On the First Eigenvalue of the Clamped Plate (*).

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Summary. – *We find a lower bound for the ratio between the first eigenvalue of any homogeneous thin plate G, which is clamped on its boundary, and the first eigenvalue of the spherical clamped plate having the same measure as G. In two dimensions, our bound is about* 0.98.

1. – Introduction and statement of results.

We are concerned with the following eigenvalue problem:

$$\begin{cases} \Delta^2 u - \lambda u = 0 & \text{in } G, \\ u = |Du| = 0 & \text{on the boundary } \partial G \text{ of } G. \end{cases} \tag{1.1}$$

Here G is any open bounded subset of n-dimensional euclidean space R^n;

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \tag{1.2}$$

is the Laplace operator and Δ^2 its square;

$$D = \text{gradient}\,. \tag{1.3}$$

We set

(1.4) $\lambda(G)$ = the first (lowest) eigenvalue of problem (1.1);

(1.5) $G^\star$ = the ball having the same n-dimensional measure as G.

Our aim is to find a lower bound for the ratio

$$\frac{\lambda(G)}{\lambda(G^\star)}\,. \tag{1.6}$$

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As is well known and easy to see,

$$\lambda(G^\star) = j_n^4\left[\frac{1}{C_n}\, m(G)\right]^{-4/n}; \tag{1.7}$$

where m stands for n-dimensional measure,

$$C_n = \pi^{n/2}/\Gamma(1 + n/2) \tag{1.8}$$

is the measure of the n-dimensional unit ball, j_n is the smallest positive root of the equation: $I'_\nu(z)J_\nu(z) - I_\nu(z)J'_\nu(z) = 0$ $(\nu = (n/2) - 1;\ I, J = \text{Bessel functions})$. For instance, in dimension $n = 2$ one has: $\lambda(G^\star)\times(\text{area } G)^2 = 1029.995\,9 \ldots$.

A conjecture has been made by a number of authors (see Payne [11]), that the ratio (1.6) is bounded from below by 1. In other words, one might expect that for homogeneous clamped plates with fixed measure the spherical plate has the lowest frequency of vibration. A result towards a proof of such a conjecture has been obtained by SZEGÖ [13] (see also [12]), who proved that the inequality:

$$\lambda(G) \geq \lambda(G^\star) \tag{1.9}$$

holds for those domains G for which a principal eigenfunction is free from nodal lines (similar, but in a sense weaker, results were obtained by HODYREVA [10], via an extension of a method by COURANT [5]). Szegö's proof, which was originally written in dimension $n = 2$, can be easily carried out in any dimension $n \geq 2$ (as well as rephrased in a completely rigorous function-theoretic setting). Unfortunately, the absence of nodal lines seems to be a crucial hypothesis for Szegö's argument, and no criteria are available for deciding whether a given domain fulfils or not such a hypothesis. As a matter of a fact, both theoretical and numerical devices have shown that clamped plates, whose principal eigenfunctions do change their sign, actually exist. COFFMAN-DUFFIN-SHAFFER [4] proved that the first eigenvalue and the principal eigenfunctions of a ring-shaped clamped plate have multiplicity two and a diametral nodal line if the outer radius is 1 and the inner radius is $< 0.001\,311\,774$ (parallel results for an infinite strip are in DUFFIN [6]). Numerical results by BAUER-REISS [1] and HACKBUSCH-HOFMANN [8] strongly indicate that the principal eigenfunction of a square clamped plate changes its sign. Indeed COFFMAN [2] has proved, by extending a method of [3], that any eigenfunction of a square clamped plate oscillates infinitely many times on any ray issuing from a corner.

Our results can be summarized in the following way. *A constant c_n exists such that the inequality*

$$\lambda(G) \geq c_n\, \lambda(G^\star) \tag{1.10}$$

holds for any bounded domain G in R^n. Such a constant is explicitly computable; the values of c_n for small dimensions n are given by the following table:

n	c_n
2	0.977 68
3	0.739 10
4	0.652 42
5	0.609 25
6	0.583 94

It will be clear from our proof that c_n is ≥ 0.5 for all n. Incidentally, our method shows that (1.10) can be replaced by (1.9) provided the nodal set of a first eigenfunction u is either empty or a locus of stationary points of u.

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2. – A symmetrization argument.

We shall use the following variational characterization of the first eigenvalue:

(2.1*a*) $$\lambda(G) = \text{minimum of the Rayleigh quotient}$$

(2.1*b*) $$\int_G (\Delta u)^2\,dx \left(\int_G u^2\,dx\right)^{-1},$$

(2.1*c*) $$\text{under the constraint:} \quad u \in W_0^{2,2}(G)\,.$$

Furthermore we set:

(2.2) $$u = \text{a principal eigenfunction}\,,$$

i.e. u will stand for any minimizer from $W_0^{2,2}(G)$ of the Rayleigh quotient (2.1*b*). Here $W_0^{2,2}(G)$ denotes the closure of $C_0^\infty(G)$ (= the collection of all infinitely differentiable functions, which vanish in a neighbourhood of $R^n \setminus G$) under the topology of the Sobolev space $W^{2,2}(R^n)$. Thus the boundary conditions: $u = |Du| = 0$ on ∂G, are included into the constraint (2.1*c*). Well-known theorems on elliptic equations ensure that u is C^∞ in (the interior of) G; moreover, no non-negligible subset of G exists where u is either constant or harmonic.

