Algorithmica (1992) 7: 121–135

Algorithmica
© 1992 Springer-Verlag New York Inc.

A Proof of the Gilbert–Pollak Conjecture on the Steiner Ratio

D.-Z. Du[1] and F. K. Hwang[2]

Abstract. Let P be a set of n points on the euclidean plane. Let $L_s(P)$ and $L_m(P)$ denote the lengths of the Steiner minimum tree and the minimum spanning tree on P, respectively. In 1968, Gilbert and Pollak conjectured that for any P, $L_s(P) \geq (\sqrt{3}/2)L_m(P)$. We provide a proof for their conjecture in this paper.

Key Words. Steiner trees, Spanning trees, Steiner ratio, Convexity, Hexagonal trees.

1. Introduction. Consider a set P of n points on the euclidean plane. A shortest network interconnecting P must be a tree, which is called a *Steiner minimum tree* and denoted by $SMT(P)$. An $SMT(P)$ may contain vertices not in P. Such vertices are called *Steiner points*, while vertices in P are called *regular points*. Computing $SMT(P)$ has been shown to be an NP-hard problem [6]. Therefore, it merits the study of approximate solutions. A spanning tree on P is just a tree with vertex set P. A shortest spanning tree on P is also called a *minimum spanning tree* on P, denoted by $MST(P)$. The *Steiner ratio* is defined to be

$$\rho = \inf\{L_s(P)/L_m(P) \mid P\},$$

where $L_s(P)$ and $L_m(P)$ are lengths of $SMT(P)$ and $MST(P)$, respectively. Since computing $MST(P)$ is fast, $MST(P)$ can be used as an approximate solution of $SMT(P)$. In this case, the Steiner ratio is a measure for the performance of $MST(P)$ as an approximation. Gilbert and Pollak [7] conjectured $\rho = \sqrt{3}/2$, and verified it for $n = 3$. The conjecture was then verified by Pollak [11] for $n = 4$, by Du, Hwang, and Yao [5] for $n = 5$, and by Rubinstein and Thomas [13] for $n = 6$. Along another line, the lower bound of ρ for general n has been pushed up from 0.5 (by Moore as reported in [7]) to 0.57 by Graham and Hwang [8], to 0.74 by Chung and Hwang [3], to 0.8 by Du and Hwang [4], and to 0.824 by Chung and Graham [2]. In either the small n exact result or the general n lower bound case,

[1] Department of Computer Science, Princeton University, and DIMACS (Center for Discrete Mathematics and Theoretical Computer Science), a National Science Foundation Science and Technology Center, supported by NSF under grant STC88-09648. Also from Institute of Applied Mathematics, Academia Sinica, Beijing, China, and supported in part by the National Natural Science Foundation of China.
[2] AT&T Bell Laboratories, Murray Hill, NJ 07974, USA.

Received April 20, 1990; revised June 22, 1990, August 9, 1990, and December 18, 1990. Communicated by F. K. Hwang.

the lack of further progress was caused by a fast growth of computation load. In this paper, we will prove $\rho = \sqrt{3}/2$ without requiring much computation.

It is well known [7] that an $SMT(P)$ must satisfy the following conditions:

(1) All leaves are regular points.
(2) Any two edges meet at an angle of at least 120°.
(3) Every Steiner point has degree exactly three.

Conditions (2) and (3) together imply that every Steiner point is incident to exactly three edges; any two of them must meet at an angle of 120°.

A tree interconnecting P and satisfying (1), (2), and (3) is called a *Steiner tree* (ST). Its topology (the graph structure of the network) is called a *Steiner topology*. An ST for n points can contain at most $n-2$ Steiner points. If an ST has exactly $n-2$ Steiner points, then it is called a *full* ST and its topology a full topology. Any ST T can be decomposed into an edge-disjoint union of smaller full ST's, which are called *full subtrees* of T. The topologies of full subtrees are called *full subtopologies* of the topology of T.

2. Inner Spanning Trees. An ST T can be determined by its topology t and at most $2n-3$ parameters, including all edge lengths of T and all angles at regular points of degree 2 in T. When writing all parameters into a vector x, the ST T is denoted by $t(x)$. Usually, every edge length is positive. However, for simplicity of discussion, we allow the edge length to be zero; in this case, $t(x)$ can be seen as a limiting ST with topology t and usual parameter vector y as y goes to x. Note that, throughout this paper, a parameter vector is said to exist for a point set and a topology if either an ST with the topology for the point set exists or its limit exists.

Consider a full topology t and a full ST T of topology t. Let $S_1 \cdots S_k$ be a path in T. The path is said to be *convex* if it contains only one or two segments or if for any $i = 1, \ldots, k-3$, the segment S_iS_{i+3} does not cross the piece $S_iS_{i+1}S_{i+2}S_{i+3}$ of the path. Two regular points are said to be *adjacent* in t if there is a convex path connecting them in T. For a Steiner topology t, two regular points are said to be *adjacent* if they are adjacent in a full subtopology of t. Given a full topology t and a parameter vector x, connecting every pair of adjacent regular points in $t(x)$, we obtain a polygon. If this polygon is simple, i.e., not self-intersecting, then it bounds an area containing the tree $t(x)$. This area is called the *characteristic area* of t at point x. If the polygon is not simple, then there must exist a pair of adjacent regular points A and B such that the segment AB intersects the tree $t(x)$. In this case, we put the tree $t(x)$ into a spiral surface (Figure 1) such that the segment AB does not intersect the tree $t(x)$ and such that AB and the convex path between A and B form a simple closed polygon bounding a simply connected region. The characteristic area of t at x is now defined to be the union of such regions. So, in general, the characteristic area is on a Riemann surface rather than a plane. All segments between adjacent regular points form a simple polygon on such a surface, bounding the characteristic area. For a Steiner topology

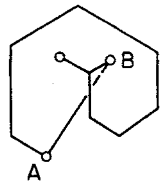

Fig. 1

t not necessarily full, the characteristic area $C(t; x)$ of t at t can be defined in a similar way. Clearly, it is the union of characteristic areas of full subtopologies of t.

