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On the Convergence of Solutions of Degenerate Elliptic Equations in Divergence Form (*).

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Summary. – *It is studied the convergence of solutions of Dirichlet problems for sequences of monotone operators of the type* $-\operatorname{div}(a_h(x, D\cdot))$, *where the functions* a_h *verify the following degenerate coerciveness assumption*

$$(a_h(x, \xi_1) - a_h(x, \xi_2) | \xi_1 - \xi_2) \geq \mu_h(x) |\xi_1 - \xi_2|^p \qquad (p \geq 2),$$

being $(\mu_h)_h$ *a sequence of function verifying a Muckenhoupt condition uniformly in* h.

0. – Introduction.

Given a sequence of Carathéodory functions $a_h \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the asymptotic behaviour, as h tends to $+\infty$, of the solutions of the equations

$$-\operatorname{div}(a_h(x, Du)) = f(x)$$

has been generally studied under equicoercive assumptions of the type

$$(a_h(x, \xi) | \xi) \geq |\xi|^p \qquad \text{for every } h \ (p > 1), \tag{0.1}$$

see for instance [1], [3], [7], [8], [12], [15], [16].

In this paper we study the case in which, instead of (0.1), each function a_h verifies a *degenerate* coerciveness condition depending on h.

One of the results proved (see Corollary 3.6) concerns, for example, the case in

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which the following conditions are assumed:

$$(0.2)\quad \begin{cases} a_h(x,0)=0\,, \\ |a_h(x,\xi_1)-a_h(x,\xi_2)| \leq L\mu_h(x)(1+|\xi_1|^p+|\xi_2|^p)^{(p-2)/p}|\xi_1-\xi_2| \qquad (p\geq 2)\,, \\ (a_h(x,\xi_1)-a_h(x,\xi_2)|\xi_1-\xi_2) \geq \mu_h(x)|\xi_1-\xi_2|^p\,, \end{cases}$$

for a.e. x in $\mathbb{R}^n$, for every ξ_1, ξ_2 in $\mathbb{R}^n$, and every $h \in \mathbb{N}$, where $(\mu_h)_h$ is a sequence of functions in the Muckenhoupt class $A_p(K)$ (see (1.3)) such that, for every cube Q of $\mathbb{R}^n$, $(\mu_h)_h$ and $(\mu_h^{1-p'})_h$ are bounded in $L^1(Q)$.

We prove the existence of a subsequence $(a_{h_r})_r$ of $(a_h)_h$, of a Carathéodory function $a_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and of a function μ_∞ in $A_p(K)$ verifying

$$(0.3)\quad \begin{cases} a_\infty(x,0)=0\,, \\ |a_\infty(x,\xi_1)-a_\infty(x,\xi_2)| \leq L'\mu_\infty(x)(1+|\xi_1|^p+|\xi_2|^p)^{(p-2)/(p-1)}|\xi_1-\xi_2|^{1/(p-1)}\,, \\ (a_\infty(x,\xi_1)-a_\infty(x,\xi_2)|\xi_1-\xi_2) \geq \mu_\infty(x)|\xi_1-\xi_2|^p\,, \end{cases}$$

for a.e. x in $\mathbb{R}^n$, for every ξ_1, ξ_2 in $\mathbb{R}^n$, such that, for every regular bounded open set Ω and f in $L^\infty(\Omega)$ the unique solutions u_r of the Dirichlet problems

$$-\operatorname{div}(a_{h_r}(x,Dv)) = f \quad \text{in } \Omega\,, \qquad v=0 \quad \text{on } \partial\Omega$$

converge weakly in $W_0^{1,1}(\Omega)$ to the unique solution u_∞ of the Dirichlet problem

$$-\operatorname{div}(a_\infty(x,Dv)) = f \quad \text{in } \Omega\,, \qquad v=0 \quad \text{on } \partial\Omega\,.$$

Moreover the weak convergence in $(L^1(\Omega))^n$ of the *momenta* $a_{h_r}(x, Du_r)$ to the *momentum* $a_\infty(x, Du_\infty)$ holds.

The above convergence result is obtained as a particular and more readable case from a general convergence result (see Theorem 3.5).

The techniques employed in this paper are classical and rely on a weighted compensated compactness type result (Theorem 1.2) proved in [6].

We finally recall that the case of homogenization, in which $a_h(x,\xi) = a(hx,\xi)$ where $a(\cdot,\xi)$ is a 1-periodic function in each variable x_i $(i = 1, 2, \ldots, n)$ is studied in [6] under less restrictive assumptions.

1. - Notations and preliminary results.

We denote by Q a generic (open or closed) cube of $\mathbb{R}^n$ $(n > 1)$ with faces parallel to the coordinates planes.

The symbols $(\cdot|\cdot)$, $|E|$, $\fint_E f\,dx$, p' indicate respectively the scalar product of $\mathbb{R}^n$, the Lebesgue measure of the set E, the mean value of f on E $\left(\text{i.e. } |E|^{-1}\int_E f\,dx\right)$ and the conjugate of p (*i.e.* $p' = p/(p-1)$).

Let $p > 1$ and let λ be a *weight* on $\mathbb{R}^n$, that is a measurable function on $\mathbb{R}^n$ such that $\lambda > 0$ a.e., λ and $\lambda^{1-p'}$ are in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, set $L^p(\Omega, \lambda) = \{u \in L^1_{\mathrm{loc}}(\Omega): u\lambda^{1/p} \in L^1(\Omega)\}$ and $W^{1,p}(\Omega, \lambda) = \{u \in W^{1,1}_{\mathrm{loc}}(\Omega): u \text{ and } |Du| \in L^p(\Omega, \lambda)\}$.

It is easy to verify that $W^{1,p}(\Omega, \lambda)$ endowed with the topology induced by the norm $\|u\|_{W^{1,p}(\Omega,\lambda)} := \|u\lambda^{1/p}\|_{L^p(\Omega)} + \||Du|\lambda^{1/p}\|_{L^p(\Omega)}$ is a reflexive and separable Banach space.

We denote by $W^{1,p}_0(\Omega, \lambda)$ the closure of $C^1_0(\Omega)$ in the topology of $W^{1,p}(\Omega, \lambda)$, by $W^{-1,p'}(\Omega, \lambda)$ its dual space and by $\langle \cdot, \cdot \rangle$ the duality bracket between $W^{-1,p'}(\Omega, \lambda)$ and $W^{1,p}_0(\Omega, \lambda)$.

We recall that (cf. Theorem 1.4 in [14] and Proposition 1.2 in [5])

(1.1) $W^{1,p}_0(\Omega, \lambda). = W^{1,p}(\Omega, \lambda). \cap W^{1,1}_0(\Omega)$ for every bounded open set Ω with Lipschitz boundary, λ in A_p (see below for the definition of A_p).

REMARK 1.1. – It easy to see that $W^{1,p}_0(\Omega, \lambda)$ continuously embeds in $W^{1,1}_0(\Omega)$ and compactly in $L^q(\Omega)$ for every $q \in [1, n/(n-1))$, hence we have that $L^n(\Omega) \subset W^{-1,p'}(\Omega, \lambda)$; moreover it can be easily proved that there exists a positive constant $c = c(p, \Omega)$ (depending only on p and Ω) such that

$$\|f\|_{W^{-1,p'}(\Omega,\lambda)} \leq c\left(\int_\Omega \lambda^{1-p'}\,dx\right)^{1/p'} \|f\|_{L^n(\Omega)}, \tag{1.2}$$

for every weight λ, on $\mathbb{R}^n$.

Given $p > 1$, $K \geq 1$ and a weight λ we say that λ is in the Muckenhoupt class $A_p(K)$ (see [11]) if

$$\left(\fint_Q \lambda\,dx\right)\left(\fint_Q \lambda^{1-p'}\,dx\right)^{p-1} \leq K \quad \text{for every cube } Q. \tag{1.3}$$

We set $A_p := \bigcup_{K\geq 1} A_p(K)$.

A_p weights verify the following higher summability property (see [4] and also [5]): for every $p > 1$ and $K \geq 1$ there exist two positive constants $c = c(p, K)$ and $\delta = \delta(p, K)$ (depending only on p and K) such that

$$\left(\fint_Q \lambda^{1+\delta}dx\right)^{1/(1+\delta)} \leq c\fint_Q \lambda\,dx\,, \qquad \left(\fint_Q \lambda^{(1-p')(1+\delta)}dx\right)^{1/(1+\delta)} \leq c\fint_Q \lambda^{1-p'}\,dx\,, \tag{1.4}$$

for every cube Q and λ in $A_p(K)$; moreover, (cf. [9]) if Ω is a bounded open set of $\mathbb{R}^n$

there exists a positive constant $c = c(p, K, \Omega)$ (depending only on p, K and Ω) such that

$$\int_\Omega |u|^p \lambda\, dx \leq c \int_\Omega |Du|^p \lambda\, dx, \tag{1.5}$$

for every λ in $A_p(K)$, u in $W_0^{1,p}(\Omega, \lambda)$.

In [6] the following result of compensated compactness type is proved (compare also with [12]).

