Annali di Matematica pura ed applicata
(IV), Vol. CLXIII (1993), pp. 161-180

Uniqueness in the Dirichlet Problem for Some Elliptic Operators with Discontinuous Coefficients (*) (**).

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Abstract. – *Uniqueness is proved for the Dirichlet problem for second order nondivergence form elliptic operators with coefficients continuous except at a countable set of points having at most one accumulation point. Moreover, gradient estimates are proved.*

1. – Introduction.

This paper deals with uniqueness for the Dirichlet problem for strongly elliptic operators in $\mathbb{R}^n$ in non-divergence form with discontinuous coefficients. More precisely we will consider operators of the form

$$L = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij}^2, \qquad \text{where } D_{ij}^2 = \frac{\partial^2}{\partial x_i \partial x_j} \tag{1}$$

and the functions a_{ij} are defined on some bounded domain $D \subset \mathbb{R}^n$ and for the moment they will only be assumed to be bounded on D. We will look for solutions to the Dirichlet problem:

$$\begin{cases} Lu = -f & \text{in } D, \\ u = \varphi & \text{on } \partial D. \end{cases} \tag{2}$$

While in the case of divergence form operators there are complete results on existence, uniqueness and regularity for solutions to the Dirichlet problem, in the non-divergence case if the coefficients a_{ij} are not continuous only a few facts are known.

Before going on let us make a few remarks about what the meaning of a solution to (2) is, since, the coefficients of L being only bounded, it is not clear at first what one may expect a solution to be, and therefore what a good uniqueness class is.

(*) Entrata in Redazione il 23 aprile 1990.

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(**) The authors are partially supported by the National Science Foundation Grant no. NSF/DMS 8421377-04.

We shall consider an example, which was originally due to GILBARG and SERRIN [6] and was then considered by PUCCI in [9]. The operator L with $a_{ij}(x) = \delta_{ij} + \alpha(x_i x_j / |x|^2)$ in the unit ball $B = B_1(0)$ has coefficients that are bounded on B and actually C^∞ on $B \setminus \{0\}$. If $\alpha > n-2$ it can be easily verified that the function $v(x) = |x|^\beta$ with $\beta = (\alpha - (n-2))/(1+\alpha)$ is a pointwise solution to $Lv = 0$ in $B \setminus \{0\}$, $v = 1$ on ∂B and is smooth outside the origin and continuous on $\overline{B}$. But the same problem admits another obvious solution which is $u(x) \equiv 1$ on B. The function v despite being «fairly smooth» has some «unwanted» characteristics, such as not satisfying the maximum principle and Harnack's inequality.

Furthermore, if we take a regularization a_{ij}^k of the coefficients of L, such that $a_{ij}^k(x) \to a_{ij}(x)$ in B, and we solve the problems $L^k u^k = \sum_{i,j=1}^n a_{ij}^k(x) D_{ij}^2 u^k = -f$ in B, $u^k = 1$ on ∂B we should reasonably expect a solution u to (2) to be a limit of a subsequence of such u^k's. This is not true for the v in Pucci's example: in fact any regularized problem has the unique solution $u^k \equiv 1$.

We observe that the above procedure actually always constructs «some» u, as a limit of a subsequence of the u^k's. This follows from the Hölder estimates which have been shown to hold for u^k independently of the regularity of a_{ij}^k (see [10]). This will be the concept of a solution to (2) which will be adopted here. A natural problem is to determine whether or not all subsequences converge to the same limit, i.e. whether this «solution» is unique.

This is precisely the question we will be dealing with here. We will prove that in fact such a limit is unique in some cases, namely when the coefficients a_{ij} have at most countably many singularities with one accumulation point in D.

We should point out that the technique used to prove uniqueness in the case of a one point discontinuity, is due to Luis Caffarelli but it has never been published by him.

Moreover when the coefficients have a single point P of discontinuity and are C^∞ outside that point we will characterize the case in which «bad» solutions (i.e. solutions of the kind in Pucci's example) exist and the case in which they do not exist. In the latter case we will show that the solution u is unique in the class $C^2(D \setminus \{P\}) \cap C(\overline{D})$ and has the property

$$\sup_{x \in D} \int_D |\nabla u(y)|^2 g(x, y)\, dy < \infty ,$$

where $g(x, y)$ is the Green's function for L in D (see Sect. 6).

In the case in which a «bad» solution $\tilde{g}$ exists, under the assumptions $0 \leq \tilde{g}(x) \leq 1$ in D, $\tilde{g}(P) = 1$ and $\tilde{g}(x) = 0$ on ∂D, we will show (Sect. 5) that there exists a unique pointwise solution u to $Lu = 0$ in $D \setminus \{P\}$ with $u = \varphi$ on ∂D and satisfying

$$\int_D \frac{|\nabla u(y)|^2}{1 - \tilde{g}(y)} g(P, y)\, dy < \infty . \tag{3}$$

Related to this problem is a result of R. BASS in [2]: he proves a uniqueness result for the Dirichlet Problem for operators L whose coefficients are homogeneous of degree zero and smooth outside the origin. The solutions that Bass considers are in the class $C^2(B\setminus\{0\}) \cap C(\overline{B})$ and satify $Lu = 0$ in $B\setminus\{P\}$ with $u = \varphi$ on ∂B and the condition $\int_D |D_{ij}^2 u|^{1+\varepsilon}(y) g(0, y)\, dy < \infty$ for some $\varepsilon > 0$ and for every $1 \leq i, j \leq n$.

2. – Preliminary results.

Throughout this paper $L = \sum_{i,j=1}^n a_{ij}(x) D_{ij}^2$ is a uniformly elliptic operator defined on a bounded smooth domain $D \subset \boldsymbol{R}^n$, i.e. the coefficients a_{ij} satisfy the uniform ellipticity condition

(4) there exist positive numbers λ and Λ such that $\lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$ $\forall x \in D$ and $\forall \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$; also $a_{ij}(x) = a_{ji}(x)$.

Let $f \in L^n(D)$ and $\varphi \in C(\partial D)$.

In order to give a definition of solution to the equations we are considering here, let's recall the uniform Hölder estimates due to Safanov ([10]), which hold for smooth solutions to $Lu = 0$:

$$\|u^k\|_{C^\alpha(B_\rho)} = \sup_{\substack{x y \in B_\rho \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C\|\varphi\|_\infty . \tag{5}$$

where C and α depend only on λ, Λ, n and ρ.

Now let $\{a_{ij}^k(x)\}$, $i,j = 1, \ldots, n$ and $k = 1, 2, \ldots, \infty$, be a regularization of the coefficients of L, i.e. a collection of smooth functions such that:

i) for each pair ij, $a_{ij}^k \to a_{ij}$ uniformly on compact subsets of $D\setminus\overline{E}$, where E is the set of points of discontinuity of a_{ij};

ii) for each k, $\{a_{ij}^k(x)\}$ satisfies (4) with the same constants as $\{a_{ij}(x)\}$.

Also let $L^k = \sum_{i,j=1}^n a_{ij}^k(x) D_{ij}^2$ which will be called a regularization of the operator L. (A regularization of L can be obtained, for example, through convolution of the coefficients of L with a smoothing kernel).

Because of the smoothness of the coefficients the problems

$$\begin{cases} L^k u^k = \sum_{i,j=1}^n a_{ij}^k(x) D_{ij}^2 u^k = -f & \text{in } D, \\ u = \varphi & \text{on } \partial D, \end{cases}$$

admit a unique solution u^k.