Given a Steiner topology t and a parameter vector x, the set of regular points on tree $t(x)$ is denoted by $P(t; x)$. A spanning tree on $P(t; x)$ is called an *inner spanning tree* for t at x if it lies in the area $C(t; x)$. It is important to remember that the vertices of an inner spanning tree for t at x all lie on the boundary of $C(t; x)$. It should also be noticed that we may have $C(t; x) \neq C(t'; x')$ for $t(x) = t'(x')$. However, for a full ST T, there exists a unique pair of a topology t and a parameter vector x such that $T = t(x)$. Thus, there is no confusion when we talk about a characteristic area or an inner spanning tree for a full ST.

Let $l(T)$ denote the length of the tree T. In this paper, we will show the following theorem:

THEOREM 1. *For any Steiner topology t and parameter vector x, there is an inner spanning tree N for t at x such that $l(t(x)) \geq (\sqrt{3}/2)l(N)$.*

Clearly, the Gilbert–Pollak conjecture is a corollary of the above theorem.

Sometimes, we speak about the inner spanning tree without mentioning an topology t and a parameter vector x when the intended t and x are clear. An inner spanning tree is called a *minimum inner spanning tree* if it has the minimum length over all inner spanning trees for the given topology at the given point.

Given a Steiner topology t, let X_t be the set of parameter vectors x such that $l(t(x)) = 1$. Note that if a component x_i of x is an angle then x_i is restricted by $120° \leq x_i \leq 240°$. It is easy to see that X_t is a compact set. Let $L_t(x)$ denote the length of the minimum inner spanning tree for $t(x)$. An important property is given in the following lemma.

LEMMA 1. *$L_t(x)$ is a continuous function with respect to x.*

Before proving Lemma 1, let us introduce some more notation. For a topology s of spanning tree, denote by $s(t; x)$ the spanning tree on the point set $P(t; x)$ with topology s. For each Steiner topology t and each parameter vector x, let $I(t; x)(MI(t; x))$ be the set of spanning tree topologies s such that $s(t; x)$ is an (a minimum) inner spanning tree for t at x. Lemma 1 is a consequence of the following two lemmas.

LEMMA 2. *If $m \in MI(t; x)$, then there exists a neighborhood of x such that for any point y in the neighborhood, $m \in I(t; y)$.*

PROOF. For contradiction, suppose such a neighborhood does not exist. Then, there is a sequence of points y_k converging to x such that $m \notin I(t; y_k)$. Thus, every $m(t; y_k)$ has at least one edge not in the characteristic area $C(t; y_k)$. Since the number of edges is finite in $m(t; x)$, there exists a subsequence of $m(t; y_k)$ each of which contains an edge not in $C(t; x)$, but these edges converge to an edge AB in $m(t; x)$. It is easy to see that AB is on the boundary of the area $C(t; x)$ and that A and B are not adjacent. (An edge between two adjacent regular points always lies in the characteristic area.) Since all vertices in an inner spanning tree lie on the boundary of $C(t; x)$, there is a regular point lying in the interior of the segment AB, contradicting the minimality of $m(t; x)$. □

LEMMA 3. *For every x, there is a neghborhood of x such that for any y in the neighborhood, $MI(t; y) \subseteq MI(t; x)$.*

PROOF. For contradiction, suppose that there is a sequence of points y_k converging to x such that for each y_k, a spanning tree topology m_k exists such that $m_k \in MI(t; y_k)\backslash MI(t; x)$. Since the number of spanning tree topologies is finite, there is a subsequence of points $y_{k'}$ such that for any $y_{k'}$, $m_{k'} = m$ for a certain spanning tree topology m. We can also assume that this subsequence lies inside of the neighborhood of x as described in Lemma 2. Thus, for each k', $l(m(t; y_{k'})) \leq l(m'(t; y_{k'}))$ for all $m' \in MI(t; x)$ since by Lemma 2 $m'(t; y_{k'}) \in I(t; y_{k'})$. Letting $k' \to \infty$, we obtain that $l(m(t; x)) \leq l(m'(t; x))$ for all $m' \in MI(t; x)$. Since $m \notin MI(t; x)$, $m(t; x)$ must not be an inner spanning tree. It follows that there exists a neighborhood of x such that for any point y in the neighborhood, $m(t; y)$ is not an inner spanning tree for $t(y)$, contradicting the existence of the subsequence of points $y_{k'}$.

PROOF OF LEMMA 1. By Lemma 3, there is a neighborhood of x in which $L_t(y) = \min_{m \in MI(t;x)} l(m(t; y))$. Thus, it is continuous at x. □

Define a function $f_t: X_t \to R$ by setting $f_t(x) = 1 - (\sqrt{3}/2)L_t(x)(= l(t(x)) - (\sqrt{3}/2)L_t(x))$. By Lemma 1, $f_t(x)$ is continuous and hence reaches the minimum in X_t. Let $F(t)$ denote the minimum value of $f_t(x)$ over all $x \in X_t$. Note that every ST is similar to an ST with length one. Thus, Theorem 1 holds iff for any Steiner topology t, $F(t) \geq 0$.