Notations and objects, which we shall deal with, are listed in the following table.

$u_+ = \max(u, 0)$ $u_- = -\min(u, 0)$	positive part of u negative part of u.
$\operatorname{sprt} u_+$ $\operatorname{sprt} u_-$	support of u_+, the closure of $\{x \in G: u(x) > 0\}$ support of u_-, the closure of $\{x \in G: u(x) < 0\}$.
$G^\star$ $(\operatorname{sprt} u_+)^\star$ $(\operatorname{sprt} u_-)^\star$ $m(G) = C_n L^n$ $m(\operatorname{sprt} u_+) = C_n a^n$ $m(\operatorname{sprt} u_-) = C_n b^n$ C_n	see (1.5) the ball having the same measure as $\operatorname{sprt} u_+$ $L =$ radius of $G^\star$ $a =$ radius of $(\operatorname{sprt} u_+)^\star$ $b =$ radius of $(\operatorname{sprt} u_-)^\star$ $(a/L)^n + (b/L)^n = 1$ ← measure of the unit n-ball.
$\alpha(t) = m\{x \in G: u(x) > t\}$ $\beta(t) = m\{x \in G: u(x) < -t\}$	distribution function of u_+ distribution function of u_-, $t \geqslant 0$.
$(\Delta u)^*_+$, $(\Delta u)^*_-$ $0 \leqslant s \to (\Delta u)^*_+(m(G) - s)$ $0 \leqslant s \to (\Delta u)^*_-(m(G) - s)$	decreasing rearrangements of $(\Delta u)_+$, $(\Delta u)_-$. increasing rearrangements of $(\Delta u)_+$, $(\Delta u)_-$.
$f(s) = (\Delta u)^*_-(s) - (\Delta u)^*_+(m(G) - s)$ $g(s) = (\Delta u)^*_+(s) - (\Delta u)^*_-(m(G) - s)$	$= (\Delta u)^*_-(s)$ if $0 \leqslant s < m\{x \in G: \Delta u(x) < 0\}$, $= -(\Delta u)^*_+(m(G) - s)$ if $m\{x \in G: \Delta u(x) \leqslant 0\} < s \leqslant m(G)$, $= 0$ otherwise; $f(s) = -g(m(G) - s)$. the signed rearrangement of Δu such that: (i) $g(s)$ decreases as s increases in $[0, m(G)]$; (ii) length $\{s \in [0, m(G)]: g(s) > t\} =$ $= m\{x \in G: \Delta u(x) > t\}$, length $\{s \in [0, m(G)]: g(s) < -t\} =$ $= m\{x \in G: \Delta u(x) < -t\}$, for every $t \geqslant 0$ (see figure).

We start our proof by applying lemmas 1 and 2 from [14] to the positive and the negative part of u. We obtain:

$$
\tag{2.3}
\begin{cases}
n^2 C_n^{2/n} \alpha(t)^{2-2/n} \leq [-\alpha'(t)] \displaystyle\int_{u(x)>t} (-\Delta u)\, dx \\
n^2 C_n^{2/n} \beta(t)^{2-2/n} \leq [-\beta'(t)] \displaystyle\int_{u(x)<-t} (\Delta u)\, dx
\end{cases}
$$

for almost every $t \geq 0$. A short proof of inequalities (2.3) is sketched in the Appendix. On the other hand, theorem 378 and a continuous version of theorem 368