THEOREM 1.2. – *Let λ be in A_p, $K \geq 1$; let $(\lambda_h)_h$ be a sequence in $A_p(K)$ and let Ω be a bounded open set.*

Consider a sequence of functions $(u_h)_h \subseteq W^{1,p}(\Omega, \lambda_h)$ and u in $W^{1,p}(\Omega, \lambda)$ such that

$$\int_\Omega (|u_h|^p + |Du_h|^p)\lambda_h dx \quad \forall h, \qquad u_h \to u \ \text{ in } L^1(\Omega)$$

and a sequence of vector functions $(a_h)_h \subseteq (L^{p'}(\Omega, \lambda_h^{1-p'}))^n$ and a in $(L^{p'}(\Omega, \lambda^{1-p'}))^n$ such that

$$\int_\Omega (|a_h|^{p'} \lambda_h^{1-p'} dx \leq c_2 \quad \forall h, \qquad -\operatorname{div}(a_h) = f \in L^n(\Omega) \ \text{ on } C_0^1(\Omega),$$

$$a_h \to a \ \text{ in } (L^1(\Omega))^n\text{-weak}.$$

Then

$$(a_h | Du_h) \to (a | Du) \quad \text{in } \mathcal{D}'(\Omega).$$

In [5] a weak compactness result for A_p weights is proved: if Q_0 is a fixed cube of $\mathbb{R}^n$ and $(\lambda_h)_h$ is a sequence in $A_p(K)$ such that $(\lambda_h)_h$ and $(\lambda_h^{1-p'})_h$ are bounded in $L^1(Q_0)$, then there exist a positive constant $c = c(n)$ (depending only on n) and two weights $\tilde{\lambda}$ and λ such that

$$\tilde{\lambda} \text{ and } \lambda \text{ are in } A_p(cK), \qquad \lambda(x) \leq \tilde{\lambda}(x) \leq K\lambda(x) \text{ for a.e. } x \in Q_0, \tag{1.6}$$

and, up to subsequences,

$$\lambda_h \to \tilde{\lambda} \quad \text{and} \quad \lambda_h^{1-p'} \to \lambda^{1-p'} \quad \text{in } L^1(Q_0)\text{-weak}. \tag{1.7}$$

REMARK 1.3. – If for every cube Q_0 of $\mathbb{R}^n$, the sequences $(\lambda_h)_h$ and $(\lambda_h^{1-p'})_h$ are bounded in $L^1(Q_0)$, then, by (1.6), (1.7) and by using a diagonal process, it can be proved (see [5]) the existence of two weights $\tilde{\lambda}$ and λ in $A_p(K)$ such that, up to subsequences, (1.6) and (1.7) hold respectively for a.e. x in $\mathbb{R}^n$ and for every cube Q_0.

We now prove the following «lower semicontinuity» type result.

LEMMA 1.4. – *Let $p > 1$, K and $\tilde{K} \geq 1$; let Ω be a bounded open set with Lipschitz* *boundary and let* $(\lambda_h)_h$ *be a sequence in* $A_p(K)$. *Let us assume that there exist two weights* $\tilde{\lambda}$ *in* $A_p(\tilde{K})$, λ *in* $A_p(K)$ *and a positive constant* c_0 *such that*

$$\lambda_h \to \tilde{\lambda} \quad \text{and} \quad \lambda_h^{1-p'} \to \lambda^{1-p'} \quad \text{in } L^1(\Omega_0)\text{-weak}, \tag{1.8}$$

$$\frac{1}{c_0}\lambda(x) \leq \tilde{\lambda}(x) \leq c_0 \lambda(x) \quad \text{for a.e. } x \in \Omega . \tag{1.9}$$

Then

(i) *if* $(u_h)_h \subseteq W_0^{1,p}(\Omega, \lambda_h)$ *is a sequence such that* $\int_\Omega |Du_h|^p \lambda_h \, dx \leq c_1 \ \forall h$, $u_h \to u$ *in* $W_0^{1,1}(\Omega)$*-weak, it follows that*

$$u \in W_0^{1,p}(\Omega, \lambda) \quad \text{and} \quad \int_\Omega |Du|^p \lambda \, dx \leq \liminf_h \int_\Omega |Du_h|^p \lambda_h \, dx .$$

(ii) *If* f *is in* $L^n(\Omega)$ *it follows that there exist two positive constants* $c_i = c_i(p, K, \Omega, c_0)$, $i = 2, 3$, *(depending only on* p, K, Ω *and* c_0*) such that*

$$c_3 \|f\|_{W^{-1,p'}(\Omega, \lambda)} \leq \liminf_h \|f\|_{W^{-1,p'}(\Omega, \lambda_h)} \leq \limsup_h \|f\|_{W^{-1,p'}(\Omega, \lambda_h)} \leq c_2 \|f\|_{W^{-1,p'}(\Omega, \lambda)} .$$

PROOF. – (i) By Hölder inequality and (1.8) it follows that

$$\int_\Omega |Du|\,|\varphi|\, dx \leq \liminf_h \left(\int_\Omega |Du_h|^p \lambda_h \, dx\right)^{1/p} \left(\int_\Omega |\varphi|^{p'} \lambda^{1-p'} dx\right)^{1/p'} \quad \forall \varphi \in C_0^0(\Omega). \tag{1.10}$$

By (1.10) and by exploiting the density of $C_0^0(\Omega)$ in $L^{p'}(\Omega, \lambda^{1-p'})$, we deduce that $|Du| \in L^p(\Omega, \lambda)$ and that

$$\int_\Omega |Du|^p \lambda \, dx \leq \liminf_h \int_\Omega |Du_h|^p \lambda_h \, dx ;$$

hence, being Ω regular, by (1.1) it turns out that u is in $W_0^{1,p}(\Omega, \lambda)$.

(ii) For every $\varepsilon > 0$ and $h \in \mathbb{N}$ there exists $v_h^{(\varepsilon)}$ in $W_0^{1,p}(\Omega, \lambda_h)$ such that

$$\|v_h^{(\varepsilon)}\|_{W_0^{1,p}(\Omega, \lambda_h)} \leq 1, \qquad \|f\|_{W^{-1,p'}(\Omega, \lambda_h)} \leq \varepsilon + \|v_h^{(\varepsilon)}\|^{-1}_{W_0^{1,p}(\Omega, \lambda_h)} \left| \int_\Omega f v_h^{(\varepsilon)} dx \right| . \tag{1.11}$$

By Hölder inequality, (1.4) and (1.11) we get that there exists a positive constant σ such that $(v_h^{(\varepsilon)})_h$ is bounded in $W_0^{1,1+\sigma}(\Omega)$ and therefore, up to subsequences, there exists $v^{(\varepsilon)}$ in $W_0^{1,1+\sigma}(\Omega)$ such that

$$v_h^{(\varepsilon)} \to v^{(\varepsilon)} \quad \text{in } W_0^{1,1}(\Omega)\text{-weak}.$$

On the other hand by (i) passing to the limit in (1.11) we get

$$\limsup_h \|f\|_{W^{-1,p'}(\Omega, \lambda_h)} \leq \varepsilon + \left(\int_\Omega |Dv^{(\varepsilon)}|^p \lambda \, dx\right)^{-1/p} \left| \int_\Omega f v^{(\varepsilon)} dx \right| . \tag{1.12}$$

By (1.5), (1.12) it follows that there exists a positive constant $c_2 = c_2(p, K, \Omega)$ for which the inequality in the right hand side in (ii) holds.

Finally, by (1.8) we have

$$\left| \int_\Omega fv\, dx \right| \|v\|^{-1}_{W_0^{1,p}(\Omega, \lambda)} \leq \liminf_h \|f\|_{W^{-1,p'}(\Omega, \lambda_h)} \qquad \forall v \in C_0^1(\Omega);$$

therefore, by (1.9) and by density of $C_0^1(\Omega)$ in $W_0^{1,p}(\Omega, \lambda)$, it follows that there exists a positive constant $c_3 = c_3(p, K, c_0)$ for which the left side in (ii) holds. ∎

Finally we recall the following result (see Lemma 7.8 in [9]).

LEMMA 1.5. – *Let δ, ρ, ϑ be real positive numbers such that $\delta + \rho + \vartheta \leq 1$. Let us assume that $(t_h)_h$, $(s_h)_h$, $(z_h)_h$ and $(w_h)_h$ are sequences in $L^1(\Omega)$ such that*

$$(s_h)_h,\ (z_h)_h \text{ and } (w_h)_h \quad \text{are non negative},$$

$$|t_h| \leq s_h^\delta z_h^\rho w_h^\vartheta \quad \text{a.e. in } \Omega, \text{ for every } h,$$

$$t_h \to t, \quad s_h \to s, \quad z_h \to z, \quad w_h \to w, \qquad \text{in } \mathcal{D}'(\Omega),$$

for some functions t, s, z and w in $L^1(\Omega)$. Then

$$|t| \leq s^\delta z^\rho w^\vartheta \quad \text{a.e. in } \Omega.$$

2. – A notion of convergence for a class of degenerate elliptic operators.

DEFINITION 2.1. – *Let p, α, β, L and K be positive constants with*

$$p > 1, \quad 0 < \alpha \leq \min\left\{\frac{p}{2}, p-1\right\}, \quad \beta \geq \max\{2, p\}, \quad L \geq 1 \quad \text{and} \quad K \geq 1. \tag{2.1}$$

If Ω is an open set, we denote by $\mathcal{M}_\Omega(p, \alpha, \beta, L, K)$ the class of the Carathéodory functions $a\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ for which there exists a positive functions λ in $A_p(K)$ and m in $L^1_{\mathrm{loc}}(\Omega)$ such that, if

$$H \equiv H(x, \xi_1, \xi_2) := m(x) + (a(x, \xi_1)|\xi_1) + (a(x, \xi_2)|\xi_2) \tag{2.2}$$

the following structure conditions hold:

(S$_1$) $H(x, \xi_1, \xi_2) > 0$,

(S$_2$) $(a(x, \xi_1) - a(x, \xi_2)|\xi_1 - \xi_2) \geq \lambda^{\beta/p}(x) H^{(p-\beta)/p}(x, \xi_1, \xi_2)|\xi_1 - \xi_2|^\beta$,

(S$_3$) $|a(x, \xi_1) - a(x, \xi_2)| \leq L\lambda^{1/p}(x) H^{(p-1-\alpha)/p}(x, \xi_1, \xi_2)(a(x, \xi_1) - a(x, \xi_2)|\xi_1 - \xi_2)^{\alpha/p}$,

for a.e. $x \in \Omega$ for every ξ_1, ξ_2 in $\mathbb{R}^n$.

When $\Omega = \mathbb{R}^n$ we denote $\mathcal{M}_{\mathbb{R}^n}(p, \alpha, \beta, L, K)$ simply by $\mathcal{M}(p, \alpha, \beta, L, K)$.

Since p, α and β **will remain fixed in the whole paper**, sometime we will write simply $\mathcal{M}_\Omega(L, K)$ and $\mathcal{M}(L, K)$ instead of $\mathcal{M}_\Omega(p, \alpha, \beta, L, K)$ and $\mathcal{M}(p, \alpha, \beta, L, K)$.