We prove Theorem 1 by contradiction. Suppose that Theorem 1 is not true and that n is the smallest number of points such that Theorem 1 does not hold. Let $F(t^*)$ be the minimum of $F(t)$ over all Steiner topologies t. Then, $F(t^*) < 0$. Some important properties of t^* are given in the following two lemmas.

LEMMA 4. *t^* is a full topology.*

PROOF. If t^* is not a full topology, then for every $x \in X_{t^*}$, the ST $t^*(x)$ can be decomposed into edge-disjoint union of several ST T_i's. Let $T_i = t_i(x(i))$ where t_i is the corresponding full subtopology of t and $x(i)$ is the parameter which T_i has under topology t_i. Since every T_i has less than n regular points, we can apply Theorem 1 to find an inner spanning tree m_i for t_i at $x(i)$ such that $l(T_i) \geq$

$(\sqrt{3}/2)l(m_i)$. Note that $\bigcup_i C(t_i; x(i)) \subseteq C(t; x)$. So, the union m of m_i is an inner spanning tree for t^* at x. Moreover,

$$l(t^*(x)) = \sum_i l(T_i) \geq (\sqrt{3}/2)l(m_i) = \; = (\sqrt{3}/2)l(m).$$

Therefore, for any $x \in X_{t^*}$, $f_t(x) \geq 0$. Thus, $F(t^*) \geq 0$, contradicting $F(t^*) < 0$. □

Since t^* is a full topology, every component of the parameter vector x in X_{t^*} is an edge length of the ST $t^*(x)$. A full topology t is said to be a *companion* of t^* if two regular points are adjacent in t iff they are adjacent in t^*. A point $x \in X_{t^*}$ is called a *minimum point* if $f_{t^*}(x) = F(t^*)$.

LEMMA 5. *Let x be a minimum point. Then, $x > 0$, that is, every component of x is positive.*

PROOF. Suppose to the contrary that x has zero components. If there is a zero component corresponding to the length of an edge incident to a regular point, then a contradiction can be derived by an argument similar to that given in the proof of Lemma 4. So, all zero components are lengths of edges between Steiner points. In this case, it is easy to find a full topology t satisfying the following conditions (see Figure 2):

(1) t is a companion of t^*.
(2) There is a tree T interconnecting the n points in the set $P(t^*; x)$, with the full topology t and with length less than $l(t^*(x))$.

If the ST of topology t for the point set $P(t^*; x)$ exists, then there exists a parameter vector y such that $P(t; y) = P(t^*; x)$. Let $h = l(t^*(x))/l(t(y))$. Clearly, $h > 1$ since $l(t(y)) \leq l(T) < l(t^*(y))$. Note that $t(hy)$ is similar to $t(y)$. Hence, $f_t(hy) = 1 - (\sqrt{3}/2)L_t(hy) = 1 - (\sqrt{3}/2)hL_t(y) = 1 - (\sqrt{3}/2)hL_{t^*}(x) < f_{t^*}(x) = F(t^*)$. Since $hy \in X_t$, we have $F(t) \leq f_t(hy) < F(t^*)$, contradicting the minimality of $F(t^*)$.

If the ST of topology t for the point set $P(t^*; x)$ does not exist, then we cannot use the above argument directly since $f_t(y)$ is undefined. (Remember that $F(t^*)$ is a minimum over all Steiner topologies. So even though T is a shorter tree, there is no contradiction to the minimality of $F(t^*)$.) Now, we consider any tree of topology t interconnecting all regular points. Such a tree can be determined by edge lengths and angles at every Steiner point. Write the lengths into a length

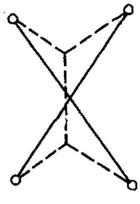

Fig. 2

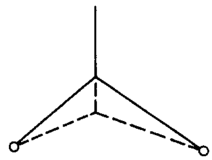

Fig. 3

vector y and the angles into an angle vector θ. Such a tree can be denoted by $t(y, \theta)$. Constructing the characteristic area for t at (y, θ) by connecting every adjacent pair of regular points, we can define the inner spanning tree and the minimum inner spanning tree for t at (y, θ) in a similar way. Let $L_t(y, \theta)$ denote the length of the minimum inner spanning tree for t at (y, θ). We can show the continuity of $L_t(y, \theta)$ by an argument similar to that in Lemmas 1–3. Restrict all angles to be in between 0° and 360° and the sum of any three angles at the same Steiner point to equal 360°. Let Y_t be the set of vectors (y, θ) with the described restrictions on θ and the restrictions $\sum y_i = 1$ and $y \geq 0$ on y. Then Y_t is compact. So, the function g defined by $g_t(y, \theta) = 1 - (\sqrt{3}/2)L_t(y, \theta)$ reaches its minimum in Y_t. We denote this minimum value by $G(t)$. By an argument similar to that in the last paragraph, we can prove that $G(t) < F(t^*)$.

Now, suppose that $g_t(y, \theta) = G(t)$. Let us study properties of the tree $t(y, \theta)$. First, we claim that two nonzero edges corresponding to two adjacent edges in t form an angle of at least 120°. In fact, if there are two such edges with an angle of less than 120°, then we can find a tree $t(y', \theta')$ such that $l(t(y', \theta')) < l(t(y, \theta))$ (see Figure 3). This will imply a contradiction to the minimality of $G(t)$. Next, we claim that y has a zero component. Indeed, if y has no zero component, then $t(y, \theta)$ must be a full ST and hence $G(t) = F(t)$. This leads to $F(t) < F(t^*)$, a contradiction.