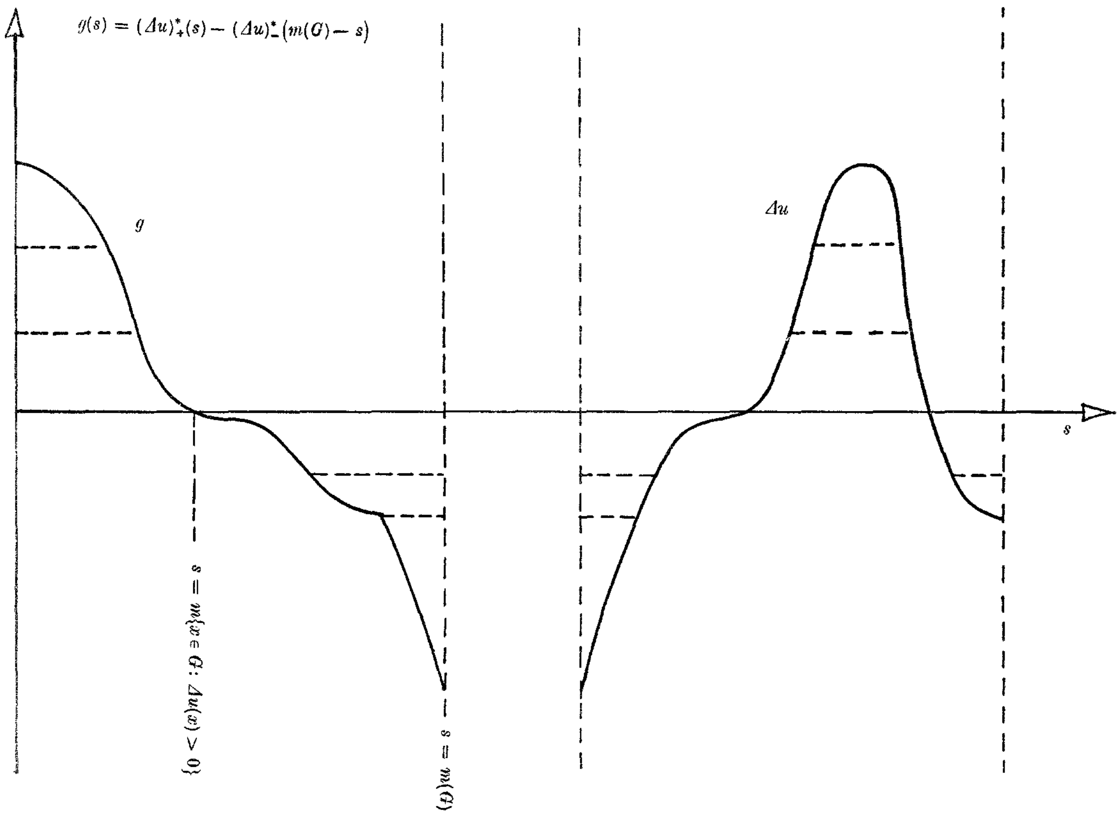

Figure 1

from [9] give:

$$\int_{u(x)>t} (-\Delta u)\, dx = \int_{u(x)>t} (\Delta u)_-\, dx - \int_{u(x)>t} (\Delta u)_+\, dx \leq$$

$$\int_0^{\alpha(t)} (\Delta u)^*_-(s)\, ds - \int_0^{\alpha(t)} (\Delta u)^*_+(m(G)-s)\, ds = \int_0^{\alpha(t)} f(s)\, ds\,,$$

and analogously:

$$\int_{u(x)<-t} \Delta u\, dx \leq \int_0^{\beta(t)} g(s)\, ds\,.$$

Then we have:

$$(2.4)\qquad \left|\begin{aligned} & n^2 C_n^{2/n} \leq [-\alpha'(t)]\alpha(t)^{-2+2/n} \int_0^{\alpha(t)} f(s)\, ds \\ & n^2 C_n^{2/n} \leq [-\beta'(t)]\beta(t)^{-2+2/n} \int_0^{\beta(t)} g(s)\, ds \end{aligned}\right.$$

for almost every $t \geq 0$.

Integrating both sides of (2.4) gives:

$$
(2.5)\qquad \begin{cases} n^2 C_n^{2/n} t \le \displaystyle\int_{\alpha(t)}^{\alpha(0)} r^{-2+2/n}\, dr \int_0^r f(s)\, ds \\[2ex] n^2 C_n^{2/n} t \le \displaystyle\int_{\beta(t)}^{\beta(0)} r^{-2+2/n}\, dr \int_0^r g(s)\, ds\,, \end{cases}
$$

for $\int \ldots (-\alpha'(t))\, dt \le \int \ldots\, (-d\alpha(t))$ and $\int \ldots (-\beta'(t))\, dt \le \ldots$ thanks to the monotonicity of α and β. Here the following property:

$$
(2.6)\qquad \int_0^r f(s)\, ds \quad \text{and} \quad \int_0^r g(s)\, ds \ge 0 \quad \text{for } 0 \le r \le m(G)
$$

plays a role. Proof of (2.6): $[0, m(G)] \ni r \to \int_0^r f(s)\, ds$ is a concave function, for $f(s)$ decreases as s increases; such a function vanishes at both ends of the interval $[0, m(G)]$, as formulas (2.12)-(2.14) below show; hence it must be nonnegative.

Inequalities (2.5) hold for all $t \ge 0$. In terms of rearrangements of u_+ and u_- they read as follows:

$$
(2.7)\qquad \begin{cases} u_+^\star(x) \equiv u_+^*(C_n|x|^n) \le v(x) \\ \qquad\qquad \text{for every } x \text{ from } (\operatorname{sprt} u_+)^\star\,, \\ u_-^\star(x) \equiv u_-^*(C_n|x|^n) \le w(x) \\ \qquad\qquad \text{for every } x \text{ from } (\operatorname{sprt} u_-)^\star\,. \end{cases}
$$