LEMMA 2.2. – *Let a be in $\mathcal{M}_\Omega(L, K)$ verifying conditions* $(S_1) \div (S_3)$ *with functions λ in $A_p(K)$, m in $L^1_{loc}(\Omega)$ and let H be the function in* (2.2). *Then there exist positive constants $c_i = c_i(p, \alpha, \beta, L)$ $(i = 1, \dots, 5)$ (depending only on p, α, β, L) such that*

$$(2.3) \qquad |a(x, \xi_1) - a(x, \xi_2)| \leq c_1 \lambda^{1/(p-\alpha)} H^{(p-1-\alpha)/(p-\alpha)} |\xi_1 - \xi_2|^{\alpha/(p-\alpha)},$$

$$\text{a.e. in } \Omega, \ \forall \xi_1, \xi_2 \in \mathbb{R}^n,$$

$$(2.4) \qquad |a(x, \xi)| \leq c_2(|a(x, 0)| + m^{1/p'}(x)\lambda^{1/p}(x) + \lambda(x)|\xi|^{p-1}),$$

$$(2.5) \qquad H \leq c_3\{m + |a(x, 0)|^{p'}\lambda^{1-p'} + \lambda(|\xi_1|^p + |\xi_2|^p)\}, \qquad \text{a.e. in } \Omega, \ \forall \xi_1, \xi_2 \in \mathbb{R}^n,$$

$$(2.6) \qquad (a(x, \xi)|\xi) \geq c_4\lambda(x)|\xi|^p - c_5(|a(x, 0)|^{p'}\lambda^{1-p'}(x) + m(x)),$$

$$\text{for a.e. } x \in \Omega \text{ for every } \xi \text{ in } \mathbb{R}^n.$$

PROOF. – The proof of the above estimates can be obtained in a standard way by using Young inequality (see, for instance [8] and [12]). ■

The following characterization of $\mathcal{M}_\Omega(L, K)$ holds.

PROPOSITION 2.3. – *Let Ω be an open set and let a: $\Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function. Then the following facts hold:*

(i) *If a verifies* $(S_1) \div (S_3)$ *with constants p, α, β, L, K satisfying* (2.1) *and functions λ in $A_p(K)$ and m in $L^1_{loc}(\Omega)$ it follows that*

$$(S_1^*) \qquad |a(x, \xi_1) - a(x, \xi_2)| \leq$$

$$\leq L_1 \lambda_*^{(1+\gamma)/p}(x)[m_*(x) + \lambda_*(x)(|\xi_1|^p + |\xi_2|^p)]^{(p-1-\gamma)/p} |\xi_1 - \xi_2|^\gamma,$$

$$(S_2^*) \qquad (a(x, \xi_1) - a(x, \xi_2)|\xi_1 - \xi_2) \geq$$

$$\geq L_2 \lambda_*^{\beta/p}(x)[m_*(x) + \lambda_*(x)(|\xi_1|^p + |\xi_2|^p)]^{(p-\beta)/p} |\xi_1 - \xi_2|^\beta,$$

for a.e. x in Ω, for every ξ_1 and ξ_2 in $\mathbb{R}^n$, where

$$(2.7) \qquad \lambda_* := \lambda, \qquad m_* := m + |a(\cdot, 0)|^{p'}\lambda^{1-p'}, \qquad \gamma = \frac{\alpha}{p - \alpha}$$

and $L_i = L_i(p, \alpha, \beta, L)$ $(i = 1, 2)$ are suitable positive constants depending only on p, α, β, L.

(ii) *If a verifies* (S_1^*) *and* (S_2^*) *with functions λ_* in $A_p(K)$ and m_* in $L^1_{loc}(\Omega)$ and positive constants γ, β and L_i $(i = 1, 2)$ such that $0 < \gamma \leq \min\{1, p - 1\}$, $\beta \geq \max\{2, p\}$ and $K \geq 1$, then a verifies* $(S_1) \div (S_3)$ *with $\lambda = c_1\lambda_*$, $m = c_2 m_* +$*

$+ c_3 |a(\cdot, 0)|^{p'} \lambda_*^{1-p'}$, $\alpha = p\gamma/\beta$, $L \geq 1$, *being* c_i $(i = 1, 2, 3)$ *and* L *suitable positive constants depending only on* p, γ, β *and* L_i.

PROOF. – i) Let us assume that $a \in \mathcal{M}_\Omega(L, K)$ verifies $(S_1) \div (S_3)$. By (2.3), (2.5) it follows that there exists a positive constant $L_1 = L_1(p, \alpha, \beta, L)$ such that

$$(2.8) \qquad |a(x, \xi_1) - a(x, \xi_2)| \leq$$
$$\leq L_1 \lambda^{1/(p-\alpha)} [m + |a(x, 0)|^{p'} \lambda^{1-p'} + \lambda(|\xi_1|^p + |\xi_2|^p)]^{(p-1-\alpha)/(p-\alpha)} |\xi_1 - \xi_2|^{\alpha/(p-\alpha)},$$
for a.e. $x \in \Omega$, for every ξ_1 and ξ_2 in $\mathbb{R}^n$.

Therefore, if we choose λ_*, m_* and γ as in (2.7), we get that (S_1^*) is satisfied at once.

On the other hand by (2.5) it follows that there exists a positive constant $L_2 = L_2(p, \alpha, \beta, L)$ such that

$$(2.9) \qquad H^{(p-\beta)/p} \geq L_2 [m + |a(x, 0)|^{p'} \lambda^{1-p'} + \lambda(|\xi_1|^p + |\xi_2|^p)]^{(p-\beta)/p}$$
$$\text{a.e. in } \Omega, \ \forall \xi_1 \xi_2 \in \mathbb{R}^n,$$

then, by (S_3) and (2.9), (S_2^*) follows.

(ii) By (S_2^*), by means of Young inequality, we deduce

$$L_2^{p/\beta} \lambda_* |\xi|^p \leq \frac{p\varepsilon^{-\beta/p}}{\beta} [(a(x, \xi)|\xi) + |a(x, 0)| |\xi|] + \frac{(\beta - p)}{\beta} \varepsilon^{(\beta-p)/\beta} (m_* + \lambda_* |\xi|^p)$$

a.e. in Ω, $\forall \xi \in \mathbb{R}^n$.

If ε is small enough, by the previous inequality, we deduce the existence of a suitable positive constant $c_* = c_*(p, \beta, L_1, L_2)$ for which the following estimate holds

$$(2.10) \qquad c_* \lambda_* |\xi|^p \leq m_* + |a(x, 0)|^{p'} \lambda_*^{1-p'} + (a(x, \xi)|\xi) \qquad \text{a.e. in } \Omega, \ \forall \xi \in \mathbb{R}^n.$$

Let us now define $m(x) := (2 + c_*) m_*(x) + 2|a(x, 0)|^{p'} \lambda_*^{1-p'}(x)$ and let H be as in (2.2), then by (2.10) (S_1) follows at once.

On the other hand by (S_2^*) it follows that

$$(2.11) \qquad L_2 \lambda_*^{\beta/p} [m_* + \lambda_*(|\xi_1|^p + |\xi_2|^p)]^{(p-\beta)/p} |\xi_1 - \xi_2|^\beta \geq$$
$$\geq L_2 c_*^{(\beta-p)/p} \lambda_*^{\beta/p} H^{(p-\beta)/p} |\xi_1 - \xi_2|^\beta \qquad \text{a.e. in } \Omega, \ \forall \xi_1 \xi_2 \in \mathbb{R}^n,$$

so, if we choose $\lambda(x) := L_2^{p/\beta} c_*^{(\beta-p)/\beta} \lambda_*(x)$, by (S_2^*) and (2.11) we get (S_2).

Finally by (S_1^*) it follows that

$$(2.12) \qquad |a(x, \xi_1) - a(x, \xi_2)| \leq L_1 c_*^{(1+\gamma-p)/p} \lambda_*^{(1+\gamma)/p} H^{(p-1-\gamma)/p} |\xi_1 - \xi_2|^\gamma$$
$$\text{a.e. in } \Omega, \ \forall \xi_1 \xi_2 \in \mathbb{R}^n,$$

and by (S_2^*) that

$$|\xi_1 - \xi_2| \leq \frac{1}{L_2} \lambda_*^{-1/p} H^{(\beta-p)/p\beta}(a(x, \xi_1) - a(x, \xi_2) | \xi_1 - \xi_2)^\beta \qquad \text{a.e. in } \Omega, \ \forall \xi_1 \xi_2 \in \mathbb{R}^n . \tag{2.13}$$

Therefore, by (2.12) and (2.13), (S_3) holds if we choose $\alpha = p\gamma/\beta$ and a suitable constant L. ■

REMARK 2.4. – Let a be in $\mathcal{M}_\Omega(L, K)$ and let us assume that $(S_1) \div (S_3)$ hold with functions λ_i in $A_p(K)$, m_i in $L^1_{loc}(\Omega)$ $(i = 1, 2)$; then (see Remark 3.1 in [6]) it can be proved that the weights λ_i are comparable, that is there exists a positive constant $c_0 = c_0(p, \alpha, \beta, L)$ for which (1.9) holds.

REMARK 2.5. – Let Ω be a bounded open set and let a be in $\mathcal{M}_\Omega(L, K)$. Let us suppose that $(S_1) \div (S_3)$ hold with functions λ in $A_p(K)$, m in $L^1(\Omega)$ and that $|a(x, 0)|^{p'} \lambda^{1-p'}$ is in $L^1(\Omega)$; then by Corollary 1.8, Chapter III in [10] and by Proposition 2.3 we deduce that, for every $f \in W^{-1, p'}(\Omega, \lambda)$ the Dirichlet problem

$$- \operatorname{div}(a(x, Dv)) = f \quad \text{in } \Omega, \qquad v \in W_0^{1,p}(\Omega, \lambda) \tag{P_a}$$

has a unique solution.

REMARK 2.6. – Let a be a function verifying $(S_1) \div (S_3)$ for some functions λ in $A_p(K)$ and m in $L^1(\Omega)$; if λ' and m' are other functions for which $(S_1) \div (S_3)$ still hold, then, by virtue of Remark 2.4, the weights λ and λ' are comparable and therefore $W_0^{1,p}(\Omega, \lambda)$ turns out to be equal to $W_0^{1,p}(\Omega, \lambda')$; this implies that problem (P_a) depends effectively on a and not on the particular choice of λ.

We now prove some properties of the operator $-\operatorname{div}(a(x, D\cdot))$ with a in $\mathcal{M}_\Omega(L, K)$.

