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# Accuracy of semi-analytical sensitivities and its improvement by the "natural method"

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**Abstract** The semi-analytical method is a very convenient tool for the computation of sensitivities in shape design. It was, however, shown in the past that this method may have serious accuracy problems, e.g. in the shape design of beams. In this paper the natural method is employed to locate the decisive defect in the incremental stiffness and to develop an alternative procedure that provides nondefective incremental stiffnesses. This approach is then extended to the more common Cartesian element description. Examples demonstrate that the error of the sensitivities in critical cases can be almost totally removed, if we employ the nondefective incremental stiffness formulation in the sensitivity computation.

## 1 Introduction

One of the main drawbacks in the semi-analytical method for the computation of shape sensitivities is the fact that in some applications large errors occur, which may even increase with a finer discretization. This was impressively demonstrated by Barthelemy and Haftka (1988). A very ingenious idea for the error elimination in these critical applications was put forward by Olhoff and Rasmussen (1990). They developed and applied correction factors for the results derived via a simple forward difference scheme and thereby obtained very accurate sensitivities. Cheng *et al.* (1990) tried to improve the accuracy by employing second order information. Most inspiring to the author was a paper by Cheng and Olhoff (1991), who used rigid body displacements to detect and correct sensitivity information. It was, in fact, this last paper and its close relation to the natural FEM-approach which has brought about the idea of employing the natural method in this field.

## 2 The natural FEM approach

Whereas the classical Cartesian finite element formulation is based on displacement modes $\boldsymbol{\omega}$, which are related to Cartesian element nodal displacements $\boldsymbol{\rho}$ so that local displacements $\mathbf{u}$ are given by

$$\mathbf{u} = \boldsymbol{\omega}\boldsymbol{\rho}\,, \tag{1}$$

the natural formulation makes use of rigid body modes $\boldsymbol{\omega}_0$ and straining modes $\boldsymbol{\omega}_N$, which are related to the generalized displacements $\boldsymbol{\rho}_0$ and $\boldsymbol{\rho}_N$

$$\mathbf{u} = [\boldsymbol{\omega}\,\boldsymbol{\omega}_N]\{\boldsymbol{\rho}_0\,\boldsymbol{\rho}_N\} = \boldsymbol{\omega}'\boldsymbol{\rho}'\,. \tag{2}$$

Of course, it is always possible to compute $\boldsymbol{\rho}'$ from $\boldsymbol{\rho}$ and vice versa

$$\boldsymbol{\rho}' = \begin{bmatrix} \boldsymbol{\rho}_0 \\ \boldsymbol{\rho}_N \end{bmatrix} = \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{a}_N \end{bmatrix} \boldsymbol{\rho} = \mathbf{a}_e\boldsymbol{\rho}\,, \tag{3}$$

$$\boldsymbol{\rho} = [\mathbf{A}_0\,\mathbf{A}_N] \begin{bmatrix} \boldsymbol{\rho}_0 \\ \boldsymbol{\rho}_N \end{bmatrix} \boldsymbol{\rho}' = \mathbf{A}_e\boldsymbol{\rho}'\,. \tag{4}$$

Obviously $\mathbf{A}_0$ contains Cartesian element nodal displacements related to rigid body modes. The traditional Cartesian element stiffness matrix $\mathbf{k}$ may be also transformed to generalized displacements

$$\mathbf{k}' = \mathbf{A}_e^t\mathbf{k}\mathbf{A}_e = \begin{bmatrix} \mathbf{A}_0^t\mathbf{k}\mathbf{A}_0 & \mathbf{A}_0^t\mathbf{k}\mathbf{A}_N \\ \mathbf{A}_N^t\mathbf{k}\mathbf{A}_0 & \mathbf{A}_N^t\mathbf{k}\mathbf{A}_N \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_N \end{bmatrix}. \tag{5}$$

Here $\mathbf{0}$ is a zero matrix and $\mathbf{k}_N$ denotes the so-called natural stiffness, which may be directly derived from the straining modes and then used to compute the Cartesian element stiffness,

$$\mathbf{k} = \mathbf{a}_e^t\mathbf{k}'\mathbf{a}_e = \mathbf{a}_N^t\mathbf{k}_N\mathbf{a}_N\,. \tag{6}$$

More information on this approach can be found in a book by Argyris and Mlejnek (1986).