Consider the subgraph of t induced by edges corresponding to zero components of y. If every connected component of the subgraph having an edge contains a regular point, then we decompose the tree $t(y, \theta)$ into edge-disjoint union of several smaller full ST's. By an argument similar to that given in the proof of Lemma 4, we can find a full topology t' with fewer regular points such that $G(t') < 0$. If there exists such a component which contains no regular point, then in a way as shown in Figure 2, we can find a full topology t', which is a companion of t, such that $G(t') < G(t)$. Repeating the above argument, we will obtain infinitely many full topologies with at most n regular points, contradicting the finiteness of the number of full topologies. □

3. Convexity. In this section, we present a key lemma. It is Lemma 7 that is obtained from the convexity of the length of a spanning tree with respect to the parameter vector.

LEMMA 6. *Let t be a full topology and s a spanning tree topology. Then $l(s(t; x))$ is a convex function with respect to x.*

PROOF. Let A and B be two regular points of the full topology t. We first show that the distance between A and B, $d(A, B)$ is a convex function of x.

Find the path in $t(x)$ which connects the points A and B. Suppose the path has k edges with lengths $x_{1'}, \ldots, x_{k'}$ and with directions $e_1, \ldots, e_k$, respectively, where $e_1, \ldots, e_k$ are unitary vectors in the order from A to B. It is easy to see that $d(A, B) = \|x_{1'}e_1 + \cdots + x_{k'}e_k\|$ where $\|\cdot\|$ is the euclidean norm and is a convex norm. Note that the part inside the norm is linear with respect to x. Thus, $d(A, B)$ is a convex function with respect to x.

Since the sum of convex function is also a convex function, it follows immediately from the above that $l(s(t; x))$ is a convex function of x. □

Lemma 5 tells us that every minimum point is an interior point of the simplex X_{t^*}. The next lemma gives another important property of a minimum point.

LEMMA 7. *Suppose that x is a minimum point and that y is a point in X_{t^*} satisfying $MI(t^*; x) \subseteq MI(t^*; y)$. Then, y is also a minimum point.*

PROOF. For any m in $MI(t^*, x)$, define $A(m) = \{z \in X_{t^*} | l(m(t^*; z)) \leq L_{t^*}(x)\}$. By Lemma 6, $A(m)$ is a convex region. We first claim that the union of all $A(m)$ for m in $MI(t^*; x)$ covers a neighborhood of x. In fact, if such a union does not cover any neighborhood of x, then in every neighborhood of x, we can find a point z such that $\min\{l(m(t^*; z)) | m \in MI(t^*; x)\} > L_{t^*}(x)$. However, by Lemma 3, we know that for z sufficiently close to x, $L_{t^*}(z) = \min\{l(m(t^*; z)) | m \in MI(t^*; x)\}$. Thus, there exists z in X_{t^*} such that $L_{t^*}(z) > L_{t^*}(x)$, so that $f_{t^*}(z) < f_{t^*}(x)$, contradicting that $f_{t^*}(x) = F(t^*)$.

Now, we show that $f_{t^*}(y) = F(t^*)$. Suppose to the contrary that $f_{t^*}(y) > F(t^*)$. Note that $MI(t^*; x) \subseteq MI(t^*; y)$. Thus, for every $m \in MI(t^*; x)$, $l(m(t^*; y)) < L_{t^*}(x)$. We claim that for all positive number c, the point $x + c(x - y)$ is not in $A(m)$ for every $m \in MI(t^*; x)$. In fact, if the point $x + c(x - y)$ for some positive c is in $A(m)$, then the point x as an interior point of the segment $[y, z]$ where $z = x + c(x - y)$ can be written as $x = \lambda y + (1 - \lambda)z$ and where $0 < \lambda = c/(1 + c) < 1$. By Lemma 6, we have

$$\begin{aligned} l(m(t^*; x)) &\leq \lambda l(m(t^*; y)) + (1 - \lambda) l(m(t^*; z)) \\ &< L_{t^*}(x), \end{aligned}$$

contradicting that $m \in MI(t^*; x)$. Finally, the fact that $x + c(x - y)$ for all $c > 0$ is not in every $A(m)$ for $m \in MI(t^*; x)$ contradicts that the union of all $A(m)$'s covers a neighborhood of x. □

Let x be a minimum point. Let y be a parameter vector for t^* but not necessarily in X_{t^*}, such that $MI(t^*; x) \subseteq MI(t^*; y)$ and $l(m(t^*; x)) = l(m(t^*; y))$ for $m \in MI(t^*; x)$. We remark that y is also a minimum point. To see this, note that there always exists a positive number h such that $hy \in X_{t^*}$. By Lemma 7, we have that $L_{t^*}(x) = L_{t^*}(hy) = h \cdot L_{t^*}(y)$. Thus, $h = 1$, so that $y \in X_{t^*}$. Therefore, y is a minimum

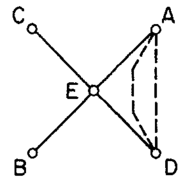

Fig. 4

point by using Lemma 7 again. In the next section, we will use this remark in some proofs.

4. Critical Structure.

Let t be a full topology and x a parameter vector. Denote by $\Gamma(t; x)$ the union of minimum inner spanning trees for $t(x)$. The following two lemmas are essentially variations of the lemmas for minimum spanning trees given by Rubinstein and Thomas [12]. They are helpful for determining the structure of $\Gamma(t; x)$.