PROPOSITION 2.7. – *Let Ω be a bounded open set, let a in $\mathcal{M}_\Omega(p, \alpha, \beta, L, K)$ and let A be the following operator*

$$A\colon W_0^{1,p}(\Omega, \lambda) \to W^{-1,p'}(\Omega, \lambda), \qquad A = -\operatorname{div}(a(x, D\cdot)).$$

Then A is continuous and invertible. Moreover the following estimates holds: there exists a positive constant $c = c(p, \alpha, \beta, L, \Omega)$ (depending only on p, α, β, L and Ω) such that, if m_* is as in (2.7) and belongs to $L^1(\Omega)$, it results

$$\|Au - Av\|_{W^{-1,p'}(\Omega,\lambda)} \leq c(\|m_*\|_{L^1(\Omega)} + \|u\|^p_{W_0^{1,p}(\Omega,\lambda)} + \|v\|^p_{W_0^{1,p}(\Omega,\lambda)})^{(p-1-\gamma)/(p-1)} \|u - v\|^{\gamma/(p-1)}_{W_0^{1,p}(\Omega,\lambda)}, \tag{2.14}$$

for every u and v in $W_0^{1,p}(\Omega, \lambda)$ with $\gamma = \alpha/(p - \alpha)$;

(2.15) $$\|A^{-1}f - A^{-1}g\|_{W_0^{1,p}(\Omega,\lambda)} \leqslant$$

$$\leqslant c(\|m_*\|_{L^1(\Omega)} + \|f\|^{p'}_{W^{-1,p'}(\Omega,\lambda)} + \|g\|^{p'}_{W^{-1,p'}(\Omega,\lambda)})^{(\beta-p)/(p(\beta-1))}\|f-g\|^{1/(\beta-1)}_{W^{-1,p'}(\Omega,\lambda)},$$

for every f and g in $W^{-1,p'}(\Omega,\lambda)$.

PROOF. – In order to get (2.14) we first observe that by Proposition 2.3 (i) there exists a positive constant $c = c(p, \alpha, \beta, L, \Omega)$ such that

(2.16) $$\|Au - Av\|_{W^{-1,p'}(\Omega,\lambda)} \leqslant \left(\int_\Omega |a(x,Du) - a(x,Dv)|^{p'}\lambda^{1-p'}dx\right)^{1/p'} \leqslant$$

$$\leqslant c\left(\int_\Omega \left[m + |a(x,0)|^{p'}\lambda^{1-p'} + \lambda(|Du|^p + |Dv|^p)\right]^{(p-1-\gamma)/(p-1)}|Du-Dv|^{\gamma p'}\lambda^{\gamma/(p-1)}dx\right)^{1/p'},$$

for every u and v in $W_0^{1,p}(\Omega,\lambda)$ with $\gamma = \alpha/(p-\alpha)$.

Then, by (1.5), Hölder inequality and (2.16), (2.14) follows at once.

By (S_1^*), (S_2^*) of Proposition 2.3 and by (2.14) A turns out to be continuous, monotone and coercive, then, by applying, for instance, Corollary 1.8, Chapter III in [10], we get at once that A is invertible.

In order to prove (2.15) let us preliminarly observe that, in general, by Hölder inequality, we have

(2.17) $$\int_\Omega |Du - Dv|^p\mu\,dx \leqslant \left(\int_\Omega \mu^{\beta/p}[r + \mu(|Du|^p + |Dv|^p)]^{(p-\beta)/p}|Du - Dv|^\beta dx\right)^{p/\beta}\cdot$$

$$\cdot\left(\int_\Omega r + \mu(|Du|^p + |Dv|^p)\,dx\right)^{(\beta-p)/\beta},$$

for every u, v in $W_0^{1,p}(\Omega,\mu)$ every positive function r in $L^1(\Omega)$ and every weight μ.

Moreover by (2.6) and Poincaré inequality in (1.5) there exists a positive constant $c_1 = c_1(p, \alpha, \beta, L, \Omega)$ such that

$$\int_\Omega |DA^{-1}f|^p\lambda\,dx \leqslant c_1\left(\|f\|_{W^{-1,p'}(\Omega,\lambda)}\|DA^{-1}f\|_{W_0^{1,p}(\Omega,\lambda)} + \int_\Omega (m + |a(x,0)|^{p'}\lambda^{1-p'})\,dx\right).$$

By applying Young inequality to the previous estimate we get the existence of a positive constant $c_2 = c_2(p, \alpha, \beta, L, K, \Omega)$ such that

$$(2.18) \qquad \|A^{-1}f\|^p_{W_0^{1,p}(\Omega,\lambda)} \leq c_2\left(\int_\Omega (m + |a(x, 0)|^{p'}\lambda^{1-p'})\,dx + \|f\|^{p'}_{W^{-1,p'}(\Omega,\lambda)}\right),$$

for every f in $W^{-1,p'}(\Omega, \lambda)$.

On the other hand, by condition (S_2^*), (1.5) and by applying (2.17) with $u = A^{-1}f$, $v = A^{-1}g$, $r = m + |a(x, 0)|^{p'}\lambda^{1-p'}$ and $\mu = \lambda$, we deduce that there exists a positive constant $c_3 = c_3(p, \alpha, \beta, L, K, \Omega)$ such that

$$(2.19) \qquad \|A^{-1}f - A^{-1}g\|_{W_0^{1,p}(\Omega,\lambda)} \leq$$

$$\leq c_3\left(\int_\Omega (m + |a(x, 0)|^{p'}\lambda^{1-p'} + \lambda|DA^{-1}f|^p + \lambda|DA^{-1}g|^p)\,dx\right)^{(\beta-p)/(p(\beta-1))} \cdot$$

$$\cdot \|f - g\|^{1/(\beta-1)}_{W^{-1,p'}(\Omega,\lambda)},$$

for every f and g in $W^{-1,p'}(\Omega, \lambda)$.

By (2.18) and (2.19), (2.15) follows at once. ∎

Now we introduce the following notion of G-convergence (see also [3], [12], [14], [15] and [16]).

DEFINITION 2.8. – *Let p, α, β, L and K be positive numbers satisfying* (2.1) *and let Ω be a bounded open set.*

Let a_h $(h = 1, 2, \ldots)$ and a be functions in $\mathcal{M}_\Omega(p, \alpha, \beta, L, K)$ verifying $(S_1) \div (S_3)$ *respectively with weights λ_h and λ in $A_p(K)$ and functions m_h and m in $L^1(\Omega)$ and such that $|a_h(x, 0)|^{p'}\lambda_h^{1-p'}$ and $|a(x, 0)|^{p'}\lambda^{1-p'}$ are in $L^1(\Omega)$.*

We say that the sequence (a_h) G-converges to a in Ω, and we write

$$a_h \xrightarrow{G} a \quad \text{in } \Omega,$$

if for every f in $L^n(\Omega)$, being u_h and u the solutions of the Dirichlet problems

$$\begin{cases} -\operatorname{div}(a_h(x, Dv)) = f & \text{in } \Omega \\ v \in W_0^{1,p}(\Omega, \lambda_h) \end{cases} \quad \text{and} \quad \begin{cases} -\operatorname{div}(a(x, Dv)) = f & \text{in } \Omega \\ v \in W_0^{1,p}(\Omega, \lambda), \end{cases}$$

it results that

$$u_h \to u \quad \text{in } W_0^{1,1}(\Omega)\text{-weak} \quad \text{and} \quad a_h(x, Du_h) \to a(x, Du) \quad \text{in } (L^1(\Omega))^n\text{-weak}.$$

The following locality property holds for G-convergence.

PROPOSITION 2.9. – *Let Ω_i $(i = 1, 2)$ be two bounded open sets with $\Omega_1 \subseteq \Omega_2$ and let $(a_h)_h$ be a sequence in $\mathcal{M}_{\Omega_2}(L, K)$.*

Let us assume that a_h satisfies $(S_1) \div (S_3)$ *with functions λ_h in $A_p(K)$, m_h in $L^1(\Omega_2)$ and that:*

(i) *there exists a cube Q_0 of $\mathbb{R}^n$ with $\bar{\Omega}_2 \subseteq \Omega_0$ such that the sequences $(\lambda_h)_h$ and $(\lambda_h^{1-p'})_h$ are bounded in $L^1(Q_0)$;*

(ii) *there exists m in $L^1(\Omega_2)$ such that $m_h \to m$ in $L^1(\Omega_2)$*-weak.

Then, if

$$a_h \xrightarrow{G} b_i \quad in\ \Omega_i\ (i = 1, 2)$$

for some functions b_i in $\mathcal{M}_{\Omega_i}(L, K)$, it follows that

$$b_1(x, \xi) = b_2(x, \xi) \quad for\ a.e.\ x \in \Omega_1\ and\ every\ \xi \in \mathbb{R}^n .$$

PROOF. – By (i) it is not restrictive to assume the existence of two weights $\tilde{\lambda}$ and λ in $A_p(cK)$ (where $c = c(n)$ is the constant appearing in (1.6)) verifying (1.6) and (1.7).

Let us suppose that $(S_1) \div (S_3)$ hold for b_i $(i = 1, 2)$ with $\lambda^{(i)}$ in $A_p(K)$ and $m^{(i)}$ in $L^1(\Omega_i)$ $(i = 1, 2)$ and set

$$A_h^{(i)} = -\operatorname{div}(a_h(x, D\cdot)) \colon W_0^{1,p}(\Omega_i, \lambda_h) \to W^{-1,p'}(\Omega_i, \lambda_h),$$

$$B^{(i)} = -\operatorname{div}(b_i(x, D\cdot)) \colon W_0^{1,p}(\Omega_i, \lambda^{(i)}) \to W^{-1,p'}(\Omega_i, \lambda^{(i)}).$$

By Definition 2.8 we get that

$$\text{(2.20)} \qquad \begin{cases} (A_h^{(i)})^{-1} f \to (B^{(i)})^{-1} f, & \text{in } W_0^{1,1}(\Omega_i)\text{-weak}, \\ a_h(x, D(A_h^{(i)})^{-1} f) \to b_i(x, D(B^{(i)})^{-1} f), & \text{in } (L^1(\Omega_i))^n\text{-weak}, \end{cases}$$

for every f in $L^n(\Omega_i)$ $(i = 1, 2)$.