## 3 Properties of analytical stiffness sensitivities

For simplicity we restrict our considerations to one design parameter $s$. A function $g$ may explicitly depend on $s$ and the nodal displacements of the assembled structure $\mathbf{r}$ which again depend implicitly on $s$. The sensitivity is then given by

$$\frac{\mathrm{d}}{\mathrm{d}_s}[g(s,\mathbf{r}(s))] = \frac{\partial g}{\partial s} + \frac{\partial g}{\partial \mathbf{r}}\frac{\partial \mathbf{r}}{\partial s}\,. \tag{7}$$

The explicit derivatives $\partial g/\partial s$ and $\partial g/\partial \mathbf{r}$ are usually trivial. The displacement derivative is obtained from

$$\frac{\partial \mathbf{r}}{\partial s} = -\mathbf{K}^{-1}\sum_{g=1}^{m}\mathbf{a}_g^t\frac{\partial \mathbf{k}_g}{\partial s}\boldsymbol{\rho}_g\,, \tag{8}$$

where $g$ denotes one of the $m$ elements in the structure, $\mathbf{K}$ is the stiffness matrix of the assembled structure and $\mathbf{a}_g$ is a Boolean matrix relating the element displacements $\boldsymbol{\rho}_g$ to the global displacements $\mathbf{r}$

$$\boldsymbol{\rho}_g = \mathbf{a}_g\mathbf{r}\,. \tag{9}$$

Finally, we need the element stiffness derivative $\partial\mathbf{k}/\partial s$ (as in Section 2, we omit for simplicity the element number $g$). In the semi-analytical approach (SA) this expression is replaced by $\Delta\mathbf{k}/\Delta s$. By (6) this derivative becomes

$$\frac{\partial \mathbf{k}}{\partial s} = \frac{\partial \mathbf{a}_N^t}{\partial s}\mathbf{k}_N\mathbf{a}_N + \mathbf{a}_N^t\frac{\partial \mathbf{k}_N}{\partial s}\mathbf{a}_N + \mathbf{a}_N^t\mathbf{k}_N\frac{\partial \mathbf{a}_N}{\partial s}\,. \tag{10}$$

From (3) and (4) we obtain readily

$$\mathbf{a}_e\mathbf{A}_e = \mathbf{I} = \begin{bmatrix} \mathbf{a}_0\mathbf{A}_0 = \mathbf{I} & \mathbf{a}_0\mathbf{A}_N = \mathbf{0} \\ \mathbf{a}_N\mathbf{A}_0 = \mathbf{0} & \mathbf{a}_N\mathbf{A}_N = \mathbf{I} \end{bmatrix} = \mathbf{I}, \tag{11}$$

where **I** denotes the unit matrix. Obviously

$$\frac{\partial \mathbf{k}}{\partial s}\mathbf{A}_0 = \mathbf{a}_N^t \mathbf{k}_N \frac{\partial \mathbf{a}_N}{\partial s}\mathbf{A}_0\,, \tag{12}$$

does not vanish, in general, unless $\partial \mathbf{a}_N/\partial s$ is zero. However if we form

$$\mathbf{A}_0^t \frac{\partial \mathbf{k}}{\partial s}\mathbf{A}_0 = \mathbf{0}\,, \tag{13}$$

then we always obtain a zero matrix. This property was also derived by Cheng and Olhoff (1991) in a different context.

## 4 Defects of semi-analytical sensitivities and a simple remedy

In the SA approach we simply replace $\partial \mathbf{k}/\partial s$ by $\Delta \mathbf{k}/\Delta s$, where

$$\Delta \mathbf{k} = \mathbf{k}(s+\Delta s) - \mathbf{k}(s) = \mathbf{k}^+ - \mathbf{k}\,. \tag{14}$$

By applying the test (13) to the incremental stiffness (14) we have

$$\mathbf{A}_0^t \Delta \mathbf{k} \mathbf{A}_0 = \mathbf{A}_0^t (\mathbf{a}_N^+)^t \mathbf{k}_N^+ \mathbf{a}_N^+ \mathbf{A}_0\,, \tag{15}$$

where the top index + again indicates the modified state. Note however that now in general

$$\mathbf{a}_N^+ \mathbf{A}_0 = \mathbf{a}_N(s+\Delta s)\mathbf{A}_0(s) \neq \mathbf{0}\,. \tag{16}$$

The traditional SA approach does not satisfy property (13). This was also stated by Cheng and Olhoff (1991) who used this result for error detection and used the rigid body forces arising from the incremental stiffness to correct the traditional SA-sensitivities.