From (5), the classical maximum principle and the theorem of Aleksandrov and

Pucci (see [1] and [9]), it follows that

$$\|u^k\|_{L^\infty(D)} + \|u^k\|_{C^\alpha(\bar{\Omega})} \leq C(\lambda, \Lambda, n, \Omega, D)\{\|\varphi\|_{L^\infty(\partial D)} + \|f\|_{L^n(D)}\}, \tag{6}$$

with Ω a compact subset of D. Observe that the above constant is independent of k and therefore Ascoli-Arzelà theorem implies that there exists a subsequence $\{u^{k_i}\}$ of $\{u^k\}$ such that $u^{k_i} \to u$ uniformly on compact subsets of D.

The discussion in Sect. 9.9 of [7] combined with a barrier argument ([7], Ch. 6) assures that the modulus of continuity of the u^k's at the boundary of D is independent of k, and therefore that the convergence $u^{k_i} \to u$ is uniform on $\bar{D}$. Moreover (6) holds for such a u. Observe, though, that there may be more than one u constructed in this way.

We are now in position to give:

DEFINITION 1. – *We will say that* $u \in C(\bar{D})$ *is a* «*good*» solution to the Dirichlet problem (2) *for* L *if there exist sequences of functions* $\{a_{ij}^k\}_{k=1}^\infty \subset C^\infty(D)$ *satisfying* i) *and* ii) *above and a sequence of functions* u^k *such that* $L^k u^k = \sum_{i,j=1}^n a_{ij}^k(x) D_{ij}^2 u^k = -f$ *in* D, $u^k = \varphi$ *on* ∂D *and* $u^k \to u$ *uniformly on* $\bar{D}$.

Also, if Ω *is an open set contained in* D, *we will say that* u *is a* «good» solution to L *in* Ω *is there exist sequences of functions* $\{a_{ij}^k\}_{k=1}^\infty \subset C^\infty(\Omega)$ *as above and a sequence of functions* u^k *such that* $L^k u^k = 0$ *in* Ω *and* $u^k \to u$ *uniformly on compact subsets of* Ω.

Since Harnack's inequality has been proved (see [10]) with constants independent of the regularity of the coefficients, it holds for «good» solutions. Moreover the strong Maximum Principle holds for «good» solutions. Precisely we have:

HARNARCK'S INEQUALITY. – *Let* u *be a non negative* «*good*» *solution to* L *in* D *and let* $\bar{B}_{2r} \subsetneq D$. *Then* $\sup_{x \in B_r} u(x) \leq \vartheta \inf_{x \in B_r} u(x)$ *where the constant* ϑ *depends only on* λ, Λ *and* n.

STRONG MAXIMUM PRINCIPLE. – *Let* u *be a* «*good*» *solution to* L *in* D. *Then if* u *has a local maximum (minimum) in* D, u *is constant in* D.

Next we are going to explain what we will mean for a Green's function for L in D and to define «normalized» Green's functions.

A well known result of ALEKSANDROV [1] and PUCCI [9] states that, if L has smooth coefficients, and u is the solution to the Dirichlet problem with homogeneous boundary data, i.e. $Lu = -f$ in D, $u = 0$ on ∂D, the following a priori estimate holds:

$$\sup_D |u| \leq C\|f\|_{L^n(D)} \quad \text{where } C = C(\lambda, n, \operatorname{diam} D), \tag{7}$$

independent of the regularity of the coefficients of L. Since we know that in this case the solution is unique, the map $f \to u(x)$ is a bounded linear functional on $L^n(D)$ for every $x \in D$. Therefore by the Riesz representation theorem there exists $g(x, \cdot)$ such that

$$(8) \qquad u(x) = \int_D g(x, y) f(y)\, dy$$

$$\text{and } \|g(x, \cdot)\|_{L^{n/(n-1)}(D)} = \sup_{\substack{f \in L^n(D) \\ \|f\| = 1}} \int_D g(x, y) f(y)\, dy \leq C(\lambda, n, \operatorname{diam} D).$$

Suppose now that the coefficients of L are not continuous and consider as before some regularization L^k of L. Let $L^k u^k = -f$ in D, $u^k = 0$ on ∂D and $g^k(x, y)$ the corresponding Green's functions. Because of (6) for every $x \in D$ there exists a subsequence, which we will still call g^k, such that $g^k(x, \cdot) \rightharpoonup g(x, \cdot)$ weakly in $L^{n/(n-1)}(D)$. Through a diagonalization process we can define $g(x, y)$ for $x \in \mathbb{Q}^n \cap D$ (where $\mathbb{Q}^n$ is the subset of points of $\mathbb{R}^n$ with rational coordinates) and then, because of the equicontinuity of $\{u^k\}$, extend it to D. Observe that this process constructs a Green's function but says nothing about the uniqueness of g.

We are now going to construct «normalized» Green's functions $\tilde{g}(x, y)$ for L. These «objects» were first introduced and thoroughly studied by P. BAUMAN in [3] and [4] in the case of an operator with continuous coefficients. Our definition will be slightly different but the results about $\tilde{g}$ that we need here will still hold. Let's first assume that L has smooth coefficients.

DEFINITION 2. – *Let B_R be a ball such that $D \subset B_{R/2}$, $P \in \overline{B}_{R/2}$ and $G(x, y)$ be the Green's function for L in B_R. Let $g(x, y)$ be the Green's function for L in D. The* «normalized» Green's function *for L in D is $\tilde{g}(x, y) = g(x, y)/G(P, y)$.*

In this case $\tilde{g}(x, y)$ is uniquely defined because the Green's function is unique when L has smooth coefficients.

For $\tilde{g}$ the following properties hold.

THEOREM 1. – *For each $x \in D$, $\tilde{g}(x, \cdot)$ satisfies the strong maximum principle in subdomains of $D \setminus \{x\}$.*

Moreover,

i) $\sup_{z \in B_r} \tilde{g}(x, z) \leq C \inf_{z \in B_r} \tilde{g}(x, z)$,

ii) $\operatorname{osc}_{z \in B_{\sigma r}} \tilde{g}(x, z) \leq C\sigma^\alpha \operatorname{osc}_{z \in B_r} \tilde{g}(x, z)$

whenever $B_{2r} \subset D \setminus \{x\}$ and $0 < \sigma < 1$, with constants C and α depending only on λ, Λ and n.

The above results were proved by BAUMAN in [3] (Theorem 2.2, 2.4 and 2.5) with constants depending on the modulus of continuity of the a_{ij}. The results of FABES and STROOK in [5] can be used to show that a constant C can be found for Theorem 1 depending only on λ, Λ and n. Observe that locally $\tilde{g}(\cdot, y)$ is a solution to $Lu = 0$.

Observe that the above result also implies that $\tilde{g}$ cannot be identically zero. To see this observe first that Theorem 1, together with the A^∞ property of $\tilde{g}(x, \cdot)$ (see BAUMAN [3]), implies that $\tilde{g}(x, y)$ is equivalent (with constants depending only on λ, Λ, n and $d(x, \partial D)$) to $\int_{B_r} g(x, z)\, dz / \int_{B_r} G(P, z)\, dz$ for $y \in B_r$ and $B_{4r} \subset D \setminus \{x\}$. It can be easily seen that this ratio is bounded from below by a constant depending again only on λ, Λ, n and $d(x, \partial D)$.

Let's now go back to the case of an operator L with discontinuous coefficients. Consider the regularized operators L^k again and let G^k be the Green's functions for L^k in B_R. We have seen that there exists a subsequence, that will be called again $\{G^k\}$, such that $G^k(P, \cdot) \rightharpoonup G(P, \cdot)$ weakly in $L^{n/(n-1)}(B_R)$.

Let now $\tilde{g}^k(x, y) = g^k(x, y)/G^k(P, y)$ and let $\Delta_m = \{(x, y) \in \overline{D} \times \overline{D} \mid |x - y| \geq 1/m\}$.