Here the left-hand sides are the *spherically symmetric rearrangements* of u_+ and u_-, and the (spherically symmetric) functions v and w are defined by:

$$
(2.8)\qquad \begin{cases} n^2 C_n^{2/n} v(x) = \displaystyle\int_{C_n|x|^n}^{C_n a^n} r^{-2+2/n}\, dr \int_0^r f(r')\, dr'\,, \\[2ex] n^2 C_n^{2/n} w(x) = \displaystyle\int_{C_n|x|^n}^{C_n b^n} r^{-2+2/n}\, dr \int_0^r g(r')\, dr'\,. \end{cases}
$$

In the derivation of (2.8) one should keep in mind the following property (together with the analogous one for u_-):

$$
\alpha(u_+^*(s)) \le s \le \alpha(u_+^*(s) - 0)\,,
$$

a consequence of the standard definition

$$
u_+^*(s) = \inf\{t \ge 0 : \alpha(t) < s\} = \sup\{t \ge 0 : \alpha(t) \ge s\}\,.
$$

Let us list some crucial properties of v and w:

(2.9a) $$v(x) = 0 \quad \text{if } |x| = a\,, \qquad w(x) = 0 \quad \text{if } |x| = b;$$

(2.9b) $$Dv(x) = Dw(x) = 0 \quad \text{if } |x| = L;$$

(2.9c) $$\int_{|x|<L} (\Delta v)^2\, dx = \int_{|x|<L} (\Delta w)^2\, dx = \int_G (\Delta u)^2\, dx;$$

hence in particular v and w are in $W^{2,2}(G^\star)$.

Properties (2.9a) are obvious. Ingredients for the proof of (2.9b) and (2.9c) are the formulas

(2.10) $$\begin{cases} nC_n \dfrac{\partial v}{\partial |x|}(x) = -\,|x|^{1-n} \displaystyle\int_0^{C_n|x|^n} f(s)\, ds \\[2ex] nC_n \dfrac{\partial w}{\partial |x|}(x) = -\,|x|^{1-n} \displaystyle\int_0^{C_n|x|^n} g(s)\, ds \end{cases}$$

(which follow at once from (2.8)), and the equations

(2.11) $$-\,\Delta v(x) = f(C_n|x|^n)\,, \qquad -\,\Delta w(x) = g(C_n|x|^n)$$

(which follow from (2.10) and the customary formula

$$\Delta v = |x|^{1-n} \frac{\partial}{\partial |x|}\left\{|x|^{n-1} \frac{\partial v}{\partial |x|}\right\}$$

for the laplacian of spherically symmetric functions). On the other hand, the definition of f implies:

(2.12a) $$\int_0^{m(G)} f(s)\, ds = \int_0^\infty (\Delta u)^*_-(s)\, ds - \int_0^\infty (\Delta u)^*_+(s)\, ds$$
$$= \int_G (\Delta u)_-\, dx - \int_G (\Delta u)_+\, dx = -\int_G \Delta u\, dx\,,$$

and:

(2.13a) $$\int_0^{m(G)} f(s)^2\, ds = \int_0^\infty [(\Delta u)^*_+(s)]^2\, ds + \int_0^\infty [(\Delta u)^*_-(s)]^2\, ds = \int_G (\Delta u)^2\, dx;$$

analogously:

(2.12b) $$\int_0^{m(G)} g(s)\, ds = \int_G \Delta u\, dx\,,$$

(2.13b) $$\int_0^{m(G)} g(s)^2\, ds = \int_G (\Delta u)^2\, dx\,.$$

From (2.11) and (2.13) one gets (2.9*c*). From (2.10) and (2.12) one gets (2.9*b*), provided the basic equation

$$\int_G (\Delta u)\, dx = 0 \tag{2.14}$$

is taken into account. Equation (2.14) is a consequence of the boundary conditions for u: in fact, Gauss-Green formulas show that the laplacian Δu of any function u from $W_0^{2,2}(G)$ must be orthogonal to any (square integrable) harmonic function. Thus the proof of (2.11*b*) consists essentially of the following remark: the gradient of a spherically symmetric $W^{2,2}$-function vanishes on the boundary of a ball if and only if the laplacian of that function has mean value zero on the same ball.

From (2.7) and (2.9*c*) we infer

$$\lambda(G)^{-1} \equiv \left(\int_G (\Delta u)^2 dx\right)^{-1} \left\{ \int_{(\text{sprt}\, u_+)^\star} (u_+^\star)^2\, dx + \int_{(\text{sprt}\, u_-)^\star} (u_-^\star)^2\, dx \right\} \leqq \tag{2.15}$$

$$\leqq \frac{\int_{|x|<a} v^2\, dx}{\int_{|x|<L} (\Delta v)^2\, dx} + \frac{\int_{|x|<b} w^2\, dx}{\int_{|x|<L} (\Delta w)^2\, dx},$$

hence we are in a position to draw the main conclusions of this section.