A look at (10) reveals a simple alternative possibility to generate directly nondefective incremental stiffnesses and subsequently accurate sensitivities. Due to (10) we have as stiffness increment

$$\Delta \mathbf{k}^* = \Delta \mathbf{a}_N^t \mathbf{k}_N \mathbf{a}_N + \mathbf{a}_N^t \Delta \mathbf{k}_N \mathbf{a}_N + \mathbf{a}_N^t \mathbf{k}_N \Delta \mathbf{a}_N\,. \tag{17}$$

Obviously this new stiffness increment, which is based on the natural formulation, satisfies automatically the property (13)

$$\mathbf{A}_0^t \Delta \mathbf{k}^* \mathbf{A}_0 = \mathbf{0}\,. \tag{18}$$

This improvement also gives dramatic benefits with respect to the sensitivity error, as we will demonstrate later.

## 5 Extension to Cartesian formulation

The direct computation of the Cartesian element stiffness is based on the relation

$$\mathbf{k} = \int_{V_p} \boldsymbol{\alpha}^t \boldsymbol{\kappa} \boldsymbol{\alpha} \det \mathbf{J}\, \mathrm{d}V_p\,, \tag{19}$$

where $\boldsymbol{\alpha}$ denotes the strain displacement matrix for the computation of the strains $\boldsymbol{\epsilon}$ from the element nodal displacement $\boldsymbol{\rho}$,

$$\boldsymbol{\epsilon} = \boldsymbol{\alpha}\boldsymbol{\rho}\,, \tag{20}$$

and the matrix $\boldsymbol{\kappa}$ contains the coefficients of Hooke's law ('material stiffness'). The integration is performed over the parameter volume $V_p$, which is related to the real volume by

$$\mathrm{d}V_p = \det \mathbf{J}\, \mathrm{d}V_p\,, \tag{21}$$

where **J** is the Jacobian relating global coordinate derivatives to parameter coordinate derivatives. It is usual to perform (19) numerically

$$\mathbf{k} = \sum_{k=1}^{\mathrm{nip}} w_k [\boldsymbol{\alpha}_k^t (\boldsymbol{\kappa} \det \mathbf{J})_k \boldsymbol{\alpha}_k]\,, \tag{22}$$

where nip indicates the number of integration points, $w_k$ the associated weight for the integrand function at integration point $P_k$ in the parameter space. Note now the direct correspondence with the natural approach:

$$\mathbf{k}_N \longrightarrow \boldsymbol{\kappa} \det \mathbf{J}, \quad \mathbf{a}_N \longrightarrow \boldsymbol{\alpha}\,. \tag{23}$$

If we apply a set of rigid body element nodal displacements as $\mathbf{A}_0$, the product $\mathbf{a}_N \mathbf{A}_0$ is zero, i.e. we have no natural displacements $\boldsymbol{\rho}_N$ and subsequently no strains. This is also the case for $\boldsymbol{\alpha}\mathbf{A}_0$, since this expression gives the strains $\boldsymbol{\epsilon}$ directly. The natural stiffness matrix $\mathbf{k}_N$ is positive definite like the expression $\boldsymbol{\kappa} \det \mathbf{J}$. Therefore, we may immediately write the improved incremental formulation in Cartesian description as

$$\Delta \boldsymbol{\kappa}^{**} = \sum_{k=1}^{\mathrm{nip}} w_k [\Delta \boldsymbol{\alpha} (\boldsymbol{\kappa} \det \mathbf{J}) \boldsymbol{\alpha} + \boldsymbol{\alpha} \Delta (\boldsymbol{\kappa} \det \mathbf{J}) \boldsymbol{\alpha} +$$

$$+ \boldsymbol{\alpha} (\boldsymbol{\kappa} \det \mathbf{J}) \Delta \boldsymbol{\alpha}]_k\,. \tag{24}$$

In the next section we demonstrate that this approach gives results which are as accurate as those obtained by the natural method.

## 6 Illustrative example

As an introductory example we consider a plane straight beam in pure bending. Cartesian and natural displacements are shown in Fig. 1. From this figure we may easily deduce the matrix

$$\mathbf{a}_N = \begin{bmatrix} 0 & +1 & 0 & -1 \\ +\frac{2}{\ell} & +1 & -\frac{2}{\ell} & +1 \end{bmatrix},$$

and the associated stiffness matrix is diagonal,

$$\mathbf{k}_N = \frac{EI}{\ell} \lceil\, 1 \quad 3 \,\rfloor\,,$$

where $EI$ denotes the bending stiffness and $\ell$ the element length of the beam. Using (6) we may verify the traditional Cartesian element stiffness.