Observe that $\tilde{g}^k(x, y)$ is Hölder continuous jointly for (x, y) in Δ_m.

We are now going to construct $\tilde{g}(x, y)$ via a diagonalization process as follows: let $\tilde{g}^{k_1}$ be a subsequence of $\tilde{g}^k$ uniformly convergent on Δ_1 and by induction let $\tilde{g}^{k_m}$ be a subsequence of $\tilde{g}^{k_{m-1}}$ subsequence of $\tilde{g}^k$ uniformly convergent on Δ_m. Taking the diagonal we obtain a sequence, call it again $\tilde{g}^k$, uniformly convergent on each Δ_m. Call the limit $\tilde{g}(x, y)$; clearly this function is defined in all $D \times D \setminus \{x = y\}$. This will be a «normalized» Green's function for L in D.

Clearly Theorem 1 holds for $\tilde{g}$ so defined. Also $\tilde{g}(\cdot, y)$ is a «good» solution in subsets of $D \setminus \{y\}$. Moreover,

THEOREM 2. – *Fix $a \in D$; then either*

i) $\lim_{x \to a} \tilde{g}(x, a) = +\infty$, *or*

ii) $\lim_{x \to a} \tilde{g}(x, a) \equiv \tilde{g}(a, a) < \infty$.

PROOF. – Without loss of generally assume $a = 0$.

Assume first that $\limsup_{x \to 0} \tilde{g}(x, 0) = +\infty$. Then there exists $\{x_j\} \subset D$, with $|x_j| \downarrow 0$ and $\tilde{g}(x_j, 0) \geq j$. Because of Harnack's inequality we have that $\tilde{g}(x, 0) \geq C \cdot j$ on $|x| = |x_j|$ and we have that $\tilde{g}(x, 0) \geq C(j+1)$ on $|x| = |x_{j+1}|$. By the maximum principle it then follows that on the anullus $|x_{j+1}| \leq |x| \leq |x_j|$ we have $\tilde{g}(x, 0) \geq C \cdot j$. As $j \to \infty$ we get $\lim_{x \to 0} \tilde{g}(x, 0) = +\infty$.

Let now $\limsup_{x \to 0} \tilde{g}(x, 0) = b < +\infty$. Then choose $r_j \downarrow 0$ so that $\tilde{g}(x, 0) < b + 1/j$ on the set where $|x| < r_j$ and $x_j \in D$ such that $|x_j| \downarrow 0$, $|x_j| < r_j$ and $\tilde{g}(x_j, 0) > b - 1/j$. Then by Harnack's inequality $0 \leq b - \tilde{g}(x, 0) + 1/j \leq \vartheta(b - \tilde{g}(x_j, 0) + 1/j) \leq \vartheta(2/j)$, when $|x| = |x_j|$. Therefore by the maximum principle, when $|x_{j+1}| \leq |x| \leq |x_j|$ we have that $0 \leq b - \tilde{g}(x, 0) + 1/j \leq \vartheta(2/j)$ or, equivalently, $-1/j \leq b - \tilde{g}(x, 0) \leq (2\vartheta - 1)(1/j)$. Now let $j \to \infty$ and the theorem is proved.

3. – Uniqueness of good solutions.

In this section we will deal with coefficients with at most countably many singularities and one accumulation point.

In order to simplify some proofs, in what follows we will always assume $0 \in D$. Moreover B_r or ∂B_r when the center is not specified will always denote balls centered in the origin.

Assume first that $a_{ij} \in C^\infty(D \setminus \{0\})$.

REMARK 1. – If for a $\tilde{g}$ we have that $\tilde{g}(0, 0) = \infty$, we can prove uniqueness for solutions u to problem (2) which satisfy $Lu = -f$ in $D \setminus \{0\}$ and belong to the class $C^2(D \setminus \{0\}) \cap L^\infty(\overline{D})$, by the following reasoning. Let u_1 and u_2 be two solutions and assume they are different at some point (say $u_1 > u_2$ there). Let $\nu = u_1 - u_2$; by maximum principle $\nu > 0$ in $D \setminus \{0\}$ and $\max_{\overline{D}} \nu(x) = \nu(0)$. May assume $0 \leq \nu(x) \leq 1$. Choose $\varepsilon > 0$ in such a way that $\tilde{g}(x, 0) > M$ on ∂B_ε. Then $w(x) = \nu(x) - (1/M)\tilde{g}(x, 0) \leq 0$ on ∂B_ε and $w(x) = 0$ on ∂D. Therefore by the maximum principle $0 \leq \nu(x) \leq (1/M)\tilde{g}(x, 0)$ in $D \setminus B_\varepsilon$. Fixing x and letting M tend to infinity we get $\nu(x) \equiv 0$ which is a contradiction.

Observe that «good» solutions in this case are in the class $C^2(D \setminus \{0\}) \cap C(\overline{D})$ and they satisfy $Lu = -f$ in $D \setminus \{0\}$, so that this argument actually gives uniqueness for «good» solutions.

REMARK 2. – Using the above reasoning we can prove that if $\tilde{g}_1$ and $\tilde{g}_2$ are two normalized Green's functions for the same operator L whose coefficients are smooth in $D \setminus \{0\}$ and $\tilde{g}_1(0, 0) = \infty$ then also $\tilde{g}_2(0, 0) = \infty$. To see this assume $\tilde{g}_2(0, 0) < \infty$ and repeat the same argument of Remark 1 replacing $\tilde{g}$ with $\tilde{g}_1$ and ν with $\tilde{g}_2$ to conclude $\tilde{g}_2(x, 0) \equiv 0$ which is a contradiction.

In view of the previous Remark, when 0 is the only point of singularity, it will make sense to distinguish the two cases $\tilde{g}(0, 0) = \infty$ and $\tilde{g}(0, 0) < \infty$.

We are now going to prove some uniqueness results for «good» solutions. The main tool in this section is Theorem 3. The proof is due to Luis Caffarelli.

THEOREM 3. – *Let L be as in* (1), *with coefficients $a_{ij} \in C^\infty(D \setminus \{0\})$ and satisfying* (4). *Let u be a «good» solutions to problem* (2), *with $\varphi \in C(\partial D)$ and $f \in C_0^\infty(D \setminus \{0\})$ and ν be a non-constant function such that $L\nu = 0$ in $D \setminus \{0\}$, $\nu \in C^2(\overline{D} \setminus \{0\}) \cap C(\overline{D})$ and $0 < \nu(x) < \nu(0)$ in $D \setminus \{0\}$. Then there exist α and C such that*

$$|u(x) - u(0)| \leq C|x|^\alpha(\nu(0) - \nu(x)),$$

where $C = C_1(\|\varphi\|_\infty + \|f\|_{L^n})$ and α depends only on λ, Λ, n and C_1 on the latter and the support of f.

PROOF. – By hypothesis f is supported away from the origin; suppose $f \equiv 0$ in B_r.

Let $\overline{u}(x) = u(x) - u(0)$ and $\overline{v}(x) = v(0) - v(x)$. Therefore $\overline{v}(x) > 0$ on $\partial B_{r/2}$ and in particular we can choose C such that $C\overline{v}(x) \pm \overline{u}(x) \geq 0$ on $\partial B_{r/2}$.

Observe that for every $s < r$ it is possible to find $x_s \in \partial B_s$ such that $\overline{u}(x_s) = 0$. If x_s did not exist we would have $\overline{u}(x) > 0$ on ∂B_s and by maximum principle $\overline{u} \geq 0$ on B_s. But $\overline{u}$ is a «good» solution in B_s for which Harnack's inequality holds in $B_{s/2}$ and $\overline{u}(0) = 0$. This implies $\overline{u}(x) \equiv 0$ in $B_{s/2}$ and therefore in $\overline{B}_s$.