THEOREM 1. – *Let p be defined by:*

$$0 < t \leq 1\,, \qquad p(t^n) = \max \frac{\int_{|x|<t} v^2\, dx}{\int_{|x|<1} (\Delta v)^2\, dx}\,; \tag{2.16}$$

where v runs in the collection of all functions having the following properties: (i) *v is endowed with square-integrable second order derivatives in the unit ball* $\{x \in R^n\colon |x| < 1\}$; (ii) $v(x) = 0$ *on the inner sphere* $|x| = t$; (iii) $Dv(x) = 0$ *on the boundary* $|x| = 1$; (iv) *v is spherically symmetric* (i.e. a function of $|x|$ only).

The following inequality holds:

$$\frac{\lambda(G^\star)}{\lambda(G)} \leqq \frac{1}{p(1)} \max\{p(t) + p(1-t)\colon 0 \leq t \leq 1\}\,. \tag{2.18}$$

Note that the right-hand side of (2.18) does not exceed 2. In fact one may infer from theorem 2 below that $p(t)$ increases as t increases from 0 to 1.

PROOF OF (2.18). – From (2.15) we get

$$\frac{1}{\lambda(G)} \leq L^4[p(a^n/L^n) + p(b^n/L^n)]\,,$$

after a straightforward dimensional analysis argument. Recall that $(a/L)^n + (b/L)^n = 1$. Furthermore $\lambda(G^\star) = L^{-4}/p(1)$, since the principal eigenfunction of a spherical clamped plate is known to be a spherically symmetric function (on the other hand, (1.7) holds and it will be clear from section 4 that $p(1) = j_n^{-4}$). Thus (2.19) follows.

3. – Variations on a theorem by Szegö.

The result from [13] can be easily recovered with the help of our previous arguments. Suppose in fact that a principal eigenfunction u has a constant (say positive) sign. Then $u \equiv u_+$, u_- is identically zero, $a = L$ and $b = 0$. Thus v vanishes on $\partial G^\star$ (together with its gradient) and we get from (2.15):

$$\frac{1}{\lambda(G^\star)} \geq \frac{\int_{G^\star} v^2\,dx}{\int_{G^\star} (\Delta v)^2\,dx} \geq \frac{\int_{G^\star} (u_+^\star)^2\,dx}{\int_G (\Delta u)^2\,dx} = \frac{\int_G u^2\,dx}{\int_G (\Delta u)^2\,dx},$$

that is:

$$\lambda(G^\star) \leq \lambda(G)\,. \tag{3.1}$$

A curious result, which we state presently, is available too. Quite the same procedure of section 2 (with a slight change: forget positive and negative part of u, apply lemmas 1 and 2 from [14] directly to u, and go ahead) leads to the following estimate:

$$u^\star \leq U\,, \tag{3.2a}$$

where $u^\star$ is the spherically symmetric rearrangement of $|u|$ and

$$U(x) = \int_{C_n|x|^n}^{m(G)} \frac{dr}{n^2 C_n^{2/n} r^{2-2/n}} \int_0^r h(s)\,ds\,, \tag{3.2b}$$

$$h(s) = (\Delta u \operatorname{sgn} u)_-^\star(s) - (\Delta u \operatorname{sgn} u)_+^\star(m(G) - s)\,. \tag{3.2c}$$

An inspection shows:

$$U(x) = 0 \quad \text{if } x \in \partial G^\star, \tag{3.3a}$$

$$n C_n^{1/n}[m(G)]^{1-1/n} |DU(x)| = \left| \int_0^{m(G)} h(s)\,ds \right| = \left| \int_G (\Delta u \operatorname{sgn} u)\,dx \right| \quad \text{if } x \in \partial G^\star, \tag{3.3b}$$

$$\int_{G^\star} (\Delta U)^2\,dx = \int_0^{m(G)} h(s)^2\,ds = \int_G (\Delta u)^2\,dx\,. \tag{3.3c}$$

Hence we have from (3.2*a*) and (3.3*c*):

$$\lambda(G) \geq \frac{\int_{G^\star} (\Delta U)^2\,dx}{\int_{G^\star} U^2\,dx}\,. \tag{3.4}$$

The right-hand side of (3.4) exceeds $\lambda(G^\star)$, provided U and DU vanish on the boundary of $G^\star$. As (3.3) shows, the latter circumstance occurs if and only if

$$\int_G (\Delta u \operatorname{sgn} u)\,dx = 0\,. \tag{3.5}$$

On the other hand, the equation

$$-\int_G (\Delta u \operatorname{sgn} u)\,dx = 2 \lim_{t\to 0} \int_{\{x\in G:\,u(x)=t\}} |Du|\,H_{n-1}(dx) \tag{3.6}$$

holds. In fact Federer's coarea formula [7] and Gauss-Green theorem give:

$$\int_{-\infty}^{+\infty} \frac{\varepsilon}{\varepsilon^2+t^2}\,dt \int_{\{x\in G:u(x)=t\}} |Du|\,H_{n-1}(dx) = \int_G |Du|^2 \frac{\varepsilon}{\varepsilon^2+u^2}\,dx = -\int_G \Delta u \,\arctan\frac{u}{\varepsilon}\,dx$$

for every $\varepsilon > 0$, since u is in $W_0^{2,2}(G)$ (i.e. Du vanishes on the boundary). Here H_{n-1} stands for the $(n-1)$-dimensional Hausdorff measure.