For every $i = 1, 2$, f and g in $L^n(\Omega_i)$ let us set $u_h^{(i)} = (A_h^{(i)})^{-1} f$, $v_h^{(i)} = (A_h^{(i)})^{-1} g$, $u^{(i)} = (B^{(i)})^{-1} f$, $v^{(i)} = (B^{(i)})^{-1} g$ and denote by H_h (respectively $H_{(i)}$) the functions in (2.2) with $m \equiv m_h$, $a \equiv a_h$ (respectively with $m \equiv m^{(i)}$, $a \equiv b_i$). By $(S_1) \div (S_3)$ we get

$$\text{(2.21)} \quad (a_h(x, Du_h^{(i)}) - a_h(x, Dv_h^{(i)}) | Du_h^{(i)} - Dv_h^{(i)}) \geqslant$$
$$\geqslant \lambda_h^{\beta/p} H_h^{(p-\beta)/p}(x, Du_h^{(i)}, Dv_h^{(i)}) |Du_h^{(i)} - Dv_h^{(i)}|^\beta ,$$

$$\text{(2.22)} \quad |a_h(x, Du_h^{(i)}) - a_h(x, Dv_h^{(i)}| \leqslant$$
$$\leqslant L \lambda_h^{1/p} H_h^{(p-1-\alpha)/p}(x, Du_h^{(i)}, Dv_h^{(i)})(a_h(x, Du_h^{(i)}) - a_h(x, Dv_h^{(i)}) | Du_h^{(i)} - Dv_h^{(i)})^{\alpha/p} ,$$

a.e. in Ω_i, for every h and i.

If we set $\delta = (p-1)/p$, $\rho = (\beta - p)/p\beta$, $\vartheta = 1/\beta$ and $t_h \equiv Du_h^{(i)} - Dv_h^{(i)}$, $s_h \equiv \lambda_h^{1-p'}$, $z_h \equiv H_h(Du_h^{(i)}, Dv_h^{(i)})$, $w_h \equiv (a_h(x, Du_h^{(i)}) - a_h(x, Dv_h^{(i)}) | Du_h^{(i)} - Dv_h^{(i)})$, the assumptions of Lemma 1.5 are satisfied, therefore taking the limit in (2.21), we get

$$(2.23)\qquad (b_i(x, Du^{(i)}) - b_i(x, Dv^{(i)} | Du^{(i)} - Dv^{(i)}) \geq$$
$$\geq \lambda^{\beta/p} H_{(i)}^{(p-\beta)/p}(x, Du^{(i)}, Dv^{(i)}) |Du^{(i)} - Dv^{(i)}|^{\beta} \quad \text{a.e. in } \Omega_i,$$

for every $u^{(i)}$ and $v^{(i)}$ in $(B^{(i)})^{-1}(L^n(\Omega_i))$ $(i = 1, 2)$.

Analogously, by applying again Lemma 1.5 with $\delta = 1/p$, $\rho = (p - 1 - \alpha)/p$, $\vartheta = \alpha/p$ and $t_h \equiv a_h(x, Du_h^{(i)}) - a_h(x, Dv_h^{(i)})$, $s_h \equiv \lambda_h$, $z_h \equiv H_h(x, Du_h^{(i)}, Dv_h^{(i)})$, $w_h \equiv (a_h(x, Du_h^{(i)}) - a_h(x, Dv_h^{(i)}) | Du_h^{(i)} - Dv_h^{(i)})$, we can take to the limit in (2.22) and get

$$(2.24)\qquad |b_i(x, Du^{(i)}) - b_i(x, Dv^{(i)})| \leq$$
$$\leq L\lambda^{1/p} H_{(i)}^{(p-1-\alpha)/p}(x, Du^{(i)}, Dv^{(i)})(b_i(x, Du^{(i)}) - b_i(x, Dv^{(i)}) | Du^{(i)} - Dv^{(i)})^{\alpha/p},$$
$$\text{a.e. in } \Omega_i,$$

for every $u^{(i)}$ and $v^{(i)}$ in $(B^{(i)})^{-1}(L^n(\Omega_i))$ $(i = 1, 2)$.

By the density of $(B^{(i)})^{-1}(L^n(\Omega_i))$ in $W_0^{1,p}(\Omega_i, \lambda^{(i)})$ and by the continuity of $b_i(x, \cdot)$ we deduce that (2.23) and (2.24) hold on the whole $W_0^{1,p}(\Omega_i, \lambda^{(i)})$.

Therefore b_i $(i = 1, 2)$ satisfies $(S_1) \div (S_3)$ with λ and m, so, by Remark 2.4, there exist positive constant c_i $(i = 1, 2)$ such that

$$(2.25)\qquad \frac{1}{c_i}\lambda(x) \leq \lambda^{(i)}(x) \leq c_i \lambda(x) \quad \text{a.e. in } \Omega_i \ (i = 1, 2);$$

moreover by (2.25) we deduce that

$$(2.26)\qquad W_0^{1,p}(\Omega_i, \lambda^{(i)}) = W_0^{1,p}(\Omega_i, \lambda) \quad (i = 1, 2).$$

Now let us set $u_h = (A_h^{(1)})^{-1} f$, $v_h = (A_h^{(2)})^{-1} g$, $u = (B^{(1)})^{-1} f$ and $v = (B^{(2)})^{-1} g$ with $f \in L^n(\Omega_1)$ and $g \in L^n(\Omega_2)$, then by (S_2) it follows that

$$(2.27)\qquad \int_{\Omega_1} (a_h(x, Du_h) - a_h(x, Dv_h) | Du_h - Dv_h) \varphi \, dx \geq 0 \quad \forall \varphi \in \mathcal{D}(\Omega_1),\ \varphi \geq 0.$$

Since $W_0^{1,p}(\Omega_1, \lambda_h) \subseteq W_0^{1,p}(\Omega_2, \lambda_h)$, by Theorem 1.2, (2.20) and (2.23), passing to the limit in (2.27), we have

$$(2.28)\qquad \int_{\Omega_1} (b_1(x, Du) - b_2(x, Dv) | Du - Dv) \varphi \, dx \geq 0 \quad \forall \varphi \in \mathcal{D}(\Omega_1),\ \varphi \geq 0.$$

for every u in $(B^{(1)})^{-1}(L^n(\Omega_1))$ and v in $(B^{(2)})^{-1}(L^n(\Omega_2))$.

Then, by $(2.21) \div (2.26)$ and by the density of $(B^{(i)})^{-1}(L^n(\Omega_i))$ in $W_0^{1,p}(\Omega_i, \lambda)$, it follows that

$$(2.29)\qquad (b_1(x, Du) - b_2(x, Dv) | Du - Dv) \geq 0 \quad \text{a.e. in } \Omega_1,$$

for every u in $W_0^{1,p}(\Omega_1, \lambda) (\subseteq W_0^{1,p}(\Omega_2, \lambda))$ and v in $W_0^{1,p}(\Omega_2, \lambda)$.

For every $t > 0$, u, v in $W_0^{1,p}(\Omega_1, \lambda)$ let us set $w := (1/t)(u - v)$, then by (2.29) we have $(b_1(x, Dv + tDw) - b_2(x, Dv) | Dw) \geq 0$ a.e. in Ω_1 and, as $t \to 0^+$, that

$$(b_1(x, Dv) - b_2(x, Dv) | Dw) \geq 0 \quad \text{a.e. in } \forall v, w \in W_0^{1,p}(\Omega_1, \lambda_1). \tag{2.30}$$

For every fixed bounded open set $\omega \subset\subset \Omega_1$, let Φ in $C_0^1(\Omega_1)$ be such that $\Phi \equiv 1$ in ω and let $v(x) := (\xi | x)\Phi(x)$, $w(x) := (\eta | x)\Phi(x)$ with ξ, η in $\mathbb{R}^n$. By (2.30) it follows that

$$(b_1(x, \xi) - b_2(x, \xi) | \eta) \geq 0 \quad \text{a.e. in } \omega, \text{ for every } \xi \text{ and } \eta \text{ in } \mathbb{R}^n$$

and hence the thesis. ∎

3. - A G-compactness result.

In this section we want to prove that from every sequence $(a_h)_h$ in $\mathcal{M}(L, K)$ it can be selected a subsequence $(a_{h_r})_r$ that G-converges to a function a in $\mathcal{M}(L', K)$ for some $L' \geq 1$.

Let Ω be a bounded open set of $\mathbb{R}^n$ with Lipschitz boundary, let p, α, β, L and K be positive constants verifying (2.1) and let, for every $h \in \mathbb{R}$, a_h be in $\mathcal{M}_\Omega(p, \alpha, \beta, L, K)$ verifying $(S_1) \div (S_3)$ with functions λ_h in $A_p(K)$ and m_h in $L^1_{loc}(\Omega)$.

Let us assume that

(3.1) $(\lambda_h)_h$ and $(\lambda_h^{1-p'})_h$ are bounded in $L^1(Q_0)$ for some cube $Q_0 \supset \bar{\Omega}$;

$$\sup_h \left(\int_\Omega |a_h(x, 0)|^{p'} \lambda_h^{1-p'} dx \right) < +\infty; \tag{3.2}$$

(3.3) $m_h \to m$ in $L^1(\Omega)$-weak, for some function m in $L^1(\Omega)$.

Let us preliminarly observe that by (3.1) and $(1.8) \div (1.10)$ there exist $c = c(n)$ and two weights $\tilde{\lambda}$ and λ in $A_p(cK)$ such that, up to subsequences,

$$\lambda_h \to \tilde{\lambda} \quad \text{and} \quad \lambda_h^{1-p'} \to \lambda^{1-p'} \qquad \text{in } L^1(Q_0)\text{-weak}; \tag{3.4}$$

$$\lambda(x) \leq \tilde{\lambda}(x) \leq K\lambda(x), \qquad \text{for a.e. } x \in Q_0. \tag{3.5}$$

Lemma 3.1. - *Let* $A_h = -\operatorname{div}(a_h(x, D\cdot)) : W_0^{1,p}(\Omega, \lambda_h) \to W^{-1,p'}(\Omega, \lambda_h)$, *then there exist a subsequence* $(A_{\sigma_h})_h$ *of* $(A_h)_h$ *and a continuous and invertible operator* $A: W_0^{1,p}(\Omega, \lambda) \to W^{-1,p'}(\Omega, \lambda)$ *such that*

$$A_{\sigma_h}^{-1} f \to A^{-1} f \quad \textit{in } W_0^{1,1}(\Omega)\text{-weak}, \quad \textit{for every } f \in L^n(\Omega). \tag{3.6}$$

Proof. - By (2.18) it follows that there exists a positive constant c_1, independent on h, such that

$$\|A_h^{-1}f\|^p_{W_0^{1,p}(\Omega,\lambda_h)} \le c_1\left(\int_\Omega (m_h + |a_h(x,0)|^{p'}\lambda_h^{1-p'})\,dx + \|f\|^{p'}_{W^{-1,p'}(\Omega,\lambda_h)}\right) \tag{3.7}$$

for every f in $L^n(\Omega)$ for every h.