LEMMA 8. *Two minimum inner spanning trees can never cross, i.e., edges meet only at vertices.*

PROOF. Suppose that AB and CD are two edges crossing at the point E (see Figure 4) and they belong to two minimum inner spanning trees W and U, respectively. Without loss of generality, assume that EA has a smallest length among the four segments EA, EB, EC, and ED. Removing the edge CD from the tree U, the remaining tree has two connected components containing C and D, respectively. Without loss of generality, assume that A is in the connected component containing C. Note that $l(AD) < l(EA) + l(ED) \leq l(CD)$. If the edge AD lies in the characteristic area, then using AD to connect the two components, we will obtain an inner spanning tree with length less than that of U, contradicting the minimality of U. If the edge AD does not lie in the characteristic area, there must exist some regular points lying inside of the triangle EAD. Consider the convex hull of those regular points and the two points A and D. The boundary of the convex hull other than the edge AD must lie in the characteristic area. This boundary contains a path from A to D. In this path, there exist two adjacent vertices which belong to different connected components of $U \setminus CD$. Connecting two such adjacent vertices, we can also obtain an inner spanning tree with length less than that of U, a contradiction. (Note: The distance of two points in a triangle is bounded by the longest edge of the triangle and hence bounded by the sum of any two edges of the triangle.) □

From Lemma 8, we can see that $\Gamma(t; x)$ divides the characteristic area $C(t; x)$ into smaller areas each of which is bounded by a polygon with vertices all being regular points. Such a polygon is called a polygon of $\Gamma(t; x)$, if it is a subgraph of $\Gamma(t; x)$.

LEMMA 9. *Every polygon of* $\Gamma(t; x)$ *has at least two equal longest edges.*

PROOF. Suppose to the contrary that $\Gamma(t; x)$ has a polygon Q with the unique longest edge e. Let m be the minimum inner spanning tree containing e. For every edge e' of Q not in m, the union of m and e' contains a cycle. If this cycle contains e, then adding e' and deleting e, we will obtain an inner spanning tree with length less than m, contradicting the minimality of m. Thus, such a cycle does not contain e. Hence, for every edge e' in Q not in m, m has a path connecting two endpoints of e' and not passing e. These paths and e form a cycle in m, a contradiction. □

When t is a full topology, the characteristic area of $t(x)$ is bounded by a polygon of n edges. Partitioning the area into $n - 2$ triangles by adding $n - 3$ diagonals, we will obtain a network with n vertices and $2n - 3$ edges. This network will be called a triangulation of $C(t; x)$. Let us first ignor the full ST $t(x)$ and consider the relationship between the vertex set and the length of edges. Note that in the previous discussion, when we say that a set P of points is given, we really mean that the distance between every two points in the set is given, that is, relative positions between those points have been given. With this understanding, we make the following observations:

(1) The vertex set (i.e., the set of regular points, $P(t; x)$) can be determined by $2n - 3$ edge lengths of the network.
(2) The $2n - 3$ edge-lengths are *independent* variables, that is, the network could vary by changing any edge-length and fixing all others.

Note that every $\Gamma(t; x)$ can be embedded in some triangulation of $C(t; x)$. Thus, all edge-lengths in $\Gamma(t; x)$ are independent.

A $\Gamma(t; x)$ is said to have a *critical structure* if $\Gamma(t; x)$ partitions $C(t; x)$ into exactly $n - 2$ equilateral triangles. Such a structure has the property that any perturbation would change the set of topologies of minimum inner spanning tree. A $\Gamma(t; x)$ with a critical structure is also said to be *critical.*

LEMMA 10. *Any minimum point* x *with the maximum number of minimum inner spanning trees has a critical* $\Gamma(t^*; x)$.

PROOF. If $\Gamma(t^*; x)$ is not critical, then one of the following must occur:

(a) $\Gamma(t^*; x)$ has a free edge, an edge not on any polygon of $\Gamma(t^*; x)$.
(b) $\Gamma(t^*; x)$ has a polygon of more than three edges.
(c) (a) and (b) do not occur, but $\Gamma(t^*; x)$ has a nonequilateral triangle.