We employ this simple element in the analysis of a cantilever beam loaded by an end moment (Fig. 2) and compute the derivative of the end displacement $v_E$ with respect to the total length $L$ of the beam. This example was also used by Cheng and Olhoff (1991). For this case we have by (7)

$$\frac{\partial v_E}{\partial s} = 0\,, \quad \frac{\partial v_e}{\partial \mathbf{r}} = [\, 0 \;\; 0 \;\; \ldots \;\; 0 \;\; 0 \;\; 1 \;\; 0 \,] = \overline{\mathbf{R}}^t\,,$$

where $\overline{\mathbf{R}}$ represents the adjoint load vector. If we substitute (8) into (7), we may term

$$\overline{\mathbf{r}} = \mathbf{K}^{-1}\overline{\mathbf{R}}\,,$$

as "adjoint" structural displacements and

$$\overline{\boldsymbol{\rho}}_g = \mathbf{a}_g \overline{\mathbf{r}}\,,$$

as "adjoint" element displacements. Noting that our shape parameter $s$ is given by

$$s = L = \mathrm{m}\ell\,,$$

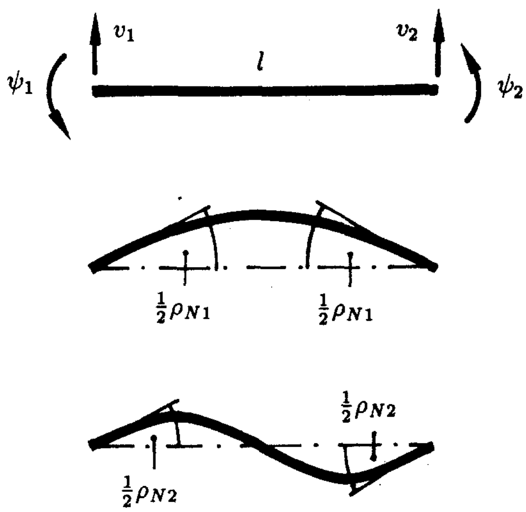


**Fig. 1.** Plane beam element in pure bending: Cartesian and natural displacements

where m is the number of equal elements, we have from (7) and (8) the considerably simplified expression

$$\frac{\partial v_e}{\partial L} = -\frac{1}{m}\sum_{g=1}^{m} \overline{\rho}_g^t \frac{\partial \mathbf{k}}{\partial \ell} \rho_g \, .$$

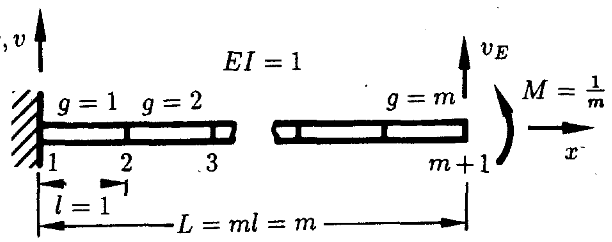


**Fig. 2.** Cantilever beam under end moment

In our application, the discretization error for the computation of displacements is zero. We also attempt to remove the solution error and to operate with exact nodal displacements just to have clear testing conditions. For this reason we use analytical results to generate the nodal displacements $v_i$ (transverse deflection) and $\psi_i$ (rotation) at node $i$. The position of this node $i$ is for element length $\ell = 1$ is

$$x_i = i - 1 \, .$$

For the adjoint load case (transverse unit load at the free end, $EI = 1$, $\ell = 1$, $L =$ m), we have

$$\overline{v}_E = (3\mathrm{m}x_i^2 - x_i^3)/6 \, , \quad \overline{\psi}_i = (2\mathrm{m}x_i - x_i^2)/2 \, ,$$

and for the real load case (moment $M = 1/\mathrm{m}$ at free end, $EI = 1$, $\ell = 1$, $L =$ m)

$$v_i = x_i^2/(2\mathrm{m}) \, , \quad \psi_i = x_i/\mathrm{m} \, .$$

By now computing the sensitivity using analytical derivatives, we have

$$\frac{\partial v_E}{\partial L} = \frac{\partial}{\partial L}\left(\frac{1}{2}\frac{ML^2}{EI}\right) = 1 \, ,$$

which is independent of the number of elements $m$.

