j) \neq S_1^1(e_0, i)] \\ \quad\quad \wedge \exists j > i\colon [j \leq x \wedge S_1^1(e_0, j) = S_1^1(e_0, i)] \\ g_i(x) \quad \text{otherwise.} \end{cases}$$

Clearly $\lambda i[S_1^1(e_0, i)]$ must be one-one, for if not, let i_0 be the smallest i such that $S_1^1(e_0, i) = S_1^1(e_0, j)$ for some $j > i$. Then by the first clause $\varphi_{e_0}^2(j, x) = \varphi_{S_1^1(e_0, j)}(x) = \varphi_{S_1^1(e_0, i_0)}(x)$ is undefined for all x, while by the second clause $\varphi_{e_0}^2(i_0, x) = \varphi_{S_1^1(e_0, i_0)}(x) = 0$ for all $x \geq j$. Thus $\lambda i[S_1^1(e_0, i)]$ is injective. Thus by definition of e_0, $\varphi_{S_1^1(e_0, i)}(x) = \varphi_{e_0}^2(i, x) = g_i(x)$ for all i and all x.

The padding lemma and the injective translation theorem are immediate consequences of Theorem 2.

COROLLARY 2. (Padding Lemma) $\forall i\colon \exists h \in \mathbf{R}_1^1$: *$h$ is injective $\wedge\, \varphi_i = \varphi_{h(j)}$ for all j.*

COROLLARY 3. (Injective Translation Theorem) *Given any two G.N.'s φ and $\bar{\varphi}$ of $\mathbf{P}_1$, we can effectively find an injective translation $t = \lambda i[S_1^1(e_0, i)]$ such that $\bar{\varphi}_i = \varphi_{t(i)}$ for all i.*

It is an easy exercise to prove the recursive isomorphism theorem for G.N.'s by the use of Corollary 3. However, we are interested in a strengthened version of this theorem.

THEOREM 3. (Isomorphism Theorem) *Let φ, $\bar{\varphi}$ be G.N.'s of $\mathbf{P}_1$ and let $t, \bar{t} \in \mathbf{R}_1$ be injective translations, i.e., $\bar{\varphi}_i = \varphi_{\bar{t}(i)}$, $\varphi_i = \bar{\varphi}_{t(i)}$ for all i. Then we can effectively find a permutation $\alpha \in \mathbf{R}_1$ such that*

(1) $\alpha(i) \leq \max\{t(j) | j \leq i\}$, $\alpha^{-1}(i) \leq \max\{\bar{t}(j) | j \leq i\}$, *and*
(2) $\bar{\varphi}_i = \varphi_{\alpha(i)}$ *for all i.*

The proof of Theorem 3 follows immediately from the following combinatorial lemma. Given a relation $A \subset N \times N$ the closure $\langle A \rangle \subset N \times N$ of A is defined by: $(i, j) \in \langle A \rangle \Leftrightarrow$ there is a finite sequence $i = i_0, i_1, \cdots, i_n = j$ such that $(i_k, i_{k+1}) \in A$ for $k = 0, \cdots, n-1$. For any partial function $f\colon N \to N$ we identify $f = \{(x, f(x)) | x \in \text{domain} f\} \subset N \times N$ and $f^{-1} = \{(f(x), x) | x \in \text{domain} f\}$. fg denotes the composition of the relations f and g.

LEMMA 2. *Let $t_0, \bar{t}_0 \in \mathbf{R}_1$ be injective functions. Then we can effectively find a permutation $\alpha \in \mathbf{R}_1$ such that (1) $\alpha \subset t_0\langle \bar{t}_0^{-1} t_0 \cup t_0 \bar{t}_0^{-1}\rangle$, and (2) $\alpha(i) \leq \max\{t_0(j) | j \leq i\}$, $\alpha^{-1}(i) \leq \max\{\bar{t}_0(j) | j \leq i\}$.*

Proof. As is usual in such proofs, graph α is enumerated in stages. Stage $2n$ is used to guarantee that $n \in$ domain α while stage $2n+1$ is used to guarantee that $n \in$ domain α^{-1}. We find it convenient to simultaneously construct two functions $t, \bar{t} \in \mathbf{R}_2$ such that the sequence $\bar{t}_i$ "approximates" α and the sequence t_i "approximates" α^{-1}. We recursively construct α, α^{-1}, t and $\bar{t}$ such that the following relations hold (let α_j be that part of α which is constructed in stages not greater than j):