By (1.2), (1.4), (3.4), (3.5), (3.7) we deduce that $(u_h)_h$, where $u_h := A_h^{-1}f$, is weakly compact in $W_0^{1,1}(\Omega)$, therefore, given f in $L^n(\Omega)$, there exists $u(f)$ in $W_0^{1,1}(\Omega)$ such that, up to subsequences, $(u_h)_h$ converges to $u(f)$ in $W_0^{1,1}(\Omega)$-weak.

By (3.4), (3.5) and Lemma 1.4 (i), it follows that $u(f) \in W_0^{1,p}(\Omega,\lambda)$; moreover, by Lemma 1.4 (ii), passing to the limit in (3.7), there exists a positive constant c_2 such that

$$\|u(f)\|_{W_0^{1,p}(\Omega,\lambda)} \le c_2(1 + \|f\|^{1/(p-1)}_{W^{-1,p'}(\Omega,\lambda)}) \quad \text{for every } f \in L^n(\Omega). \tag{3.8}$$

Let X be a countable and dense subset of $L^n(\Omega)$; then, by means of a diagonal process, the existence of an increasing sequence $(\sigma_h)_h$ can be deduced such that, for every f in X, $(A_{\sigma_h}^{-1}f)_h$ converges in $W_0^{1,1}(\Omega)$-weak to some function in $W_0^{1,p}(\Omega,\lambda)$. Let us define the operator

$$B: X \to W_0^{1,p}(\Omega,\lambda), \qquad Bf := \lim_h A_{\sigma_h}^{-1}f \qquad (\text{in } W_0^{1,1}(\Omega)\text{-weak}),$$

we want to prove that there exists an operator (that for simplicity we still denote by B) B: $W^{-1,p'}(\Omega,\lambda) \to W_0^{1,p}(\Omega,\lambda)$ such that:

$$\|Bf - Bg\|_{W_0^{1,p}(\Omega,\lambda)} \le \tag{3.9}$$

$$\le c_3(1 + \|f\|_{W^{-1,p'}(\Omega,\lambda)} + \|g\|_{W^{-1,p'}(\Omega,\lambda)})^{(\beta-p)/((p-1)(\beta-1))}\|f-g\|^{1/(\beta-1)}_{W^{-1,p'}(\Omega,\lambda)}$$

for every f and g in $W^{-1,p'}(\Omega,\lambda)$ and some constant c_3;

$$Bf = \lim_h A_{\sigma_h}^{-1}f \quad (\text{in } W_0^{1,1}(\Omega)\text{-weak}), \text{ for every } f \text{ in } L^n(\Omega); \tag{3.10}$$

$$B \text{ is invertible}. \tag{3.11}$$

Let f and g be in X, we clearly have that

$$\langle f-g, Bf-Bg\rangle = \lim_h \langle f-g, A_{\sigma_h}^{-1}f - A_{\sigma_h}^{-1}g\rangle. \tag{3.12}$$

On the other side let us observe that, by (2.18), (2.19), (3.2) and (3.3), it follows that there exists a positive constant c_4, independent on h, such that

$$\|A_{\sigma_h}^{-1}f - A_{\sigma_h}^{-1}g\|^\beta_{W_0^{1,p}(\Omega,\lambda)} \le$$

$$\le c_4(1 + \|f\|_{W^{-1,p'}(\Omega,\lambda_h)} + \|g\|_{W^{-1,p'}(\Omega,\lambda_h)})^{(\beta-p)/(p-1)}\langle f-g, A_{\sigma_h}^{-1}f - A_{\sigma_h}^{-1}g\rangle$$

for every f and g in X, for every h in $\mathbb{N}$.

By (3.4), (3.5), (1.5) and Lemma 1.4 (ii), taking to the limit in the previous inequality we get that there exists a positive constant c_5 such that

$$(3.13)\quad \|Bf-Bg\|_{W_0^{1,p}(\Omega,\lambda)} \leqslant$$
$$\leqslant c_5^{1/(\beta-1)}(1+\|f\|_{W^{-1,p'}(\Omega,\lambda)}+\|g\|_{W^{-1,p'}(\Omega,\lambda)})^{(\beta-p)/((p-1)(\beta-1))}\|f-g\|^{1/(\beta-1)}_{W^{-1,p'}(\Omega,\lambda)},$$

for every f and g in X.

Since X is dense also in $W^{-1,p'}(\Omega,\lambda)$, by (3.13) it follows that B can be extended to the whole $W^{-1,p'}(\Omega,\lambda)$ and that (3.13) still holds on the whole $W^{-1,p'}(\Omega,\lambda)$, hence (3.9) follows.

Let us now prove (3.10).

Let f and g be in $L^n(\Omega)$, by (1.4) and (3.12) it follows that there exists a positive constant c_6, independent on h, such that

$$(3.14)\quad \|A_{\sigma_h}^{-1}f-A_{\sigma_h}^{-1}g\|_{W_0^{1,p}(\Omega,\lambda_{\sigma_h})}\leqslant c_6\left(\int_\Omega \lambda_{\sigma_h}^{1-p'}dx\right)^{1/(p'(\beta-1))}\cdot$$
$$\cdot\left[1+(\|f\|_{L^n(\Omega)}+\|g\|_{L^n(\Omega)})\left(\int_\Omega \lambda_{\sigma_h}^{1-p'}dx\right)^{1/p'}\right]^{(\beta-p)/((p-1)(\beta-1))}\|f-g\|^{1/(\beta-1)}_{L^n(\Omega)},$$

for every f and g in $L^n(\Omega)$, for every h.

Moreover let us observe that, if $f\in L^n(\Omega)$ and $g\in X$, we can write $Bf-A_{\sigma_h}^{-1}f=$ $=(Bf-Bg)+(Bg-A_{\sigma_h}^{-1}g)+(A_{\sigma_h}^{-1}g-A_{\sigma_h}^{-1}f)$, then by (3.1), (3.8), (3.14), being X dense in $L^n(\Omega)$, (3.9) follows at once.

Let us now prove (3.11).

Let f and g be in $L^n(\Omega)$ and set $u_h=A_{\sigma_h}^{-1}f$, $v_h=A_{\sigma_h}^{-1}g$. By (2.16) and (2.17) it follows that there exists a positive constant c_7, independent on h, such that if $m_h^*(x):=$ $=m(x)+|a_h(0,x)|^{p'}\lambda_h^{1-p'}(x)$, $\gamma=\alpha/(p-\alpha)$, then

$$(3.15)\quad \|f-g\|_{W^{-1,p'}(\Omega,\lambda_h)}=\|A_{\sigma_h}u_h-A_{\sigma_h}v_h\|_{W^{-1,p'}(\Omega,\lambda_h)}\leqslant$$
$$\leqslant c_7\left(\int_\Omega[m_h^*+\lambda_h(|Du_h|^p+|Dv_h|^p)]\,dx\right)^{(p-1-\gamma)/p}\left(\int_\Omega |Du_h-Dv_h|^p\lambda_h\,dx\right)^{\gamma/p}$$

for every h.

On the other side, by (2.19), (S_2^*) of Proposition 2.3 and by (3.15), we get that there exists a positive constant c_8, independent on h, such that

$$(3.16)\quad \|f-g\|^{p/\gamma}_{W^{-1,p'}(\Omega,\lambda_h)}\leqslant$$
$$\leqslant c_8(1+\|f\|_{W^{-1,p'}(\Omega,\lambda_h)}+\|g\|_{W^{-1,p'}(\Omega,\lambda_h)})^{\delta p'}\langle f-g,A_{\sigma_h}^{-1}f-A_{\sigma_h}^{-1}g\rangle^{p/\beta},$$

for every f and g in $L^n(\Omega)$, for every h,

with $\delta=(p-1-\gamma)/\gamma+(\beta-p)/\beta$.

By (3.4), (3.5), (3.9) and by Lemma 1.4, taking the limit in (3.16), we get that for some positive constant c_9, independent on h,

$$\|f-g\|_{W^{-1,p'}(\Omega,\lambda)} \leq c_9(1+\|f\|_{W^{-1,p'}(\Omega,\lambda)}+\|g\|_{W^{-1,p'}(\Omega,\lambda)})^{\vartheta p'}\langle f-g, Bf-Bg\rangle^{p/\beta}, \tag{3.17}$$

for every f and g in $L^n(\Omega)$.

Moreover, by the density of $L^n(\Omega)$ in $W^{-1,p'}(\Omega,\lambda)$ and by the continuity of B, it follows that (3.17) holds on the whole $W^{-1,p'}(\Omega,\lambda)$.

Therefore, since B: $W^{-1,p'}(\Omega,\lambda)\to W_0^{1,p}(\Omega,\lambda)$ is continuous, monotone and coercive, (3.11) soon follows, for instance, from Corollary 1.8, Ch. III in [10].

Finally if we take $A:=B^{-1}$: $W_0^{1,p}(\Omega,\lambda)\to W_0^{1,p'}(\Omega,\lambda)$ the thesis follows. ■

Lemma 3.2. – *Let $(A_h)_h$ and $(\sigma_h)_h$ be as in Lemma 3.1. Then there exist a subsequence $(\delta_h)_h$ of $(\sigma_h)_h$ and a continuous operator M: $W^{-1,p'}(\Omega,\lambda)\to(L^{p'}(\Omega,\lambda^{1-p'}))^n$ such that*

$$a_{\delta_h}(x, DA_{\delta_h}^{-1}f)\to Mf \quad \textit{in } (L^1(\Omega))^n\textit{-weak}, \ \textit{for every } f \textit{ in } L^n(\Omega). \tag{3.18}$$

Proof. – Let us set for simplicity $\lambda_h\equiv\lambda_{\sigma_h}$, $m_h\equiv m_{\sigma_h}$, $a_h\equiv a_{\sigma_h}$, $A_h\equiv A_{\sigma_h}$ and define the operators M_h: $W^{-1,p'}(\Omega,\lambda_h)\to(L^{p'}(\Omega,\lambda_h^{1-p'}))^n$, $M_hf:=a_h(x, DA_h^{-1}f)$.