We will show that in each case, the number of minimum inner spanning trees can be increased. First, assume that (a) occurs. Embedding $\Gamma(t^*; x)$ into a triangulation of $C(t^*; x)$ we can find a triangle containing the free edge e. Let e' be an edge of the triangle not in $\Gamma(t^*; x)$ such that in a minimum inner spanning tree containing e, removing e and adding e' will result in another inner spanning tree. (Such an edge e' must exist for, if the triangle has only one edge not in $\Gamma(t^*; x)$ then this edge must have the desired property; if the triangle has two edges not in $\Gamma(t^*; x)$, then the one which lies between the two connected components of the minimum inner spanning tree after removing e meets the requirement.) Clearly, $l(e) < l(e')$. Now we decrease the length of e' and fix all other edge-lengths in the triangulation. Let l be the length of the shrinking e'. At the beginning, $l = l(e')$. At the end, $l = l(e) < l(e')$. For each l, denote by $\bar{P}(l)$ the corresponding set of regular points. Then $\bar{P}(l(e')) = P(t^*, x)$. Consider the set L of all $l \in [l(e), l(e')]$ satisfying the condition that there is a minimum point y such that $\bar{P}(l) = P(t^*; y)$. Since the set of minimum points is a closed set and contains the point x, the set L is nonempty and closed. Therefore, there exists a minimal element l^* in L. Let $m \in MI(t^*; x)$ and $m' \in MI(t^*; y)$. Since both x and y are minimum points for t^*, $l(m(t^*; x)) = l(m'(t^*; y))$. Furthermore, since e is a free edge, decreasing e' does not effect the length of any edge in $\Gamma(t^*; x)$. Hence, $m \in MI(t^*; y)$ and $MI(t^*; x) \subseteq MI(t^*; y)$. Suppose $MI(t^*; y) = MI(t^*; x)$. Clearly, $l^* \neq l(e)$ since, when $l^* = l(e)$, dropping e and adding e' will give one more minimum inner spanning tree. By Lemma 5, y has no component being zero. This means that there exists a neighborhood of l^* such that for l in it, the ST of full topology t^* exists for the point set $\bar{P}(l)$. Thus, there exists an $l < l^*$ such that $\bar{P}(l) = P(t^*; z)$ for some length vector z. From the proof of Lemma 1, we know that there exists a neighborhood of y such that for y' in it, $MI(t^*; y') \subseteq MI(t^*; y)$. Thus, z can be chosen also to satisfy that $MI(t^*; z) \subseteq MI(t^*; y)$. Note that for every $m, m' \in MI(t^*; x)$, $l(m(t^*; z)) = l(m'(t^*; z))$ and that $MI(t^*; y) = MI(t^*; x)$. It follows that $MI(t^*; z) = MI(t^*; x)$. By the remark we made at the end of Section 3, z is also a minimum point, a contradiction to the assumption that l^* is minimum. Therefore, $MI(t^*; x) \subset MI(t^*; y)$ and $I(t^*; x)$ is not critical.

For the other two cases, we can give similar proofs by decreasing the length of an edge not in $\Gamma(t^*; x)$ in case (b) and by increasing the length of all shortest edges in $\Gamma(t^*; x)$ in case (c). □

Now, in order to derive a contradiction to $F(t^*) < 0$, it suffices to show that for any ST $t(x)$ with critical $\Gamma(t; x)$, Theorem 1 holds for $t(x)$.

Note that a critical $\Gamma(t; x)$ contains $n - 2$ equilateral triangles which form a framework fixing all regular points. Let a be the length of an edge of the equilateral triangles in the $\Gamma(t; x)$. If we divide the plane into a union of disjoint equilateral trangles with edge length a, then all regular points in a critical $\Gamma(t; x)$ can be placed on the lattice points. The following lemma is easy to prove.

LEMMA 11. *The minimum spanning tree for n lattice points has length at least* $(n - 1)a$. *For the point set* $P(t; x)$ *with critical* $\Gamma(t; x)$, *the minimum inner spanning tree has length exactly* $(n - 1)a$.

PROOF. The first part is obvious. The second part follows immediately from the fact that any minimum inner spanning tree is a minimum spanning tree of the graph $\Gamma(t; x)$. □

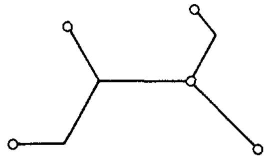

Fig. 5

By Lemma 11, we can see that for the point set $P(t; x)$ with critical $\Gamma(t; x)$, every minimum inner spanning tree is a minimum spanning tree in the plane. Thus, to show that Theorem 1 holds for $t(x)$, it suffices to verify the truth of Gilbert–Pollak conjecture for the point set $P(t; x)$ with critical $\Gamma(t; x)$.

5. Minimum Hexagonal Trees. We now study a different kind of trees. Given three directions (Figure 5) each two of which meet an an angle of 120°, a shortest network interconnecting a given set P of points and having edges all parallel to the three directions is called a *minimum hexagonal tree* on P. Let $L_h(P)$ denote the length of the minimum hexagonal tree for P. Weng [15] showed the following lemma. For convenience of the reader, we also include a proof here.

LEMMA 12. $L_s(P) \geq (\sqrt{3}/2)L_h(P)$.

PROOF. First, we note that if a triangle ABC has the angle at A not less than 120°, then $l(BC) \geq (\sqrt{3}/2)(l(AB) + l(AC))$. (For a proof, see [4] and [11].) Now, each edge of the Steiner minimum tree can be replaced by two edges meeting at an angle of 120° and parallel to the given directions. Therefore, the lemma holds. □

Let the three directions of a hexagonal tree be parallel to the edges in the critical $\Gamma(t; x)$, respectively. Let T be a minimum hexagonal tree for a given set P of points. A point on T but not in P is called a *junction* if the point is incident to at least three lines. Since only three possible slopes exist for lines, there are exactly three lines meeting at a junction. In a hexagonal tree, an edge is a path between two vertices (regular points or junctions). Thus, an edge can contain several straight segments. An edge is called a *straight edge* if it contains only one straight segment, and is called a *nonstraight edge* if it is not a straight edge. Any two segments adjacent to each other in an edge meet at an angle of 120° since if they meet at an angle of 60° then we can shorten the edge easily. Note that an edge with more than two straight segments can always be replaced by an edge with at most two straight segments. Thus, we assume in the following that every edge has at most two straight segments. When we talk about an edge of a junction, its *first segment* is the segment incident to the junction. The other segment, if it exists, is the *second segment* of the edge. A hexagonal tree for n points is said to be *full* if it contains exactly $n - 2$ junctions. Any hexagonal tree can be decomposed into edge-disjoint union of smaller full hexagonal trees. Such a full hexagonal tree will be said to be a full hexagonal subtree of the hexagonal tree.