(1) For every j: t_j and $\bar{t}_j$ are injective

$$t_{j+1} \subset t_j \langle \bar{t}_j t_j \cup t_j^{-1} \bar{t}_j^{-1}\rangle, \qquad \bar{t}_{j+1} \subset \bar{t}_j \langle t_j^{-1} \bar{t}_j^{-1} \cup \bar{t}_j t_j\rangle$$

(2) For every j: $\alpha_j \subset t_{j+1} \cap \bar{t}_{j+1}^{-1} \wedge j \in \text{domain } \alpha_{2j} \wedge j \in \text{domain } \alpha_{2j+1}^{-1}$

(3) For every n and every z: if $t_{n+1}(z) > t_n(z)$ then there exists $y < z$ such that $t_n(y) = t_{n+1}(z)$; if $\bar{t}_{n+1}(z) > \bar{t}_n(z)$ then there exists $y < z$ such that $\bar{t}_n(y) = \bar{t}_{n+1}(z)$.

(4) For every n and every z: $t_{n+1}(z) < t_n(z)$ implies $\alpha(z) = t_{n+1}(z)$; $\bar{t}_{n+1}(z) > \bar{t}_n(z)$ implies $\alpha^{-1}(z) = \bar{t}_{n+1}(z)$.

(1) and (2) imply that α is bijective and $\alpha \subset t_0\langle \bar{t}_0 t_0 \cup t_0^{-1} \bar{t}_0^{-1}\rangle$. Part (2) of the lemma follows from (3) and (4).

Construction of α, t, $\bar{t}$.

Stage n, n even. Calculate $m := \min\{j | j \notin \text{domain } \alpha_{n-1}\}$ and check whether there exists $j < t_n(m)$: $\bar{t}_n(j) = m$. In this case set

$$\alpha(m) := \min\{j < t_n(m) : \bar{t}_n(j) = m\}.$$

Then set $\bar{t}_{n+1} := \bar{t}_n$ and $t_{n+1} := \sigma^{\alpha(m)}_{t_n(m)} t_n$, where σ^a_b permutes the values a and b:

$$\sigma^a_b(z) = \begin{cases} a \text{ if } z = b \\ b \text{ if } z = a \\ z \text{ otherwise.} \end{cases}$$

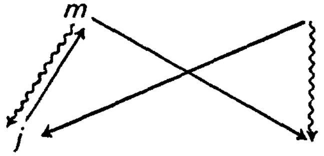
If no such j exists, then set

$$\alpha(m) := t_n(m), \qquad t_{n+1} := t_n, \qquad \bar{t}_{n+1} := \sigma^m_{t_n \alpha(m)} \bar{t}_n.$$

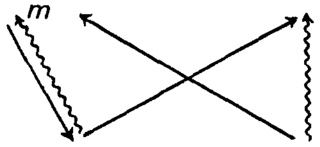
Go on to stage $n+1$.

Stage n, n odd. Proceed as in the case n even but interchange the role of α, α^{-1} and t, $\bar{t}$.

This completes the construction. The relations (1)–(4) are all proved by trivial induction involving straightforward checking of all cases of the construction. We illustrate stage n of the construction with n even. The arrows ↑, ↓, ⇡, ⇣ mean t_n, $\bar{t}_n$, t_{n+1}, $\bar{t}_{n+1}$:



there exists j



no j exists

Theorem 3 yields the following special isomorphism theorem for optimal Gödel numberings.

THEOREM 4. (Linear Isomorphism Theorem) *Let φ and $\bar{\varphi}$ be optimal Gödel numberings of* $\mathbf{P}_1$, *Then we can effectively find a permutation $\alpha \in \mathbf{R}_1$ such that α and α^{-1} are linearly bounded and $\bar{\varphi}_i = \varphi_{\alpha(i)}$ for all i.*

Proof. It follows from Lemma 1 and Theorem 2 that there exist injective functions t and $\bar{t}$ which are linearly bounded such that $\bar{\varphi}_i = \varphi_{\bar{t}(i)}$ and $\varphi_i = \bar{\varphi}_{t(i)}$ for all i. Hence Theorem 4 follows from Theorem 3.

Theorem 2 also implies that there cannot exist a Gödel numbering φ such that K_φ is minimal:

THEOREM 5. *Let $\varphi \in \mathbf{P}_2$ be an optimal G.N. of* $\mathbf{P}_1$. *Then we can effectively construct an optimal G.N. $\bar{\varphi}$ such that* $\lim\sup_n K_{\bar{\varphi}}(\varphi_n)/n < 1$.