Then, by (2.4), (1.4), (3.2), (3.3) and (3.7), it follows that, given $f\in L^n(\Omega)$, the sequence $(M_hf)_h$ is weakly compact in $(L^1(\Omega))^n$. Therefore, if X is a countable and dense subset of $L^n(\Omega)$, by means of a diagonal process, we can assume that, for every $f\in X$, the sequence $(M_hf)_h$ converges, up to subsequences, in $(L^1(\Omega))^n$-weak to a function Mf.

Let us prove that

$$Mf\in(L^{p'}(\Omega,\lambda^{1-p'}))^n \qquad \text{for every } f\in X. \tag{3.19}$$

By Hölder inequality and (3.4) we get

$$\int_\Omega |Mf|\,|\varphi|\,dx \leq \liminf_h \left(\int_\Omega |M_hf|^{p'}\lambda_h^{1-p'}\,dx\right)^{1/p'}\left(\int_\Omega |\varphi|^p\lambda\,dx\right)^{1/p} \qquad \forall\varphi\in C_0^0(\Omega), \tag{3.20}$$

hence, by (3.5) and (3.20), (3.19) follows at once.

We now want to prove that M can be extended to a continuous operator on the whole $W^{-1,p'}(\Omega,\lambda)$ and that

$$Mf=\lim_h a_h(x, DA_h^{-1}f) \quad (\text{in } (L^1(\Omega))^n\text{-weak}), \ \text{for every } f\in L^n(\Omega). \tag{3.21}$$

By (2.16)÷(2,19), (3.2), (3.3), (3.14) and by (3.4), (3.5) and Lemma 1.4 it follows that there exists a positive constant c_3, independent on h, such that, if

$\delta = (p(\beta - \gamma - 1))/((p-1)(\beta - 1))$ and $\gamma = \alpha/(p - \alpha)$,

$$\liminf_h \left(\int_\Omega |M_h f - M_h g|^{p'} \lambda_h^{1-p'} dx \right)^{1/p'} \leqslant \tag{3.22}$$

$$\leqslant c_3 (1 + \|f\|_{W^{-1,p'}(\Omega,\lambda)} + \|g\|_{W^{-1,p'}(\Omega,\lambda)})^\delta \|f - g\|^{\gamma p'/(\beta-1)}_{W^{-1,p'}(\Omega,\lambda)},$$

for every f and g in X.

On the other hand, by using arguments similar to the ones employed in the proof of (3.19), it can be proved that

$$\|Mf - Mg\|_{(L^{p'}(\Omega,\lambda^{1-p'}))^n} \leqslant \liminf_h \left(\int_\Omega |M_h f - M_h g|^{p'} \lambda_h^{1-p'} dx \right)^{1/p'}; \tag{3.23}$$

hence, by (3.22) and (3.23), it follows that

$$\|Mf - Mg\|_{(L^{p'}(\Omega,\lambda^{1-p'}))^n} \leqslant c_3 (1 + \|f\|_{W^{-1,p'}(\Omega,\lambda)} + \|g\|_{W^{-1,p'}(\Omega,\lambda)})^\delta \|f - g\|^{\gamma p'/(\beta-1)}_{W^{-1,p'}(\Omega,\lambda)}, \tag{3.24}$$

for every f and g in X.

By the density of X in $W^{-1,p'}(\Omega, \lambda)$ and (3.24) M can be extended to an operator, still denoted by M, defined on the whole $W^{-1,p'}(\Omega, \lambda)$, moreover (3.24) holds on the whole $W^{-1,p'}(\Omega, \lambda)$.

Let us now prove (3.21).

Let f be in $L^n(\Omega)$ and g in X; since we can write

$$Mf - a_h(x, DA_h^{-1} f) =$$
$$= (Mf - Mg) + (Mg - a_h(x, DA_h^{-1} g)) + (a_h(x, DA_h^{-1} g) - a_h(x, DA_h^{-1} f))$$

by (3.15), (3.24) and by the density of X in $L^n(\Omega)$, (3.21) follows at once and so the thesis follows. ∎

Now we can prove a partial G-compactness result.

PROPOSITION 3.3. – *Let Ω be a bounded open set of $\mathbb{R}^n$ with Lipschitz boundary and let a_h be in $\mathcal{M}_\Omega(p, \alpha, \beta, L, K)$ $(h = 1, 2, \ldots)$ verifying* $(S_1) \div (S_3)$ *with functions λ_h in $A_p(K)$ and m_h in $L^1(\Omega)$.*

Let us assume that (3.1) ÷ (3.3) *hold, then there exist a subsequence $(a_{h_r})_r$ of (a_h) and a function a in $\mathcal{M}_\Omega(p, \alpha, \beta, (cK)^{1/p} L, cK)$ (where $c = c(n)$ is a positive constant depending only on n) such that*

$$a_{h_r} \xrightarrow{G} a \quad \text{in } \Omega.$$

PROOF. – Let us first observe that by (1.6), (1.7), (3.1) we can assume that (3.4) and (3.5) hold.

Let A_h be as in Lemma 3.1, then by Lemma 3.1 and Lemma 3.2 we can assume that there exist two continuous operators $A: W_0^{1,p}(\Omega, \lambda) \to W^{-1,p'}(\Omega, \lambda)$ and $M: W^{-1,p'}(\Omega, \lambda) \to (L^{p'}(\Omega, \lambda^{1-p'}))^n$ with A invertible such that, up to subsequences, (3.6) and (3.18) hold with $h \equiv \sigma_h \equiv \delta_h$.

Moreover let us observe that by (3.9) it follows that

$$Y := A^{-1}(L^n(\Omega)) \quad \text{is dense in } W_0^{1,p}(\Omega, \lambda). \tag{3.25}$$

Let us define the operator $\tilde{M} := M \circ A$ and, for given u and v in Y, let $u_h = A_h^{-1}Au$, $v_h = A_h^{-1}Av$ be in $W_0^{1,p}(\Omega, \lambda_h)$.

If H_h is the function in (2.2) with $m \equiv m_h$, $a \equiv a_h$, then by $(S_1) \div (S_3)$ we get

$$|Du_h - Dv_h| \leqslant \tag{3.26}$$

$$\leqslant \lambda_h^{-1/p} H_h^{(\beta-p)/p\beta}(x, Du_h, Dv_h)(a_h(x, Du_h) - a(x, Dv_h)|Du_h - Dv_h)^{1/\beta},$$

$$|a_h(x, Du_h) - a(x, Dv_h)| \leqslant \tag{3.27}$$

$$\leqslant L\lambda_h^{1/p} H_h^{(p-1-\alpha)/p}(x, Du_h, Dv_h)(a_h(x, Du_h) - a(x, Dv_h)|Du_h - Dv_h)^{\alpha/p},$$

a.e. in Ω, for every h.

Set $\delta = (p-1)/p$, $\rho = (\beta - p)/p\beta$, $\vartheta = 1/\beta$ and $t_h \equiv Du_h - Dv_h$, $s_h = \lambda_h^{1-p'}$, $z_h \equiv H_h(x, Du_h, Dv_h)$, $w_h \equiv (a_h(x, Du_h) - a_h(x, Dv_h)|Du_h - Dv_h)$, then by (3.4), (3.6), (3.19) and Theorem 1.2, the assumptions of Lemma 1.5 are fulfilled hence, taking the limit in (3.26) we get

$$|Du - Dv| \leqslant \tag{3.28}$$

$$\leqslant \lambda^{-1/p}[m + (\tilde{M}u|Du) + (\tilde{M}v|Dv)]^{(\beta-p)/p\beta}(\tilde{M}u - \tilde{M}v|Du - Dv)^{1/\beta} \quad \text{a.e. in } \Omega,$$

for every u and v in Y.

Analogously, by applying again Lemma 1.5 with $\delta = 1/p$, $\rho = (p-1-\alpha)/p$, $\vartheta = \alpha/p$, $t_h \equiv a_h(x, Du_h) - a_h(x, Dv_h)$, $s_h \equiv \lambda_h$, $w_h \equiv (a_h(x, Du_h) - a_h(x, Dv_h)|Du_h - Dv_h)$ and taking the limit as $h \to +\infty$ in (3.27), we get

$$|\tilde{M}u - \tilde{M}v| \leqslant \tag{3.29}$$

$$\leqslant L\tilde{\lambda}^{1/p}\Big[m + (\tilde{M}u|Du) + (\tilde{M}v|Dv)\Big]^{(p-1-\alpha)/p}(\tilde{M}u - \tilde{M}v|Du - Dv)^{\alpha/p} \quad \text{a.e. in } \Omega,$$

for every u and v in Y.

By (3.25) we get also that (3.28) and (3.29) hold on the whole $W_0^{1,p}(\Omega, \lambda)$; moreover from (3.5), (3.28) and (3.29) it follows that

$$|\tilde{M}u - \tilde{M}v| \leqslant \tag{3.30}$$

$$\leqslant L^{p/(p-\alpha)}\tilde{\lambda}^{p-\alpha}[m + (\tilde{M}u|Du) + (\tilde{M}v|Dv)]^{(p-1-\alpha)/(p-\alpha)}|Du - Dv|^{\alpha/(p-\alpha)} \quad \text{a.e. in } \Omega,$$

for every u and v in $W_0^{1,p}(\Omega, \lambda)$.

We now construct a function a for which $A = -\operatorname{div}(a(x, D\cdot))$.

Let $(\omega_j)_j$ be an increasing sequence of open sets of $\mathbb{R}^n$ such that $\bar{\omega}_j \subset \Omega$ for every j and $\bigcup_{j=1}^{\infty} \omega_j = \Omega$ and let $(\Phi_j)_j$ be a sequence of functions in $C_0^1(\Omega)$ such that $\Phi_j \equiv 1$ in ω_j for every j.