Consider a minimum hexagonal tree. Suppose a junction has two nonstraight edges. Then these two edges have segments in the same direction. Flip the edges if necessary to line up these two segments, then the second segments of these two edges as well as the first segment of the third edge are three segments each lying completely on one side of the line just constructed. Therefore one side has the majority of the three segments and we can move the line either to shorten the tree or to decrease the number of nonstraight edges. Hence, there exists a minimum hexagonal tree such that each of its junctions has at most one nonstraight edge.

In the next lemma, we consider the lattice which divides the plane into equilateral triangles. Given three directions parallel to the edges of the equilaterals, we study hexagonal trees for lattice points.

LEMMA 13. *For any set of n lattice points, there is a minimum hexagonal tree whose junctions are all lattice points.*

PROOF. Suppose that the lemma is false. Then there exists a set of points such that every minimum hexagonal tree contains a junction which is not a lattice point. Call such a set *bad*. Let P be a smallest bad set. Then every minimum hexagonal tree for P must be full with no junction on a lattice point. (Otherwise, a smaller bad set exists.) Consider a minimum hexagonal tree T with the property that each of its junctions has at most one nonstraight edge. Note that there exists a junction J which is adjacent to two regular points A and B, or T contains a cycle. Let C be the third vertex adjacent to J. If C is a regular point, then it is easy to show that J is a lattice point, or J can be moved to a lattice point since at least two of the three edges JA, JB and JC are straight. Hence, C is a junction. We will show that J is a lattice point, for otherwise one of the following two things can happen: a junction can be moved to a regular point, or to another junction. Since the latter movement cannot last forever and two junctions joining together would result in a shortening of the tree, we obtain a contradiction.

Let us first consider the case that both edges AJ and JB are straight. If AJ and JB are in different directions, then J is a lattice point. Hence, they are in the same direction. Let e be a line through C parallel to AB. If C has a nonstraight edge with an endpoint lying on the AB side of e, then we can shorten the tree. If C has a straight edge overlapping e, then we can move edge JC such that either J or C meets a regular point or a junction other than J and C. Hence, we may assume that the two vertices adjacent to C are on the other side of e away from AB. Now, we can move C further away from J (Figure 6).

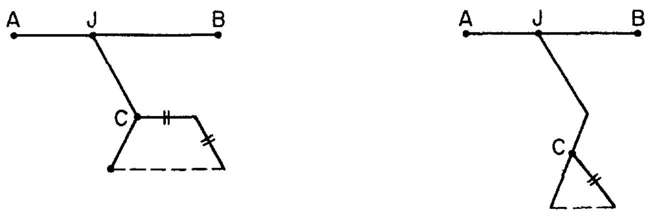

Fig. 6

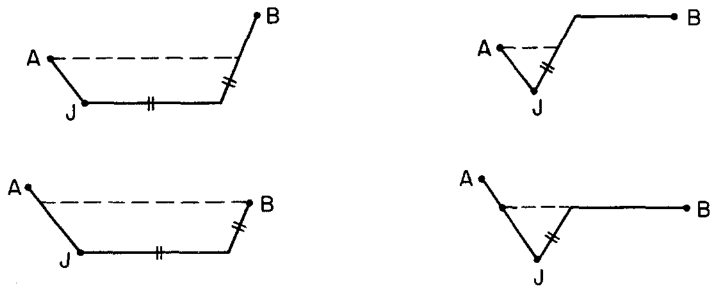

Fig. 7

Second, we consider the case that AJ is a straight edge and JB is a nonstraight edge with a segment in the same direction as AJ. Flip JB, if necessary, to line up the two first segments of AJ and JB. Let BD be the first segment of JB. Then D must be a lattice point. Now, we can use D to replace B and go back to the first case.

Third, if AJ is a straight edge and JB is a nonstraight edge without a segment in the same direction as AJ, then J can be moved either to A or to a lattice point (Figure 7). Since other cases are symmetric to the above three cases, the lemma is proved. □

Consider the lattice containing the critical $\Gamma(t; x)$. Since all regular points are lattice points, by Lemma 13, there is a minimum hexagonal tree with junctions all being lattice points. By Lemma 11, the length of the minimum hexagonal tree is at least $(n-1)a$, the length of a minimum spanning tree. Note that a minimum spanning tree, in this case, is a hexagonal tree. Thus, a minimum spanning tree is also a minimum hexagonal tree. By Lemma 12, we have

$$L_s(P) \geq (\sqrt{3}/2)L_h(P) = (\sqrt{3}/2)L_m(P).$$

Theorem 1 is proved.

6. A Remark on Characteristic Area. When the full ST $t(x)$ has two nonadjacent edges crossing each other, it is hard to determine the surface on which the characteristic area is defined. Here, we give an alternative treatment.

First, we notice that for any two edges on $t(x)$, we can give a system of linear inequalities to describe the sufficient and necessary condition for these two edges to intersect each other. For example, suppose that $A_1 \cdots A_7$ is a convex path in $t(x)$ and $x_i = d(A_i, A_{i+1})$. Then A_1A_2 and A_6A_7 intersect each other iff $x_1 + x_2 \geq x_4 + x_5 \geq x_2$ and $x_6 + x_5 \geq x_2 + x_3 \geq x_5$.

Thus, if we delete the x, at which $t(x)$ has nonadjacent edges intersecting, from X_t, then the closure $\hat{X}_t$ of remaining points is still a polytope, but not necessarily convex. Note that for any point x in the interior of $\hat{X}_t$ $t(x)$ cannot have nonadjacent edges intersecting. However, for x on the boundary of $\hat{X}_t$ $t(x)$ can have nonadjacent edges intersecting; this happens only if there is a regular point touching an edge or another regular point. We now describe how to modify previous arguments to accommodate the changes in boundary and in convexity when X_t (X_{t^*}) is replaced by $\hat{X}_t$ ($\hat{X}_{t^*}$).