Proof. We construct an injective, linearly bounded $h \in \mathbf{R}_1$ such that $\varphi_{h(n)} = \varphi_0$ for all n. Hence by enumerating the functions $(\varphi_i | i \in N - h(N) - \{0\})$ in their natural order of succession one gets an optimal G.N. $\bar{\varphi}$ such that $\lim\sup_n K_{\bar{\varphi}}(\varphi_n)/n < 1$.

It is well known that there exist recursive enumerations ψ of $\mathbf{P}_1$ such that ψ is not a Gödel numbering of $\mathbf{P}_1$ [6]. Some constructions of such a ψ imply that programs relative to ψ get excessively long. Now we give an example of a recursive enumeration ψ of $\mathbf{P}_1$ such that all p.r. functions $g \in \mathbf{P}_1$ have short programs relative to ψ but ψ cannot be a Gödel numbering of $\mathbf{P}_1$.

THEOREM 6. *There exists $\psi \in \mathbf{P}_2$ such that*

(*1*) $\forall g \in \mathbf{P}_2 : \exists c : \forall h \in \mathbf{P}_1 : \|K_\psi(h)\| \leq \|K_g(h)\| + c$;
(*2*) *ψ is not a Gödel numbering of* $\mathbf{P}_1$.

Proof. Let φ be any optimal Gödel numbering of $\mathbf{P}_1$. Calculate ψ as follows:

$$\psi(2i, x) = \begin{cases} \varphi_i(x) & \text{if } x \neq 0 \vee \varphi_i(x) \text{ is even} \\ \varphi_i(x) - 1 & \text{if } x = 0 \wedge \varphi_i(x) \text{ is odd} \\ \text{undefined if } \varphi_i(x) \text{ is undefined.} \end{cases}$$

$$\psi(2i+1, x) = \begin{cases} \varphi_i(x) & \text{if } x \neq 0 \vee \varphi_i(x) \text{ is odd} \\ \varphi_i(x) + 1 & \text{if } x = 0 \wedge \varphi_i(x) \text{ is even} \\ \text{undefined if } \varphi_i(x) \text{ is undefined.} \end{cases}$$

Obviously $K_\psi(h) \leq 2K_\varphi(h) + 1$ for all $h \in \mathbf{P}_1$. This proves part (1) of Theorem 6. Furthermore, the construction of ψ implies:

$$\text{(R)} \qquad \begin{cases} i \text{ even} \Rightarrow \psi_i(0) \text{ even} \vee \psi_i(0) \text{ undefined} \\ i \text{ odd} \Rightarrow \psi_i(0) \text{ odd} \vee \psi_i(0) \text{ undefined.} \end{cases}$$

Now suppose ψ is a Gödel numbering. Then via a standard application of the recursion theorem we construct an integer i such that

$$\psi_i(x) = \begin{cases} 0 & x \neq 0 \\ 0 & x = 0 \wedge i \text{ odd} \\ 1 & x = 0 \wedge i \text{ even.} \end{cases}$$

This yields a contradiction to the relation (R).

3. Optimal Enumerations. In a Gödel numbering we enumerate the set of all p.r. functions and describe functions in $\mathbf{P}_1$ by programs. In the same sense we can use any algorithm to enumerate (describe) the outputs by the inputs. We consider enumerations of N. A surjective $\beta \in \mathbf{P}_1$ is called a recursive enumeration (en.) of N. We define the program complexity $K_h(a)$ of integer a relative to $h \in \mathbf{P}_1$ by $K_h(a) = \min\{n | h(n) = a\}$. If $h(n) = a$, we call n a program that computes a.

Definition 3. $\beta \in \mathbf{P}_1$ is an optimal en. of N if and only if

$$\forall g \in \mathbf{P}_1 : \exists h \in \mathbf{R}_1 : h \text{ is l.b.} \wedge g = \beta h.$$

This means that for any rec. en. $g \in \mathbf{P}_1$ there exists $h \in \mathbf{R}_1$ which transforms a program for n relative to g into a program for n relative to β and the size of the program increases at most linearly. This implies

COROLLARY 4. *Let $\beta \in \mathbf{P}_1$ be any optimal en. of N. Then for every $h \in \mathbf{P}_1$ there exists $c \in N$ such that $\|K_\beta(n)\| \leq \|K_h(n)\| + c$ for all n.*

THEOREM 7. *There exists an optimal en. $\beta \in \mathbf{P}_1$ of N.*

Proof. Define $\beta g(n, i) = \varphi(n, i)$ where φ is any G.N. of $\mathbf{P}_1$ and $g(n, i) = 2^n + i2^{n+1} - 1$.