For every $\xi \in \mathbb{R}^n$ let us define the functions $\varphi_j^{(\xi)}$ by $\varphi_j^{(\xi)}(x) := \Phi_j(x)(\xi|x)$ $(x \in \mathbb{R}^n)$ and let a be the function defined by $a(x, \xi) := (\tilde{M}\varphi_j^{(\xi)})(x)$ if $x \in \omega_j$. By (3.30) it follows that a is well defined since $(\tilde{M}\varphi_j^{(\xi)}) \equiv (\tilde{M}\varphi_i^{(\xi)})$ a.e. in ω_j, if $i > j$. Moreover, by (3.5), (3.28) ÷ (3.30) it follows that $a \in \mathcal{M}_\Omega((cK)^{1/p}L, cK)$.

In particular, by (3.30), $a(x, \cdot)$ turns out to be a continuous function on $\mathbb{R}^n$, for a.e. x in Ω.

Then, in order to get the thesis it is sufficient to prove that

$$\tilde{M}u = a(x, Du) \quad \text{a.e. in } \Omega, \text{ for every } u \in W^{1,p}(\Omega, \lambda);$$

but this can be proved by the Minty trick (see, for instance, proof of the Theorem 1.1 in [8]). ∎

REMARK 3.4. – If we replace condition (3.1) with the following one: for every cube Q of $\mathbb{R}^n$ $(\lambda_h)_h$ and $(\lambda_h^{1-p'})_h$ are bounded in $L^1(Q)$, then, by Remark 1.3, it follows that $\lambda \in A_p(K)$ and that (3.5) holds on the whole $\mathbb{R}^n$. Therefore by (3.29) we get that $a \in \mathcal{M}_\Omega(K^{1/p}L, K)$.

Now we can prove the main result of this paper.

THEOREM 3.5. – *Let p, α, β, L and K be constants verifying* (2.1). *Let a_h $(h = 1, 2, \ldots)$ be functions in $\mathcal{M}(p, \alpha, \beta, L, K)$ and assume that each a_h verifies* (S_1) ÷ (S_3) *with functions λ_h in m_h and $L^1_{loc}(\mathbb{R}^n)$.*

Moreover let us assume that:

(i) *for every cube Q of $\mathbb{R}^n$ the sequences $(\lambda_h)_h$ and $(\lambda_h^{1-p'})_h$ are bounded in $L^1(Q)$;*

(ii) *for every cube Q of $\mathbb{R}^n$ there exists a positive constant $c = c(Q)$ (depending only on Q) such that*

$$\int_Q |a_h(x, 0)|^{p'} \lambda_h^{1-p'} dx \leq c(Q) \quad \textit{for every } h;$$

(iii) *there exists a function m in $L^1_{loc}(\mathbb{R}^n)$ such that $m_h \to m$ in $L^1(Q)$-weak for every cube Q of $\mathbb{R}^n$.*

Then there exist a subsequence $(a_{h_r})_r$ of $(a_h)_h$ and a function a in $\mathcal{M}(p, \alpha, \beta, LK^{1/p}, K)$ such that

$$a_{h_r} \xrightarrow{G} a \quad \textit{in } \Omega,$$

for every bounded open set Ω of $\mathbb{R}^n$ with Lipschitz boundary.

PROOF. – Let us begin to observe that, by Remark 1.3, we can assume that there exist two weights $\tilde{\lambda}$ and λ in $A_p(K)$ for which (3.4) and (3.5) hold for every cube Q.

For every $j \in \mathbb{N}$, let $Q_j = (-j, j)^n$, then by Proposition 3.3, Remark 3.4, it follows that there exist a subsequence $(a_h^{(1)})_h$ of $(a_h)_h$ and a function $a^{(1)}$ in $\mathcal{M}_{Q_1}(LK^{1/p}, K)$ such that $a_h^{(1)} \xrightarrow{G} a^{(1)}$ in Q_1.

Analogously, by applying again Proposition 3.3 to the sequence $(a_h^{(1)})_h$, we get the existence of a subsequence $(a_h^{(2)})_h$ of $(a_h^{(1)})_h$ and of a function $a^{(2)}$ in $\mathcal{M}_{Q_2}(LK^{1/p}, K)$ such that

$$a_h^{(2)} \xrightarrow{G} a^{(2)} \quad \text{in } Q_2 . \tag{3.31}$$

On the other side we have also that

$$a_h^{(2)} \xrightarrow{G} a^{(1)} \quad \text{in } Q_1 , \tag{3.32}$$

then, by Proposition 2.9, it follows that $a^{(1)}(x, \xi) = a^{(2)}(x, \xi)$ for a.e. $x \in Q_1$, for every $\xi \in \mathbb{R}^n$.

By repeating the above construction for every $j \in \mathbb{N}$, we get a sequence $(a_h^{(j)})_h$ and a function $a^{(j)}$ in $\mathcal{M}_{Q_j}(LK^{1/p}, K)$ such that

$$a_h^{(j)} \xrightarrow{G} a^{(j)} \quad \text{in } Q_j , \tag{3.33}$$

$$a^{(j)}(x, \xi) = a^{(i)}(x, \xi) \text{ for a.e. } x \in Q_j, \textit{ every } \xi \in \mathbb{R}^n, \ j \in \mathbb{N} \text{ and every } 1 \leq i \leq j . \tag{3.34}$$

Therefore if we define $a\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ as

$$a(x, \xi) := a^{(j)}(x, \xi) \quad \text{if } x \in Q_j, \ \xi \in \mathbb{R}^n ;$$

by (3.34) it follows that a is well defined and that $a \in \mathcal{M}(LK^{1/p}, K)$.

Now let us consider the diagonal sequence $\tilde{a}_h \equiv a_h^{(h)}$; clearly, it follows that

$$\tilde{a}_h \xrightarrow{G} a \quad \text{in } Q_j, \text{ for every } j . \tag{3.35}$$

On the other hand, if Ω is a regular bounded open set of $\mathbb{R}^n$, by Proposition 3.3 there exist a subsequence $(\tilde{a}_{h_r})_r$ of $(\tilde{a}_h)_h$ and a function $a^{(\Omega)} \in \mathcal{M}_\Omega(L(cK)^{1/p}, cK)$ such that

$$\tilde{a}_{h_r} \xrightarrow{G} a^{(\Omega)} \quad \text{in } \Omega . \tag{3.36}$$

Let $j_0 \in \mathbb{N}$ be such that $\bar{\Omega} \subset Q_{j_0}$, by (3.42), (3.43) and by Proposition 2.9 we get that $a(x, \xi) = a^{(\Omega)}(x, \xi)$ for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^n$, hence the thesis follows. ■

As a particular case, by Theorem 3.5, we deduce the following corollary.

COROLLARY 3.6. – *Let* $p \geq 2$, $K \geq 1$ *and let* $a_h: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ $(h = 1, 2, \dots)$ *be Carathéodory functions verifying* (0.2) *with* $L \geq 1$ *and* μ_h *in* $A_p(K)$ *for every* h *such that, for every cube* Q, $(\mu_h)_h$ *and* $(\mu_h^{1-p'})_h$ *are bounded in* $L^1(Q)$.

Then there exist a subsequence $(a_{h_r})_r$ *of* $(a_h)_h$, *a weight* μ_∞ *in* $A_p(K)$ *and a Carathéodory function* $a_\infty: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *verifying* (0.3) *with* μ_∞ *and with a suitable positive constant* L' *such that, for every regular bounded open set* Ω *and for every* f *in* $L^\infty(\Omega)$, *the solutions* u_r *of the problems*

$$(\mathrm{P_r}) \qquad -\operatorname{div}(a_h(x, Dv)) = f \quad in\ \Omega, \qquad v \in W_0^{1,p}(\Omega, \mu_{h_r}) \quad (r = 1, 2, \dots)$$

converge in $W_0^{1,1}(\Omega)$-weak *to the solution* u_∞ *of the problem*

$$-\operatorname{div}(a_\infty(x, Dv)) = f \quad in\ \Omega, \ v \in W_0^{1,p}(\Omega, \mu_\infty).$$

Moreover the weak convergence in $(L^1(\Omega))^n$ *of the momenta* $(a_{h_r}(x, Du_r))_r$ *to the momentum* $a_\infty(x, Du_\infty)$ *holds.*

PROOF. – By Proposition 2.3 (ii) it follows that a_h verifies $(S_1) \div (S_3)$ with function

$$(3.37) \quad \lambda_h(x) = c_1 \mu_h(x), \quad m_h(x) = c_2 \mu_h(x), \quad \alpha = 1, \quad \beta = p \quad \text{and} \quad \tilde{L} \geq 1,$$

being c_i $(i = 1, 2)$ and $\tilde{L}$ suitable positive constants independent on h; hence $(a_h)_h$ is contained in $\mathcal{M}(p, 1, p, \tilde{L}, K)$.

By (3.37), (1.6) and (1.7) we can apply Theorem 3.5 and get the existence of a subsequence $(a_{h_r})_r$ of $(a_h)_h$ and of a function a_∞ in $\mathcal{M}(p, 1, p, \tilde{L}K^{1/p}, K)$ verifying $(S_1) \div (S_3)$ with functions λ in $A_p(K)$ and m in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$(3.38) \qquad a_{h_r} \xrightarrow{G} a_\infty \quad \text{in } \Omega, \qquad \text{for every regular bounded open set } \Omega.$$

Moreover, by Proposition 2.3 (i), it follows that a_∞ verifies (S_1^*) and (S_2^*) with $\lambda_*(x) = c_4 \lambda(x)$, $m_*(x) = c_5 \lambda(x)$, $\gamma = 1/(p-1)$, $\beta = p$ for suitable positive constants L_i $(i = 1, 2)$.

For every $r \in \mathbb{N}$ and every regular bounded open set Ω, let w_r be the (unique) solutions of the problems in $(\mathrm{P_r})$ relative to $f \equiv 0$; then, since by (0.2) $a_{h_r}(x, 0) \equiv 0$, it turns out that $w_r = 0$ a.e. in Ω. By (3.38) it follows that

$$0 \equiv a_{h_r}(x, Dw_r) \to a(x, 0) \quad \text{in } (L^1(\Omega))^n\text{-weak},$$

and, being Ω an arbitrary regular bounded open set, that $a(x, 0) = 0$ a.e. in $\mathbb{R}^n$.

Finally, if we take $\mu_\infty(x) := (1/L_2)\lambda(x)$, the thesis soon follows. ∎

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