Lemmas 1–4 remain unchanged. For the proof of Lemma 5, we need to consider the new boundary. Note that if x is an new boundary point, then $t(x)$ has a regular point touching an edge or another regular point. In the former case, we can decompose $t(x)$ at the touching point to obtain two trees each with less than n regular points. In the latter case, we can reduce the number of regular points by one. In either case, an contradiction is achieved by an argument similar to the one used in the proof of Lemma 4.

Lemma 6 is alright. But Lemma 7 has to be modified as follows.

LEMMA 7′. *Suppose that x is a minimum point and that y is a point in $\hat{X}_{t^*}$ such that the segment $[x, y] \subset \bar{X}_{t^*}$ and $MI(t^*; x) \subseteq MI(t^*; y)$. Then y is also a minimum point.*

In proof of Lemma 7, we keep X_{t^*} in the definition of $A(m)$ so that $A(m)$ is still convex. In the remark at the end of Section 3, we require y to satisfy that for any w in $[x, y]$ $t^*(w)$ exists and does not have nonadjacent edges intersecting. Finally, we claim that in the proof of Lemma 8, we can choose z so close to y such that for any w in $[y, z]$ $t(x)$ exists and does not have nonadjacent edges intersecting. So, we have no trouble for the current case to work.

7. Discussions. The method used in this paper can also be applied to determining Steiner ratio in other normed plane or space. For example, the following theorem can be obtained in a similar way.

THEOREM 2. *In the plane with L_p-norm $\|x\|_p = (|x_1|^p + |x_2|^p)^{1/p}$, the Steiner ratio is achieved by the vertex set of a network which is a union of $n - 2$ equilateral triangles where n is the number of vertices of the network.*

Liu and Du [10] showed that for $1 < p < \infty$, properties of minimum Steiner trees in the L_p-plane are similar to those in the euclidean plane. They believe that the Steiner ratio will be achieved by four points when $p \neq 2$. In fact, for $p = 1$, Hwang [9] has proved that the Steiner ratio is 2/3, which is achieved by four points and cannot be achieved by three points.

In a space of dimension more than two, the critical structure is more complicated. For example, in three-dimensional euclidean space, all polytopes with only equilaterally triangular facets are candidates. Recently, W. D. Smith [14] showed that in the euclidean space of dimension d, $3 \leq d \leq 9$, the Steiner ratio is not achieved by the vertex set of regular simplex as conjectured in [1]. This suggests that determining the Steiner ratio in higher-dimensional space is much more difficult but an interesting topic for further research.

Acknowledgments. The authors wish to thank Dr. M. Bern, Dr. E. N. Gilbert and Dr. W. D. Smith for their helpful comments. The first author also wishes to thank Professor R. L. Graham, Professor R. Tarjan, Professor A. C. Yao and Professor F. F. Yao for their encouragement on his research.

References

[1] F. R. K. Chung and E. N. Gilbert, Steiner trees for the regular simplex, *Bull. Inst. Math. Acad. Sinica*, **4**, 313–325, 1976.
[2] F. R. K. Chung and R. L. Graham, A new bound for euclidean Steiner minimum trees, *Ann. N.Y. Acad. Sci.*, **440**, 328–346, 1985.
[3] F. R. K. Chung and F. K. Hwang, A lower bound for the Steiner tree problem, *SIAM J. Appl. Math.*, **34**, 27–36, 1978.
[4] D. Z. Du and F. K. Hwang, A new bound for the Steiner ratio, *Trans. Amer. Math. Soc.*, **278**, 137–148, 1983.
[5] D. Z. Du, F. K. Hwang, and E. N. Yao, The Steiner ratio conjecture is true for five points, *J. Combinatorial Theory, Ser. A*, **32**, 396–400, 1985.
[6] M. R. Garey, R. L. Graham and D. S. Johnson, The complexity of computing Steiner minimal trees, *SIAM J. Appl. Math.*, **32**, 835–859, 1977.
[7] E. N. Gilbert and H. O. Pollak, Steiner minimal trees, *SIAM J. Appl. Math.*, **16**, 1–29, 1966.
[8] R. L. Graham and F. K. Hwang, Remarks on Steiner minimal trees, *Bull. Inst. Math. Acad. Sinica*, **4**, 177–182, 1976.
[9] F. K. Hwang, On Steiner minimal trees with rectilinear distance, *SIAM J. Appl. Math.*, **30**, 104–114, 1976.
[10] Z. C. Liu and D. Z. Du, On Steiner minimal trees with L_p distance, *Algorithmica*, to appear.
[11] H. O. Pollak, Some remarks on the Steiner problem, *J. Combinatorial Theory, Ser. A*, **24**, 278–295, 1978.
[12] J. H. Rubinstein and D. A. Thomas, A variational approach to the Steiner network problem, *Proceedings of NATO Workshop on Topological Networks*, Copenhagen, Denmark, 1989.
[13] J. H. Rubinstein and D. A. Thomas, The Steiner ratio conjecture for six points, *J. Combinatoria Theory, Ser. A.*, to appear.
[14] W. D. Smith, How to find Steiner minimal trees in euclidean d-space, *Algorithmica*, to appear.
[15] J. F. Weng, Steiner problem in hexagonal metric, unpublished manuscript.