In conclusion, one can assert that (3.1) holds if the nodal set $\{x\in G\colon u(x)=0\}$ of a principal eigenfunction is either empty, or included in $\{x\in G\colon Du(x)=0\}$.

4. – A one-dimensional problem.

THEOREM 2. – *Let p be defined by* (2.16) *and let $0<t\leq 1$. Then*

$$t^{1/n}p(t)^{-1/4} \tag{4.1a}$$

is the smallest positive root z of the equation:

$$tP(z) = 1\,. \tag{4.1b}$$

Here:

$$P(z) = 1 - \frac{m+1}{z}\left\{\frac{I_{m+1}(z)}{I_m(z)} + \frac{J_{m+1}(z)}{J_m(z)}\right\} = 1 - \frac{m+1}{z}\left\{\frac{I'_m(z)}{I_m(z)} - \frac{J'_m(z)}{J_m(z)}\right\} = 1 - \frac{m+1}{z}\left\{\frac{2}{zI_m(z)\,J_m(z)}\int_0^z tI_m(t)\,J_m(t)\,dt\right\} \tag{4.2}$$

and $m = n/2 - 1$.

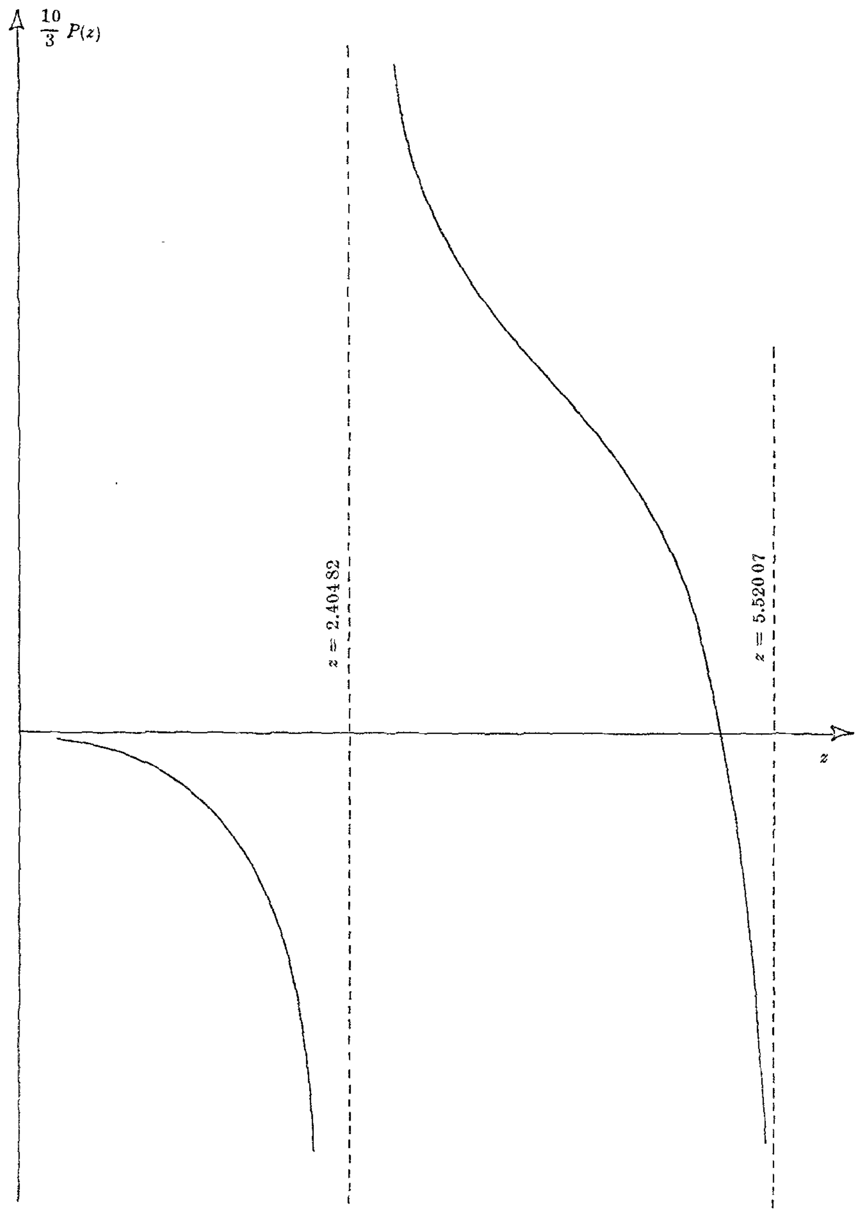

Figure 2

As is easy to check, $P(z) < 1$ if $0 < z < j_{m,1}$; $P(z)$ decreases monotonically from $+\infty$ to $-\infty$ if z increases from $j_{m,1}$ to $j_{m,2}$ ($j_{m,k} =$ k-th positive zero of J_m). The behaviour of $P(z)$ for $n = 2$ is shown in fig. 2.