Let now $w_1^{\pm}(x) = C\overline{v}(x) \pm \overline{u}(x)$. We have that $w_1^{\pm}(x) \geq 0$ on $\partial B_{r/2}$ and $w_1^{\pm}(0) = 0$; hence, again by maximum principle, $w_1^{\pm}(x) \geq 0$ in $B_{r/2}$. Also, by Harnack's inequality, applied to $w_1^{\pm}$ in the anullus $A_1 = B_{r/2} \setminus B_{r/4}$ we have

$$w_1^{\pm}(x) \geq \vartheta w_1^{\pm}(x_{r/2}) = C\vartheta\overline{v}(x_{r/2}) \geq C\vartheta^2\overline{v}(x) \qquad \text{for } x \in A_1 .$$

So in particular $w_2^{\pm}(x) = C(1 - \vartheta^2)\overline{v}(x) \pm \overline{u}(x) \geq 0$ on $\partial B_{r/4}$.

Let now $w_j^{\pm}(x) = C(1 - \vartheta^2)^{j-1}\overline{v}(x) \pm \overline{u}(x)$ and $A_j = B_{r/2^j} \setminus B_{r/2^{j+1}}$. Having established that $w_j^{\pm}(x) \geq 0$ in $\partial B_{r/2^j}$ and observing that $w_j^{\pm}(0) = 0$ for every j, again by the maximum principle we can conclude that $w_j^{\pm}(x) \geq 0$ in $B_{r/2^j}$ and by Harnack's inequality that $w_j^{\pm}(x) \geq \vartheta w_j^{\pm}(x_j) = C\vartheta\overline{v}(x_j) \geq C\vartheta^2\overline{v}(x)$ in A_j, where $x_j = (x_{r/2^j})$. In other words $|u(0) - u(x)| \leq C(1 - \vartheta^2)^j(v(0) - v(x))$ when $r2^{-j-1} \leq |x| < r2^{-j}$.

Set $\alpha = -\log(1 - \vartheta^2)/\log 2$ and the theorem is proved.

As an immediate consequence of Theorem 3 we have the following uniqueness theorem:

THEOREM 4. – *Let L and φ and f be as in Theorem* 3; *then problem* (2) *has a unique «good» solution.*

PROOF. – Let u_1 and u_2 be two possible «good» solutions to the given problem. If they are different at some point in D (say $u_1 > u_2$ there), then by the maximum principle applied to $v = u_1 - u_2$ (which satisfies $Lv = 0$ in $B \setminus \{0\}$) we must have $v(x) > 0$ in $D \setminus \{0\}$ and $\max\limits_{\overline{D}} v(x) = v(0)$. But then v satisfies the hypothesis for Theorem 3 and by applying the same with u first equal to u_1 then to u_2 we conclude that $|v(x) - v(0)| = o(|v(x) - v(0)|)$; this can only happen if $v(x) \equiv v(0)$ in a neighborhood of the origin which is a contradiction; therefore we conclude $u_1 \equiv u_2$.

REMARK 3. – From now on let $\tilde{g}(x) = \tilde{g}(x, 0)$. Then if $\tilde{g}(0) = \lim\limits_{x \to 0} \tilde{g}(x, 0) < \infty$, $\tilde{g}(x)$ satisfies the hypothesis for v in Theorem 3. This case, by Bauman's paper [4], corresponds to the case in which the point zero has positive capacity, i.e. when we can solve the Dirichlet problem for L in the punctured set $D \setminus \{0\}$. So in this case we can actually have functions which are pointwise solutions in $D \setminus \{0\}$ and that do not satisfy the maximum principle and Harnack's theorem across the origin. Theorem 3 in this case tells us that a «good» solution has a better regularity than one of these other solutions.

DEFINITION 2. – *Given an operator L and a smooth domain D we will say that the* Dirichlet problem for L in D is uniquely solvable *if for any $\varphi \in C(\partial D)$ and $f \in L^n(D)$ there exists a unique «good» solution u to problem* (2).

When this happens the mapping $\varphi \to u(x)$, where u is the «good» solution to

$$\begin{cases} Lu = 0 & \text{in } D, \\ u = \varphi & \text{on } \partial D, \end{cases}$$

is a positive linear functional on $C(\partial D)$ for every $x \in D$; hence by the Riesz representation theorem there exists a unique measure $\omega^x_{L,D}(dQ)$ define on ∂D such that

$$u(x) = \int_{\partial D} \varphi(Q)\, \omega^x_{L,D}(dQ);$$

(in the following when it is clear to which operator and domain we are refering to, we will simply write $\omega^x(dQ)$).

Also observe that if ω^x_k's are the harmonic measures for the operators L^k in D, where the $\{L^k\}$ is a regularization of L, then $\omega^x_k \rightharpoonup \omega^x$ weakly. Therefore for every $w \in C(\overline{D})$ the function $u(x) = \int_{\partial D} w(Q)\, \omega^x(dQ)$ is the unique «good» solution to the problem

$$\begin{cases} Lu = 0 & \text{in } D, \\ u = w & \text{on } \partial D. \end{cases}$$

LEMMA 5. – *If u_1 and u_2 are two «good» solutions on a domain D in which there is uniqueness, then $\nu = u_1 - u_2$ is a «good» solution.*

PROOF. – $\nu(x) = \int_{\partial D} (u_1 - u_2)(Q)\, \omega^x(dQ)$.

THEOREM 6. – *Let D be a smooth domain and L an operator with bounded coefficients. Then the Dirichlet problem for L in D is uniquely solvable if and only if for every $x \in D$ there exists $0 < r < d(x, \partial D)$ such that the Dirichlet problem for L in $B_r(x)$ is uniquely solvable.*

PROOF. – Let u_1 and u_2 be two possible distinct «good» solutions to problem (2) in D. Since $u_1 \equiv u_2$ on ∂D, $u_1 - u_2$ must attain positive maximum or a negative minimum at some $x_0 \in D$. Assume $u_1(x_0) > u_2(x_0)$. Let $\nu(x) = [u_2(x) - u_2(x_0)] - [u_1(x) - u_1(x_0)]$ (observe that $\nu \geq 0$ in D) and let r_0 be such that the Dirichlet problem for L in $B_{r_0}(x_0)$ is uniquely solvable. In view of the discussion about harmonic measure, the previous Lemma and the uniqueness assumption in $B_{r_0}(x_0)$, $\nu(x) = \int_{\partial B_{r_0}(x_0)} \nu(Q)\, \omega^x_{r_0}(dQ)$ (with $\omega^x_{r_0}$ the harmonic measure for L in $B_{r_0}(x_0)$) is the (unique) «good» solution to the problem $Lu = 0$ in $B_{r_0}(x_0)$, $u = \nu$ on $\partial B_{r_0}(x_0)$. Since $\nu(x) \geq 0$ and $\nu(x_0) = 0$ by the Harnack's inequality follows that $\nu \equiv 0$ in $B_{r_0}(x_0)$.

Now let $x_1 \in \partial B_{r_0}(x_0)$ and let r_1 be such that the Dirichlet problem for L in $B_{r_1}(x_1)$ is uniquely solvable. Repeat the same argument to conclude that $\nu \equiv 0$ in $B_{r_0}(x_0) \cup B_{r_1}(x_1)$. Since every point has a neighborhood in which there is uniqueness for the Dirichlet problem for L, it is clear that we can keep repeating the same argument to conclude that $\nu \equiv 0$ in D. But $u_1(x) = u_2(x)$ on ∂D and this implies $u_1(x_0) = u_2(x_0)$.