This compares with the definition of universal functions in Rogers [4]:

Definition 4. $\beta \in \mathbf{P}_1$ is universal if and only if there exists $f \in \mathbf{R}_2$ such that $\beta f = \varphi$, where φ is some G.N. of $\mathbf{P}_1$.

A proof attributed to Blum and given in Rogers [4] proves that if β and $\bar{\beta}$ are universal, then there exists a recursive permutation α such that $\bar{\beta} = \beta\alpha$. We shall prove that for any two optimal en. there exists a recursive permutation α such that $\bar{\beta} = \beta\alpha$ with both α and α^{-1} linearly bounded. We shall use the following

LEMMA 3. *A function β is an optimal en. if and only if β is a universal function with a linearly bounded encoder $f \in \mathbf{R}_2$, i.e., $\beta f = \varphi$ and $\lambda x[f(i, x)]$ is l.b. for all i.*

The proof is analogous to that of Lemma 1. We next prove a result analogous to Theorem 2 with a proof analogous to that proof.

The statement and proof of Lemma 3 and Theorem 8 are due to the referee. He pointed out this way to prove Corollaries 5 and 6 and Theorem 8.

THEOREM 8. *Let β be any universal function. Then for every $g \in \mathbf{P}_1$ we can effectively find an integer e_0 such that $g(i) = \beta f(e_0, i)$ for all i and $\lambda i[f(e_0, i)]$ is injective.*

Proof. Via a standard application of the recursion theorem, we can effectively find an integer e_0 such that

$$\varphi_{e_0}(i) = \begin{cases} \text{undefined if } \exists j < i: f(e_0, i) = f(e_0, j) \\ 0 \qquad \text{if } \forall j < i: f(e_0, i) = f(e_0, j) \wedge \exists j > i: [f(e_0, i) = f(e_0, j) \\ \qquad\quad \text{and } g(i) \text{ has not yet converged in less than } j \text{ steps} \\ g(i) \qquad \text{otherwise} \end{cases}$$

Note that $\lambda i[f(e_0, i)]$ must be injective, for if not, let i_0 be the smallest i such that $f(e_0, i) = f(e_0, j)$ for some $j > i$. Then $\varphi_{e_0}(j) = \beta f(e_0, j) = \beta f(e_0, i_0)$ is undefined, but $\beta f(e_0, i_0) = \varphi_{e_0}(i_0)$ is definitely defined. Furthermore, since f is injective, by definition of e_0, $\beta f(e_0, i) = \varphi_{e_0}(i) = g(i)$ for all i.

As with Gödel numberings this theorem immediately yields a padding lemma for universal functions as well as an injective translation theorem. We state these corollaries for optimal enumerations.

COROLLARY 5. (Padding Lemma) *Let β be an optimal en. of N. Then for every i we can effectively find an injective, linearly bounded $t \in \mathbf{R}_1$ such that $\beta(i) = \beta t(j)$ for all j.*

COROLLARY 6. (Injective Translation Theorem) *Given any two optimal en. β and $\bar{\beta}$ of N, we can effectively find an injective, linearly bounded $t \in \mathbf{R}_1$ such that $\bar{\beta} = \beta t$.*

Corollary 6 and Lemma 2 immediately yield a special isomorphism theorem for optimal enumerations.

THEOREM 9. (Linear Isomorphism Theorem) *Let β and $\bar{\beta}$ be any optimal enumerations of N. Then we can effectively find a permutation $\alpha \in \mathbf{R}_1$ such that α and α^{-1} are linearly bounded and $\bar{\beta} = \beta\alpha$.*

Obviously Theorems 5 and 6 also hold with respect to optimal enumerations. The analogous result to Theorem 6 is particularly interesting.

THEOREM 10. *There exists $\psi \in \mathbf{P}_1$ such that*

(1) $\forall g \in \mathbf{P}_1: \exists c: \forall n: \|K_\psi(n)\| \leq \|K_g(n)\| + c$, and
(2) ψ is not a universal function.

Kolmogoroff and Solomonoff only required property (1) for their concept of optimal (universal, resp.) algorithm, although their examples are optimal enumerations in our sense. In view of Theorem 10, our concept of optimal enumerations seems to be the more natural concept.