If we now use the traditional incremental stiffness as obtained by direct differentiation in the SA-method, then the errors increase rapidly with the element number $m$ and end up with 250% error for 100 elements (see Table 1). This error occurs in spite of a very small modification ($0.0001L$) and exact nodal displacements, which is of course now a well-known fact. If we, however, employ the natural method as proposed in (17) for the computation of $\Delta\mathbf{k}^*$, we obtain almost exact sensitivities (the rounding-off error is still inherent in the result).

Finally, we test the Cartesian approach. In our case we have only one strain component ($\varepsilon_{xx}$) and the material stiffness is simply given by Young's modulus $E$. Further it is usual to pre-integrate over the cross-sectional area of the beam. After this operation the strain is replaced by the curvature and the material stiffness by the bending stiffness. Introducing the parameter coordinate $\zeta$

$$\zeta = \frac{2\hat{x}}{\ell} \, ,$$

by (19) we have (Fig. 3)

$$\mathbf{k} = \int_{-1}^{+1} \boldsymbol{\alpha}^t \kappa \boldsymbol{\alpha} \frac{\ell}{2} \, \mathrm{d}\zeta \, ,$$

with

$$\kappa = EI \, ,$$

and

$$\boldsymbol{\alpha} = \frac{4}{\ell^2}\frac{d^2\omega}{d\zeta^2} = \frac{1}{2\ell^2}\left[6\zeta \quad \ell(3\zeta - 1) \quad -6\zeta \quad \ell(3\zeta + 1)\right] \, .$$

Numerical integration with a two point scheme will employ the integration points (Fig. 3)

$$\zeta_k = \pm\sqrt{3}/3 \, .$$

We now form $\Delta\mathbf{k}^{**}$ as proposed in (24) and again obtain very accurate sensitivities, as can be seen in Table 1.

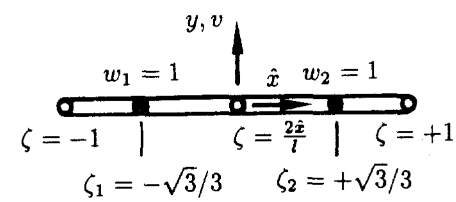


**Fig. 3.** Local coordinates and integration points for plane beam element

## 7 Conclusions

In the SA approach for the computation of sensitivities, we usually employ incremental Cartesian stiffnesses, which are obtained by direct differentiation. If the design variable affects the shape, e.g. the length of a beam, this incremental stiffness is defective and leads to errors which increase with the number of elements. This defect in the incremental stiffness can be verified by the natural approach. The natural method also leads to a new formula for the computation of an incremental stiffness that is not defective. The same approach can also be extended to the more common Cartesian element description. It is demonstrated, that the use of these

**Table 1.** Cantilever beam under end moment: sensitivities of tip deflection with respect to total length

| Number of elements | Semi-analytic sensitvities with | | | | | |
|---|---|---|---|---|---|---|
| | $\Delta\mathbf{k}$ (traditional) | | $\Delta\mathbf{k}^*$ (natural) | | $\Delta\mathbf{k}^{**}$ (Cartesian) | |
| | value | error[%] | value | error[%] | value | error[%] |
| 1 | 0.999650 | -0.03 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 5 | 0.993652 | -0.63 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 10 | 0.974908 | -2.51 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 15 | 0.943667 | -5.63 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 20 | 0.899930 | -10.01 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 25 | 0.843697 | -15.63 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 30 | 0.774967 | -22.50 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 35 | 0.693742 | -30.63 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 40 | 0.600020 | -40.00 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 45 | 0.493802 | -50.62 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 50 | 0.375087 | -62.49 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 55 | 0.243877 | -75.61 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 60 | 0.100170 | -89.98 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 65 | -0.056033 | -105.60 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 70 | -0.224733 | -122.47 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 75 | -0.405928 | -140.59 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 80 | -0.599620 | -159.96 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 85 | -0.805808 | -180.58 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 90 | -1.024493 | -202.45 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 95 | -1.255673 | -225.57 | 0.999900 | -0.01 | 0.999800 | -0.02 |
| 100 | -1.499350 | -249.94 | 0.999900 | -0.01 | 0.999800 | -0.02 |

new incremental stiffnesses almost completely removes the above error in sensitivities. Although specific and detailed considerations are given only for the simple case of a plane beam, the theory presented is general and should apply also to other element types. Further examples willl be presented in a forthcoming paper.

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