COROLLARY. – *Let L be an operator with coefficients $a_{ij} \in C^\infty(D \setminus \{x_1, \ldots, x_m\})$. Then the Dirichlet problem for L in D is uniquely solvable.*

REMARK 4. – Let $\operatorname{supp} f \subseteq D \setminus B_r(0)$ and let $\overline{u}(x)$ and $\overline{\nu}(x)$ be as in the proof of Theorem 3. We can repeat the same argument in there if the following conditions hold:

a) on each $\partial B_s(0)$ there exists x_s such that $\overline{u}(x_s) = u(0) - u(x_s) = 0$, for $s < r$;

b) if $C\overline{\nu}(x) \pm \overline{u}(x) \geq 0$ on ∂B_s implies that the same is ≥ 0 on B_s and satisfies a Harnack's inequality on $B_{r/2^j} \setminus B_{r/2^{j+1}}$.

THEOREM 7. – *Let $\{x_m\}_{m=1}^\infty \subseteq D$ and $x_m \to 0$ (in D), let L be an operator with coefficients $a_{ij} \in C^\infty(D \setminus \{x_1, \ldots, 0\})$. Then the Dirichlet problem for L in D is uniquely solvable for data $f \in C_0^\infty(D \setminus \{x_1, \ldots, 0\})$ and $\varphi \in C(\partial D)$.*

PROOF. – Suppose u_1 and u_2 are two «good» solutions to (2). If $u_1(x_0) > u_2(x_0)$ for some $x_0 \in D$, then by the Corollary to Theorem 6 and by Lemma 5 $\max_{\overline{D}}[u_1(x) - u_2(x)] = u_1(0) - u_2(0)$. Assume $u_1(0) > u_2(0)$ and $f \equiv 0$ on $B_r(0)$. Let's now prove that *a*) and *b*) in Remark 4 are satisfied, with $\overline{u}(x) = u_1(x) - u_1(0)$ (or $u_2(x) - u_2(0)$) and $\overline{\nu}(x) = \nu(x) - \nu(0) = [u_1(x) - u_2(x)] - [u_1(0) - u_2(0)]$.

Suppose $\overline{u}(x) > 0$ on ∂B_s, $s < r$, and let ω^x_δ be the harmonic measure for L in $B_s \setminus \overline{B}_\delta(0)$ (which is well defined in view of Theorem 6, since in $B_s \setminus \overline{B}_\delta(0)$ the coefficients of L have only finitely many points of discontinuity); then

$$\overline{u}(x) = \int_{\partial(B_s \setminus \overline{B}_\delta(0))} \overline{u}(Q)\, d\omega^x_\delta = \int_{\partial B_s(0)} \overline{u}(Q)\, d\omega^x_\delta + o(1) \quad \text{as } \delta \to 0.$$

Hence $\overline{u}(x) \geq 0$ in B_s and $\overline{u}(0) = 0$; from Harnack's inequality $\overline{u}(x) \equiv 0$ in $B_{s/2}$ and therefore in B_s. So *a*) holds.

Finally if $C\overline{\nu}(x) \pm \overline{u}(x) \geq 0$ on ∂B_s a similar argument shows that this function is ≥ 0 in B_s. Moreover because of Lemma 5 this function is a «good» solution on anulli and therefore satisfies Harnack's inequality there.

Now assume as in Theorem 3 that $f \equiv 0$ on $B_r(0)$; repeating the same argument there we conclude that $|u_1(x) - u_1(0)| \leq C|x|^\alpha(\nu(0) - \nu(x))$ and the analogous for u_2; as in Theorem 4 we obtain that $|\overline{\nu}(x)| = o(|\overline{\nu}(x)|)$ as $x \to 0$ and conclude the uniqueness.

(Observe that if for some $\tilde{g}$ we have $\tilde{g}(0, 0) = \infty$ the same argument in Remark 1 applies.)

As a consequence of Theorems 4 and 7, and the Corollary to Theorem 6 we also have uniqueness for the Green's function in those case. We will state and prove this result in the case of Theorem 7.

THEOREM 8. – *Let L be the operator in Theorem 7. Then there exists a unique Green's function for L in D.*

PROOF. – Let $g_1(x, y)$ and $g_2(x, y)$ be two Green's functions for L in D, i.e. suppose there exist two sequences of operators L_1^k and L_2^k converging to L (in the sense of Definition 1) such that for every $x \in D$, the corresponding Green's functions $g_1^k(x, \cdot) \rightharpoonup g_1(x, \cdot)$ and $g_2^k(x, \cdot) \rightharpoonup g_2(x, \cdot)$ weakly in $L^{n/(n-1)}(D)$. Let $f \in C_0^\infty(D \setminus \{x_1, \ldots, 0\})$; then $u_1(x) = \int_D f(y) g_1(x, y)\, dy = \lim_{k\to\infty} \int_D f(y) g_1^k(x, y)\, dy$ and $u_2(x) = \int_D f(y) g_2(x, y)\, dy = \lim_{k\to\infty} \int_D f(y) g_2^k(x, y)\, dy$ are «good» solutions to $Lu = -f$ in D, $u = 0$ on ∂D. Because of the uniqueness Theorem 7 we have $\int_D f(y)[g_1(x, y) - g_2(x, y)]\, dy = 0$ for every $f \in C_0^\infty(D \setminus \{x_1, \ldots, 0\})$ and therefore $g_1(x, \cdot) = g_2(x, \cdot)$ a.e.

4. – Properties of normalized Green's functions.

In this paragraph we are going to study the behaviour of normalized Green's functions $\tilde{g}(x, y)$ which were introduced in Section 2. We will need these properties to study the regularity properties of the gradient of «good» solutions. B_R and P are as in Section 2.

Assume first that the coefficients of L are $C^\infty(B)$. By $f \sim g$ we will mean that two constants C_1 and C_2 exist such that $f(x) \leq C_1 g(x) \leq C_2 f(x)$.

THEOREM 9. – *Let $B_{4r}(x) \subset B_R$ and $g_{4r}(x, y)$ denote the Green's function for L in $B_{4r}(x)$. Then if $\partial B_r(x)$*

$$\frac{g_{4r}(x, y)}{G(P, y)} \sim \frac{r^2}{\omega(B_r(x))} \sim \frac{r^2}{\omega(B_r(y))} \qquad \textit{where } \omega(E) = \int_E G(P, y)\, dy$$

and the equivalence constants depend only on λ, Λ and n.

PROOF. – The proof follows exactly the one of Bauman for Lemma 2.1 in [4].

THEOREM 10. – *Let $g_r(x, y)$ denote the Green's function for L in $B_r(0)$ $r \leq 1$. Then*

$$\int_{|x-y|}^{r} \frac{s}{\omega(B_s(x))} ds \sim \frac{g_r(x, y)}{G(P, y)} \sim \int_{|x-y|}^{r} \frac{s}{\omega(B_s(y))} ds \quad \text{for } x, y \in B_{r/2}(0),$$

with equivalence constants depend only on λ, Λ and n.

PROOF. – The proof is a slight modification of the proof of Theorem 2.3 in [4].

Let's now go back to the general case where L has just bounded coefficients. Observe that, because of the way we constructed a $\tilde{g}$ for a general operator, it is possible to find a subsequence of regularized operators L^k such that the corresponding $\tilde{g}^k \to \tilde{g}$ uniformly on compact subsets of $D \times D \setminus \{x = y\}$ and at the same time $G^k(x, \cdot) \rightharpoonup G(x, \cdot)$ weakly in $L^{n/(n-1)}(D)$.

Therefore if we let $r < d(\partial D, a)/2$ and $\tilde{g}_r(x, y) = \tilde{g}_{B_r(a)}(x, y)$, then Theorem 9 tells us that

$$\tilde{g}_r(x, a) \sim \int_{|x-a|}^{r} \frac{s}{\omega(B_s(a))} ds \quad \text{if } |x - a| < r/2.$$

We are now going to state and prove a theorem which describes the continuity of $\tilde{g}(x, a)$ at $x = a$ when $\tilde{g}(a, a) < \infty$.