Proof. Let $\beta \in \mathbf{P}_1$ be an optimal en. of N. Calculate ψ as follows:

$$\psi(2i) = \begin{cases} \beta(i) & \text{if } \beta(i) \text{ is even} \\ \beta(i) - 1 & \text{if } \beta(i) \text{ is odd} \\ \text{undefined} & \text{if } \beta(i) \text{ is undefined,} \end{cases}$$

$$\psi(2i+1) = \begin{cases} \beta(i) & \text{if } \beta(i) \text{ is odd} \\ \beta(i) - 1 & \text{if } \beta(i) \text{ is even} \\ \text{undefined} & \text{if } \beta(i) \text{ is undefined.} \end{cases}$$

Obviously $K_\psi(n) \leq 2K_\beta(n)+1$ for all n. This proves part (1) of Theorem 10. Furthermore, the construction implies

$$\text{(R)} \qquad \begin{cases} i \text{ even} \Rightarrow \psi(i) \text{ even} \vee \psi(i) \text{ is undefined} \\ i \text{ odd} \Rightarrow \psi(i) \text{ odd} \vee \psi(i) \text{ is undefined.} \end{cases}$$

Now suppose that ψ is an optimal en. of N with encoder $f \in \mathbf{R}_2$. Via a standard application of the recursion theorem we find an integer j such that

$$\psi f(j, x) = \varphi_j(x) = \begin{cases} 0 \text{ if } f(j, x) \text{ is odd} \\ 1 \text{ if } f(j, x) \text{ is even.} \end{cases}$$

This contradicts relation (R).

We now establish a relation between optimal Gödel numberings and optimal enumerations.

PROPOSITION 1. *Let $\varphi \in \mathbf{P}_2$ be any Gödel numbering of $\mathbf{P}_1$ and let $\beta \in \mathbf{P}_1$ be any optimal en. of N. Then $\psi = \lambda i, x[\varphi_{\beta(i)}(x)]$ is an optimal Gödel numbering of $\mathbf{P}_1$.*

The proof is entirely obvious. The same idea yields the following result.

PROPOSITION 2. *Let $g \in \mathbf{P}_1$ be surjective and let $\beta \in \mathbf{P}_1$ be any optimal en. of N. Then $g\beta$ is an optimal en. of N.*

4. The Relative Frequency of Gödel Numbers. We shall only consider optimal Gödel numberings, but the results hold for optimal enumerations as well. We define the lower (upper) relative frequency of Gödel numbers of $h \in \mathbf{P}_1$ relative to $g \in \mathbf{P}_2$:

$$L_g(h) = \liminf_n n^{-1} \sum_{i=1}^{n} \delta^h_{g_i}, \qquad U_g(h) = \limsup_n n^{-1} \sum_{i=1}^{n} \delta^h_{g_i},$$

where δ denotes the Kronecker symbol. The upper (lower) relative frequency of Gödel numbers essentially does not depend on the optimal G.N. that has been chosen.

THEOREM 11. *Let φ be any optimal G.N. Then for every $g \in \mathbf{P}_2$ there is an integer c such that for all $h \in \mathbf{P}_1$:*

$$L_\varphi(h) > c^{-1} L_g(h), \qquad U_\varphi(h) > c^{-1} U_g(h).$$

Proof. This obviously holds for the optimal G.N. that has been constructed in the proof of Theorem 1. By the linear isomorphism theorem the assertion holds for any optimal G.N.

Next we prove that p.r. functions with short programs have a high relative frequency of programs. It is an open problem whether the converse also holds.

THEOREM 12. *Let $f \in \mathbf{R}_1$ be such that $\sum_n f(n)^{-1} < \infty$ and let φ be any optimal G.N. of $\mathbf{P}_1$. Then there is an integer c such that $L_\varphi(h) \cdot f(K_\varphi(h)) \geq c^{-1}$ for all $h \in \mathbf{P}_1$.*

Proof. We can easily construct a recursive function $g \in \mathbf{R}_1$ and an integer c such that for all j:

$$\liminf_n \frac{1}{n} \sum_{i=1}^{n} \delta_j^{g(i)} \geq f(j)^{-1} c^{-1}.$$

Define $\psi \in \mathbf{P}_2$ by $\psi(i, x) = \varphi_{g(i)}(x)$. Hence $L_\psi(h) f(K_\varphi(h)) \geq c^{-1}$ for all $h \in \mathbf{P}_1$. Therefore Theorem 12 follows from Theorem 11.