Using theorem 2, the functions

$$p(t) \quad \text{and} \quad q(t) = p(t) + p(1-t) \tag{4.3}$$

have been evaluated numerically for $2 \leq n \leq 6$. The graphs of p and q are plotted in fig. 3 for $n = 2$. Other results are summarized in the following table.

n	$10^3 \times p(1)$	$10^3 \times \max q(t)$ [$= 10^3 \times q(0.5)$]
2	9.582 08	9.800 81
3	4.206 63	5.691 54
4	2.212 36	3.391 00
5	1.298 76	2.131 72
6	0.822 09	1.407 83

PROOF OF THEOREM 2. – Since all trial functions involved in (2.16) are spherically symmetric, we are faced by the following one-dimensional problem:

$$\frac{\int_0^1 [u'' + ((n-1)/r)u']^2 r^{n-1}\,dr}{\int_0^t u^2 r^{n-1}\,dr} = \text{minimum}\,,$$

under the constraints:

$$\int_0^1 [u^2 + (u'')^2 + (u'/r)^2] r^{n-1}\,dr < \infty\,,$$

$$u(r) = 0 \quad \text{at } r = t\,, \qquad u'(r) = 0 \quad \text{at } r = 1\,.$$

Let:

$$\begin{cases} u = \text{a minimizing function}\,, \\ \mu^4 = \text{the minimum value of the relevant functional}\,. \end{cases}$$

The pair u, μ must satisfy the Euler equation of the problem, that is:

$$\int_0^1 \left[u'' + \frac{n-1}{r}u'\right]\left[\varphi'' + \frac{n-1}{r}\varphi'\right] r^{n-1}\,dr = \mu^4 \int_0^t u\varphi r^{n-1}\,dr$$

for all test functions φ such that:

$$\begin{cases} \int_0^1 [\varphi^2 + (\varphi'')^2 + (\varphi'/r)^2] r^{n-1}\,dr < \infty\,, \\ \varphi(t) = \varphi'(1) = 0\,. \end{cases}$$

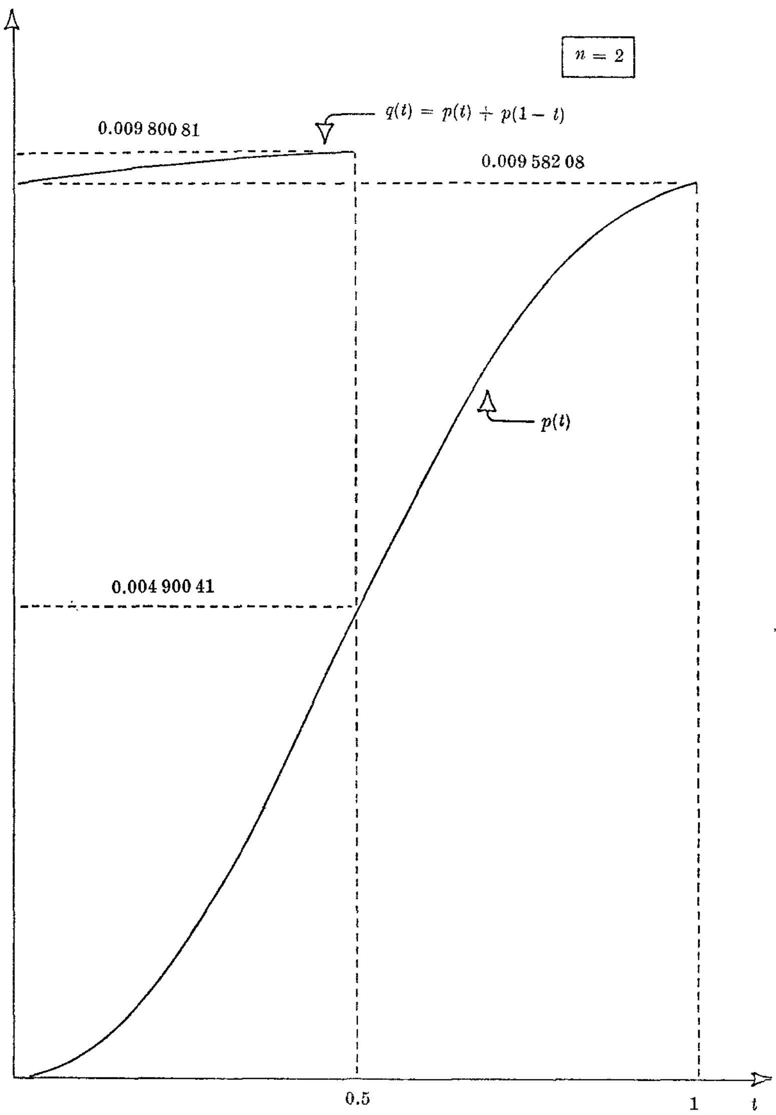

Figure 3

Appropriate choices of φ show that the minimizing function u satisfies the following differential equation:

$$\begin{cases} \left(\dfrac{d^2}{dr^2}+\dfrac{n-1}{r}\dfrac{d}{dr}-\mu^4\right)^2 u=0 & \text{if } 0<r<t\,, \\ \left(\dfrac{d^2}{dr^2}+\dfrac{n-1}{r}\dfrac{d}{dr}\right)^2 u=0 & \text{if } t<r<1\,, \end{cases}$$

together with the following boundary conditions:

$$\begin{cases} u \quad \text{is smooth near} \quad r = 0\,, \\ u(r) = 0 \quad \text{at} \quad r = t\,, \\ u' \quad \text{and} \quad u'' + (n-1)(u'/r) \quad \text{vanish at} \quad r = 1\,. \end{cases}$$

Integrations give:

$$u(r) = \begin{cases} A(\mu r)^{-m} \begin{vmatrix} I_m(\mu t) & J_m(\mu t) \\ I_m(\mu r) & J_m(\mu r) \end{vmatrix} & \text{if } 0 \le r \le t\,, \\ B\left[\dfrac{r^2 - t^2}{2} + \dfrac{r^{2-n} - t^{2-n}}{n-2}\right] & \text{if } t < r \le 1\,; \end{cases}$$

where A, B are constants (the last term must be replaced by $\ln(t/r)$ when $n = 2$). For determining A and B (i.e. the ratio A/B, since u is defined up to multiplicative constants), we have to keep in mind that u', u'', u''' have no discontinuities across the interface $r = t$. This leads to a system of linear homogeneous equations in A, B which can be arranged in the following form:

$$\begin{cases} (1 - t^{-n})\, t^2 A + W(\mu t)(\mu t)^{-m} B = 0 \\ [1 + (n-1)\, t^{-n}]\, t^2 A + [(\mu t)\, W'(\mu t) - 2m W(\mu t)](\mu t)^{-m} B = 0 \end{cases}$$

where:

$$W(z) = I_m(z)\, J_{m+1}(z) + I_{m+1}(z)\, J_m(z) = I'_m(z)\, J_m(z) - I_m(z)\, J'_m(z) = \frac{2}{z}\int_0^z s I_m(s)\, J_m(s)\, ds\,.$$

Putting the determinant of the coefficients equal to zero gives the equation:

$$t^{-n} = 1 - n\left[1 + (\mu t)\,\frac{W'(\mu t)}{W(\mu t)}\right]^{-1},$$

which allows one to determine μ. The claimed assertions easily follow.

Appendix.

In section 2 we used the following lemma.

LEMMA. – *Let u be a* (real-valued) *function from $W_0^{1,2}(G)$ such that Δu is in $L^1(G)$. The following inequality*

$$n^2 C_n^{2/n} [\alpha(t)]^{2-2/n} \le [-\alpha'(t)] \int_{\{x \in G:\, u(x) > t\}} (-\Delta u)\, dx \tag{A.1a}$$

holds for almost every $t > 0$. *Here*:

$$\alpha(t) = m\{x \in G : u(x) > t\}\ , \tag{A.1b}$$

and G is any open subset of euclidean n-space.

We present here a short proof of inequality (A.1). For the sake of simplicity, we restrict ourselves to the case (which is enough for our purposes) where u is infinitely differentiable. We refer to [14] for a more exhaustive proof.

Cauchy-Schwarz inequality yields:

$$\left(\frac{1}{h} \int\limits_{t<u(x)\leq t+h} |Du|\, dx\right)^2 \leq \frac{\alpha(t) - \alpha(t+h)}{h}\ \frac{1}{h} \int\limits_{t<u(x)\leq t+h} |Du|^2\, dx \tag{A.2}$$

for every $h > 0$. Letting $h \to 0$, we get:

$$[H_{n-1}\{x \in G : u(x) = t\}]^2 \leq [-\alpha'(t)] \int\limits_{\{x \in G :\, u(x) = t\}} |Du| H_{n-1}(dx) \tag{A.3}$$

for almost every t. In the derivation of (A.3) from (A.2) we have applied Federer coarea formula [7] and the fact that α is almost everywhere differentiable. H_{n-1} stands for $(n-1)$-dimensional measure.

On the other hand, Gauss-Green formulas yield:

$$\int\limits_{\{x \in G :\, u(x) = t\}} |Du| H_{n-1}(dx) = \int\limits_{\{x \in G :\, u(x) > t\}} (-\Delta u)\, dx \tag{A.4}$$

for every t such that

$$\partial\{x \in G : u(x) > t\} = \{x \in G : u(x) = t\}\ , \tag{A.5}$$

an equation which holds for almost every $t > 0$ thanks to Sard's theorem and the vanishing of u on the boundary of G. Thus (A.3) and (A.4) imply (A.1) via (A.5) and the isoperimetric theorem.

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