THEOREM 11. – *Let $\tilde{g}$ be a normalized Green's functions constructed above. Fix $a \in D$ and assume $\tilde{g}(a, a) < \infty$. Then*

$$0 \leq \tilde{g}(a, a) - \tilde{g}(x, a) \sim \int_{0}^{|x-a|} \frac{s}{\omega(B_s(a))} ds \quad \text{for } |x - a| \leq \frac{d(a, \partial D)}{2}$$

and $\tilde{g}(x, a) < \tilde{g}(a, a)$ for every $x \in D \setminus \{a\}$.

Observe that, since we deal with the case $\tilde{g}(a, a) < \infty$, it only makes sense in the context of operators with discontinuous coefficients (in fact when the coefficients are continuous $\tilde{g}(a, a) < \infty$ can occur only in dimension two: see BAUMAN [4]).

PROOF. – Observe that since $\tilde{g}_r(x, a) \leq \tilde{g}(x, a)$ also $\lim_{x \to a} \tilde{g}_r(x, a) = \tilde{g}_r(a, a) < \infty$. Then, if we let $x \to a$, from the observation above we get $\tilde{g}_r(a, a) \sim \int_0^r s/(\omega(B_s(a)))\, ds$. So we will only need to prove that $\tilde{g}(a, a) - \tilde{g}(x, a) \sim \tilde{g}_r(a, a)$ with $|x - a| = r$.

Let $m_r = \min_{|x-a|=r} [\tilde{g}(a, a) - \tilde{g}(x, a)]$ and $M_r = \max_{|x-a|=r} [\tilde{g}(a, a) - \tilde{g}(x, a)]$. Then clearly

$$\frac{\tilde{g}(a, a) - \tilde{g}(x, a)}{M_r} \leqslant 1 \leqslant \frac{\tilde{g}(a, a) - \tilde{g}(x, a)}{m_r} \quad \text{for } |x - a| = r. \tag{9}$$

Consider now the function $w(x) = [\tilde{g}(a, a) - \tilde{g}(x, a)] - [\tilde{g}_r(a, a) - \tilde{g}_r(x, a)]$. We claim that there exists x_0 such that $|x_0 - a| = r$ and $w(x_0) = 0$. To see this observe that $w(x)$ is a «good» solution throughout $B_r(a)$. (This follows from the fact that if $\tilde{g}^k$ converges to $\tilde{g}$ in $D \times D \setminus \{x = y\}$, being $\tilde{g}^k(x, y) \geqslant \tilde{g}_r^k(x, y)$ in $B_r \times B_r$, we can assume that up to a subsequence, $\tilde{g}_r^k$ converges to $\tilde{g}_r$ in $B_r \times B_r \setminus \{x = y\}$. Moreover $\tilde{g}^k(x, a) - \tilde{g}_r^k(x, a)$ is a «good» solution since the two normalized Green's functions have the same pole). If $w(x)$ is never zero on $\partial B_r(a)$ it has constant sign (say it is > 0) and therefore by the maximum principle it will be $\geqslant 0$ in all $B_r(a)$ in which case Harnack's inequality will imply $w(x) \equiv 0$ (since $w(a) = 0$). Then we have $\tilde{g}(a, a) - \tilde{g}(x_0, a) = \tilde{g}_r(a, a)$ which combined with (9) gives $m_r \leqslant \tilde{g}_r(a, a) \leqslant M_r$. Since $m_r \sim M_r$ our claim and the theorem are proved.

In order to prove the results in the next section we will need the following Caccioppoli type theorem.

THEOREM 12. – *Let $u \in C^\infty(D)$ and $Lu = 0$ and $w \geqslant 0$ such that $L^* w \leqslant 0$ in the weak sense (i.e. $\int_D wL\varphi \leqslant 0$ for every $\varphi \in C_0^\infty(D)$, $\varphi \geqslant 0$).*

Then $\int_{B_R} w(y) |\nabla u|^2(y)\, dy \leqslant C/R^2 \int_{B_{2R}} w(y)\, u^2(y)\, dy$ where $B_{2R} \subsetneq D$ and the constant C depends only on λ, Λ and n.

PROOF. – First of all we have that

$$L(u^2) = 2u\,Lu + 2 \sum_{i,j=1}^n a_{ij} D_i u\, D_j u = 2 \sum_{i,j=1}^n a_{ij} D_i u\, D_j u \geqslant \lambda |\nabla u|^2.$$

In what follows C is a constant that depends only on n.

Let $\varphi \in C_0^\infty(B_{2R})$, be such that $0 \leqslant \varphi \leqslant 1$, $\varphi \equiv 1$ on B_R, $|\nabla \varphi| \leqslant C/R$ and $|D_{ij}^2 \varphi| \leqslant C/R^2$. (Observe that φ^2 satisfies the same estimates). Then using the hypothesis on φ and on w, the ellipticity of L and Hölder's inequality it follows that

$$\lambda \int_{B_{2r}} w(y)\, \varphi^2 |\nabla u|^2 dy \leqslant \int_{B_{2R}} w(y)\, L(u^2)\, \varphi^2 dy =$$

$$= \left(\text{using } L(u^2 \varphi^2) = u^2 L\varphi^2 + \varphi^2 L u^2 + 4u\varphi \sum_{i,j=1}^n a_{ij} D_i u\, D_j \varphi \right) =$$

$$= \int_{B_{2R}} w(y)\, L(u^2\varphi^2)\, dy - \int_{B_{2R}} w(y)\, u^2 L(\varphi^2)\, dy - 4 \int_{B_{2R}} w(y)\, u\varphi \sum_{i,j=1}^{n} a_{ij} D_i u\, D_j \varphi\, dy \leqslant$$

$$\leqslant \frac{C}{R^2} \int_{B_{2R}} w(y)\, u^2(y)\, dy + 4\Lambda \int_{B_{2R}} w(y)\, u\varphi\, |\nabla u|\, |\nabla \varphi|\, dy \leqslant$$

$$\leqslant \frac{C}{R^2} \int_{B_{2R}} w(y)\, u^2(y)\, dy + 4\Lambda \left(\int_{B_{2R}} w(y)\, \varphi^2\, |\nabla u|^2 dy \right)^{1/2} \left(\int_{B_{2R}} w(y)\, u^2\, |\nabla \varphi|^2 dy \right)^{1/2} \leqslant$$

$$\leqslant \frac{C}{R^2} \int_{B_{2R}} w(y)\, u^2(y)\, dy + 4\Lambda\varepsilon \int_{B_{2R}} w(y)\, \varphi^2\, |\nabla u|^2 dy + \frac{4\Lambda C}{\varepsilon R^2} \int_{B_{2R}} w(y)\, u^2(y)\, dy\,.$$

And choosing ε small enough and observing that

$$\int_{B_{2R}} w(y)\, |\nabla u|^2(y)\, dy \leqslant \int_{B_{2R}} w(y)\, \varphi^2\, |\nabla u|^2 dy$$

the theorem is proved.

5. – The case of one point discontinuity: $\tilde{g}(0) < \infty$.

In this paragraph we are going to prove more results about solutions in the case of coefficients with a single point of discontinuity. Recall that $\tilde{g}(x) \equiv \tilde{g}(x, 0)$ and $\tilde{g}(0) = \lim_{x \to 0} \tilde{g}(x)$.