Let φ be any optimal G.N. of $\mathbf{P}_1$. We consider the set

$$W_\varphi = \{ i \in N \mid i = K_\varphi(\varphi_i) \}.$$

W_φ is the set of lowest Gödel numbers relative to φ. We can prove that there exist arbitrarily large gaps in W_φ.

THEOREM 13. *Let φ be any optimal G.N. and let $\epsilon > 0$. Then there exist infinitely many n such that*

$$\exists j < n : \forall k < n^{1-\epsilon} : (j + k \notin W_\varphi).$$

(This means that for infinitely many n there exist gaps of size $n^{1-\epsilon}$ under the Gödel numbers of W_φ that are less than n.)

Proof. Let $f \in \mathbf{R}_1$ be any function such that $\sum_n f(n)^{-1} < \infty$. Then we can easily find a bijective $g \in \mathbf{R}_2$ and $c \in N$ such that $g(i, n) \leq c \cdot f(i) \cdot n$ for all i and all $n > 0$. Let $h \in \mathbf{P}_3$ be given then we define $\psi \in \mathbf{P}_2$ by $\psi(g(i, n), x) = h(n, i, x)$. Hence $K_\psi h_{n,i} \leq c \cdot f(i) \cdot n$ for all i and all $n > 0$. Therefore there exists $\bar{c}$ such that $K_\varphi h_{n,i} \leq \bar{c} \cdot f(i) \cdot n$ for all i and all $n > 0$.

Now set $h_{n,i} = \varphi_{2^n + i}$ and $f(i) = \ulcorner i^{1+\epsilon} \urcorner$, where $\ulcorner\ \urcorner$ denotes the lowest integer greater than. Hence $K_\varphi(\varphi_{2^n+i}) \leq \bar{c} \cdot i^{1+\epsilon} \cdot n$ for all i and all $n > 0$. This implies $K_\varphi(\varphi_{2^n+i}) \leq 2^n$ for $i < (2^n/n\bar{c})^{1/1+\epsilon}$. Hence $2^n + i \notin W_\varphi$ for $i < (2^n/n\bar{c})^{1/1+\epsilon}$. This proves the theorem.

THEOREM 14. *Let φ be any optimal G.N. of $\mathbf{P}_1$ and let $\beta \in \mathbf{P}_1$ be any optimal en. of N. Then there exists an integer c such that $\|K_\varphi(\varphi_i)\| \leq \|K_\beta(i)\| + c$ for all i.*

Proof. Define $\psi \in \mathbf{P}_2$ by $\psi(j, x) = \varphi(\beta(j), x)$. Since φ is optimal G.N. there exists c such that $\|K_\varphi(\varphi_i)\| \leq \|K_\psi(\varphi_i)\| + c$. Obviously $K_\psi(\varphi_i) \leq K_\beta(i)$. Hence $\|K_\varphi(\varphi_i)\| \leq \|K_\beta(i)\| + c$, which proves Theorem 14.

We shall give a more explicit interpretation of Theorem 14 which states that the binary notations of numbers in W_φ are asymptotically random.

Let X^* be the set of all finite binary sequences. Relative to a partial algorithm $A: X^* \to X^*$ we define the Kolmogorov program complexity of $x \in X^*$ by $K_A(x) = \min\{|y| : A(y) = x\}$. Here $|y|$ denotes the length of sequences y. It is known from [2] that there is a partial algorithm A such that for any partial algorithm $B: X^* \to X^*$ there is an integer c such that $K_A(x) \leq K_B(x) + c$ for all $x \in X^*$. Let A be such a universal algorithm. Those finite binary sequences x for which $K_A(x)$ is not much smaller than $|x|$ have random behaviour [8]. We now prove that the binary notations of numbers in W_φ are asymptotically random.

THEOREM 15. *Let φ be any optimal G.N. of $\mathbf{P}_1$. Then there exists $c \in N$ such that for all $i \in W_\varphi$: $K_A(\mathrm{bn}(i)) \geq \|i\| - c$. Here* bn($i$) *is the binary notation of i.*

Proof. It follows from Theorem 14 that for all $i \in W_\varphi$

$$\|K_\beta(i)\| \geq \|K_\varphi(\varphi_i)\| - c = \|i\| - c.$$

Hence $\|K_\beta(i)\| \geq |\mathrm{bn}(i)| - c$ for all $i \in W_\varphi$. Since the universal algorithm A can be chosen such that $K_A \mathrm{bn}(i) = \|K_\beta(i)\|$, Theorem 15 holds.

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