THEOREM 13. – *Let L be as in Theorem 3, $B = B_1(0)$ and assume $\tilde{g}(0) = 1$. Then there exists a unique $u \in C^2(B \setminus \{0\}) \cap C(\overline{B})$ satisfying*

$$\text{(i)} \quad \begin{cases} Lu = 0 & \text{in } B \setminus \{0\}\,, \\ u = \varphi & \partial B\,, \end{cases}$$

$$\text{(ii)} \quad \int_B \frac{|\nabla u|^2}{1 - \tilde{g}(y)}\, g(0, y)\, dy < \infty\,,$$

where the constant C depends only on λ, Λ and n.

PROOF. – We are going to show that the u that satisfies (i) and (ii) is exactly the «good» solution to $Lu = 0$ in B, $u = \varphi$ on ∂B. More precisely we are going to show in Proposition 1 that if u is a «good» solution then (i) and (ii) are satisfied. Moreover such a u is unique. Suppose u_1 and u_2 are two functions satisfying (i); then by the maximum principle $u_1(x) - u_2(x) = c\, \tilde{g}(x)$. We will show in Proposition 2 that for $u = \tilde{g}$ the integral in (ii) is infinite.

PROPOSITION 1. – *If u is a «good» solution to $Lu = 0$ in B, $u = \varphi$ on ∂B, with L as in Theorem 13 then u satisfies condition* (i) *in Theorem 13 and moreover*

$$\int_B \frac{|\nabla u|^2}{1-\tilde{g}(y)} g(0, y)\, dy \leq C\|\varphi\|_\infty^2 ,$$

where the constant C depends only on λ, Λ and n.

PROOF. – Condition (i) is satisfied since the coefficients of L are smooth in $B\setminus\{0\}$. Recall from Theorem 3 that

$$(1-\tilde{g}(x)) \sim \int_0^{|x|} \frac{s}{\omega(B_s(0))} ds, \qquad \text{where } \omega(E) = \int_E G(P, y)\, dy .$$

Let $L^k = a_{ij}^k(x) D_{ij}^2$ be a smooth approximation of L and let ω^k be the analogue of ω for L^k. Define $\gamma^k(r) = \int_0^r s/(\omega^k(B_s(0)) + \delta)\, ds$ and for every $\varepsilon > 0$ the auxiliary functions

$$v_\varepsilon^k(x) = \begin{cases} \dfrac{1}{\gamma^k(|x|)} & \text{for } |x| > \varepsilon \\[2ex] \dfrac{1}{2[\omega^k(B_\varepsilon) + \delta](\gamma^k(\varepsilon))^2}[\varepsilon^2 - |x|^2] + \dfrac{1}{\gamma^k(\varepsilon)} & \text{for } |x| \leq \varepsilon . \end{cases}$$

It is easy to verify that $v_\varepsilon^k \in C^{1,1}(\overline{B})$ and that for $|x| \leq \varepsilon$ it is $v_\varepsilon^k(x) \geq 1/\gamma^k(\varepsilon)$.

The following estimates also hold:

for $|x| \leq \varepsilon$

$$|\nabla v_\varepsilon^k| \leq \frac{\varepsilon}{\omega^k(B_\varepsilon) + \delta}[\gamma^k(\varepsilon)]^{-2} \qquad \text{and} \qquad |Lv_\varepsilon^k| \leq \frac{C}{\omega^k(B_\varepsilon) + \delta}[\gamma^k(\varepsilon)]^{-2}$$

and for $|x| > \varepsilon$

$$|\nabla v(x)_\varepsilon^k| \leq \frac{|x|}{\omega^k(B_{|x|}) + \delta}[\gamma^k(x)]^{-2},$$

$$Lv_\varepsilon^k \geq -L\gamma^k[\gamma^k(x)]^{-2} \qquad \text{and} \qquad |L\gamma^k(x)| \leq \frac{1}{\omega^k(B_{|x|}) + \delta}.$$

The latter is a consequence of the fact that if v is an adjoint solution, then $\int_{B_r} v(y)\, dy \sim r \int_{\partial B_r} v(y)\, dy$. (To see this, integrate by parts $\int_{B_r} v(y) L(|y|^2 - r^2)\, dy$).

Also observe that $\gamma^k(2r) \leq 4\gamma^k(r)$.

Now let $L^k u^k = 0$ in B, $u^k = \varphi$ on ∂B; then

$$\int_{|y| \geq \varepsilon} \frac{|\nabla u^k|^2}{\gamma^k(y)} g^k(0, y)\, dy \leq C \int_B L^k(u^k - u^k(0))^2 \nu_\varepsilon^k(y) g^k(0, y)\, dy =$$

$$= C \left\{ \int_B L^k[(u^k - u^k(0))^2 \nu_\varepsilon^k](y) g^k(0, y)\, dy - \int_B (u^k - u^k(0))^2(y) L^k \nu_\varepsilon^k g^k(0, y)\, dy - \right.$$

$$\left. - \int_B 4(u^k - u^k(0))\, a_{ij}^k(y) D_i u^k D_j \nu_\varepsilon^k g^k(0, y)\, dy \right\} = C\{I + II + III\}.$$

Using the representation formula for solutions we obtain:

$$|I| \leq \int_{\partial B} |u^k(Q) - u^k(0)|^2 \nu_\varepsilon^k(Q) \frac{\partial g^k}{\partial \mathrm{n}_Q}(0, y)\, d\sigma(Q) \leq \frac{4}{\displaystyle\int_0^1 \frac{s}{\omega^k(B_s) + \delta}\, ds} \|\varphi\|_\infty^2,$$

where $\partial/\partial \mathrm{n}_Q$ is the exterior conormal derivative.

Also by using the previous estimates:

$$II \leq \sum_{j=0}^{K(\varepsilon)} \int_{B_{1/2^j} \setminus B_{1/2^{j+1}}} |u^k(y) - u^k(0)|^2 \frac{[\gamma^k(|y|)]^{-2}}{\omega^k(B_{|y|})} g^k(0, y)\, dy +$$

$$+ \int_{B_\varepsilon} |u^k(y) - u^k(0)|^2 \frac{[\gamma^k(|y|)]^{-2}}{\omega^k(B_\varepsilon)} g^k(0, y)\, dy \leq$$

$$\leq \sum_{j=0}^{K(\varepsilon)} \sup_{B_{1/2^j}} |u^k(y) - u^k(0)|^2 \frac{[\gamma^k(2^j)]^{-2}}{\omega^k(B_{1/2^j})} g^k(0, B_{1/2^j}) +$$

$$+ \sup_{B_\varepsilon} |u^k(y) - u^k(0)|^2 \frac{[\gamma^k(\varepsilon)]^{-2}}{\omega^k(B_\varepsilon)} g^k(0, B_\varepsilon),$$

where $g(x, E) = \int_E g(x, y)\, dy$ and $K(\varepsilon)$ is chosen in such a way that $2^{-K(\varepsilon)-1} < \varepsilon$.

Finally by Theorem 12, together with the doubling property of $g(x, \cdot)$ (see [3]) Schwartz's inequality and previous estimates we also have that $III \leq$ last term of the previous chain of inequalities.

Letting now k tend to infinity and applying Theorem 3 and Theorem 10 we obtain that:

$$\int_{|y|\geq\varepsilon}\frac{|\nabla u|^2}{\gamma(y)}g(0,y)\,dy\leq$$

$$\leq C\left\{\frac{1}{\int_0^1\frac{s}{\omega(B_s)+\delta}ds}+\sum_{j=1}^{K(\varepsilon)}2^{-2j\gamma}\frac{[1-\tilde g(2^{-j})]^2}{\left[\int_0^{2^{-j}}\frac{s}{\omega(B_s)+\delta}ds\right]^2}+\varepsilon^{2\gamma}\frac{[1-\tilde g(\varepsilon)]^2}{\left[\int_0^{\varepsilon}\frac{s}{\omega(B_s)+\delta}ds\right]^2}\right\}\|\varphi\|^2.$$

Letting then δ tend to 0 we conclude

$$\int_{|y|\geq\varepsilon}\frac{|\nabla u|^2}{(1-\tilde g(y))}g(0,y)\,dy\leq C\|\varphi\|_\infty^2\left\{\frac{1}{\tilde g(1)}+\sum_{j=0}^{K(\varepsilon)}2^{-2j\gamma}+\varepsilon^{2\gamma}\right\}.$$

Finally let ε tend to 0 and the Proposition is proved.

PROPOSITION 2. – *In the hypothesis of Theorem* 13 *we have that*

$$\int_B\frac{|\nabla\tilde g|^2}{1-\tilde g(y)}g(0,y)\,dy=\infty\,.$$

PROOF. – Remember $\tilde g(y)\equiv\tilde g(y,0)$. Let $E=\{t\in(0,1)\,|\,t$ is a critical value for $\tilde g\}$. By Sard's theorem $|E|=0$. For $t\in(0,\infty)\setminus E$, $\tilde g^{-1}(t)=\{y\in B\,|\,\tilde g(y)=t\}$ is a smooth C^∞ manifold. Therefore the following computations are justified: integrate by parts observing that the normal to the set $\tilde g(x)=t$ is given by $(\nabla\tilde g/|\nabla\tilde g|)(x)$ and recalling that $\tilde g(x)=0$ on ∂B and obtain

$$0=\int_{0<\tilde g(x)<t}g(0,x)L\tilde g\,dx=$$

$$=\int_{0<\tilde g(x)<t}D_i[a_{ij}(x)g(0,x)D_j\tilde g]\,dx-\int_{0<\tilde g(x)<t}D_i[a_{ij}(x)g(0,x)]D_j\tilde g\,dx=$$

$$=\int_{\tilde g(x)=t}a_{ij}(x)D_j\tilde g\frac{D_i\tilde g}{|\nabla\tilde g|}g(0,x)\,dS_x-\int_{0<\tilde g(x)<t}D_j[D_i[a_{ij}(x)g(0,x)]\tilde g(x)]\,dx=$$

$$= \int_{\tilde{g}(x)=t} a_{ij}(x) D_j \tilde{g} \frac{D_i \tilde{g}}{|\nabla \tilde{g}|} g(0, x)\, dS_x + \int_{|x|=1} x_j D_j [a_{ij}(x) g(0, x)] \tilde{g}(x)\, dx -$$

$$- \int_{\tilde{g}(x)=t} D_i (a_{ij}(x) g(0, x)) \tilde{g}(x) \frac{D_j \tilde{g}}{|\nabla \tilde{g}|} dS_x .$$

Hence,

$$\int_{\tilde{g}(x)=t} a_{ij}(x) D_j \tilde{g} \frac{D_i \tilde{g}}{|\nabla \tilde{g}|} g(0, x)\, dS_x = t \int_{\tilde{g}(x)=t} D_i [a_{ij}(x) g(0, x)] \frac{D_j \tilde{g}}{|\nabla \tilde{g}|} dS_x . \tag{11}$$

Also,

$$0 = \int_{0 < \tilde{g}(x) < t} D^2_{ij} [a_{ij}(x) g^k(0, x)]\, dx =$$

$$= \int_{|x|=1} x_j D_i [a_{ij}(x) g(0, x)]\, dx + \int_{\tilde{g}(x)=t} D_i [a_{ij}(x) g(0, x)] \frac{D_j \tilde{g}}{|\nabla \tilde{g}|} dS_x .$$

By developing the derivatives this implies that the integral on the right hand side of (11) equals 1 and therefore

$$\int_{1-\tilde{g}(x)=t} a_{ij}(x) D_j \tilde{g} \frac{D_i \tilde{g}}{|\nabla \tilde{g}|} g(0, x)\, dS_x = 1 - t \quad \text{for a.e. } t \in [0, 1] .$$

By applying the co-area formula ([8]) we conclude that

$$\int_{\varepsilon \le 1 - \tilde{g}(x) \le 1} a_{ij}(x) D_i \tilde{g} D_j \tilde{g} \frac{g(0, x)}{1 - \tilde{g}}\, dx = \int_{\varepsilon}^{1} \frac{1-t}{t}\, dt = -\ln \varepsilon - 1 + \varepsilon$$

and letting $\varepsilon \to 0$ the theorem is proved.

6. – The case of one point discontinuity: $\tilde{g}(0) = \infty$.

THEOREM 14. – *Let L and B be as in Theorem 3 and assume $\tilde{g}(0, 0) = \infty$. Then there exists a unique $u \in C^2(B \setminus \{0\})$ satisfying*

(i) $\begin{cases} Lu = 0 & \textit{in } B \setminus \{0\} , \\ u = \varphi & \textit{on } \partial B , \end{cases}$

(ii) $\sup \int_B |\nabla u|^2 g(x, y)\, dy < \infty .$

PROOF. – We are going to show as in the previous section that this u is precisely the «good» solution to $Lu = 0$ in B, $u = \varphi$ on ∂B.

If u is the «good» solution to this problem it is easy to seen that (ii) is verified, since if $u^k \to u$ with u^k solutions to the regularized operators, the following holds independently of k

$$\int_B |\nabla u^k|^2 g^k(x, y)\, dy \leq C \int_B L^k(u^k(y)^2)\, g^k(x, y)\, dy \leq$$

$$\leq C \int_{\partial B} |u^k(Q)|^2 \frac{\partial g^k}{\partial \mathfrak{n}_Q}(x, y)\, d\sigma(Q) \leq C\|\varphi\|_\infty^2 .$$

Now assume that $u \in C^2(B \setminus \{0\})$ and $Lu = 0$ in $B \setminus \{0\}$ and (ii) holds.

If u is bounded u is the «good» solution by Remark 1.

If u is unbounded and has constant sign we can repeat the same argument in the proof of Proposition 2 to the integral $\int_{t < u(y) < s} g(x_0, y)\, Lu\, dy$ and obtain that

$$\int_{u(y) = t} a_{ij}(y) \frac{D_j u\, D_i u}{|\nabla u|} g(x_0, y)\, dS_y = \int_{u(y) = s} a_{ij}(y) \frac{D_j u\, D_i u}{|\nabla u|} g(x_0, y)\, dS_y = C,$$

for $t < s$ and x_0 such that $u(x_0) \leq t$. Then by the coarea formula again $\int_{u(y) > t} |\nabla u|^2 g(x_0, y)\, dy \sim \int_t^\infty C\, ds = \infty$.

Finally assume u changes sign and $\limsup_{x \to 0} \int_B |\nabla u|^2 g(x, y)\, dy < \infty$; we will show that in this case u is bounded and therefore we are back in the first case. First of all by the maximum principle for every $s < 1$ there exists x_s with $|x_s| = s$ and $u(x_s) = 0$.

Let $w(x) = u^2(x) + \int_{t/2 \leq |y| \leq (3/2)t} g(x, y)\, L[u^2(y)]\, dy$. On $t/2 < |x| < (3/2)t$ we have that $Lw = L(u^2) - L(u^2) = 0$, $w \in C^\infty(B)$. Clearly $0 \leq u^2(x) \leq w(x)$ in the same anulus. Therefore by Harnack's inequality (applied to w on $|x| = t$) we have

$$u^2(x) \leq w(x) \leq \vartheta w(x_t) =$$

$$= \vartheta \int_{t/2 \leq |y| \leq (3/2)t} g(x_t, y)\, L[u^2(y)]\, dy \leq C \int_{t/2 \leq |y| \leq (3/2)t} g(x_t, y)\, |\nabla u|^2\, dy$$

and the last term is bounded by hypothesis.

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