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Limiting Distributions of the Number of Pure Strategy Nash Equilibria in N-Person Games[1]

By I. Y. Powers[2]

Abstract: We study the number of pure strategy Nash equilibria in a "random" n-person non-cooperative game in which all players have a countable number of strategies. We consider both the cases where all players have strictly and weakly ordinal preferences over their outcomes. For both cases, we show that the distribution of the number of pure strategy Nash equilibria approaches the Poisson distribution with mean 1 as the numbers of strategies of two or more players go to infinity. We also find, for each case, the distribution of the number of pure strategy Nash equilibria when the number of strategies of one player goes to infinity, while those of the other players remain finite.

1 Introduction

The Nash equilibrium (N.E.) solution concept (Nash (1951)) is often used to solve n-person non-cooperative games because of the appealing notion of stability that it embodies. A N.E. solution specifies strategies for all players, such that each player achieves his most preferred payoff, given the N.E. strategies adopted by the other players. Any n-person game in which each player has a countable number of strategies can be represented in matrix form, as in the following example, for n = 2:

Player 2

	$s_1 \backslash s_2$	1	2	3	4
Player 1	1	(21,32)	(43,56)	(31,27)	(24,11)
	2	(12,46)	(56,34)	(18,54)	(28,45)
	3	(14,38)	(25,30)	(42,10)	(30,39)

payoffs in dollars

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2 Imelda Yeung Powers, 842 Mandy Lane, Camp Hill, PA 17011, U.S.A.

In this game, s_k (k = 1, 2) denotes the strategy of Player k, and the ordered pairs denote, componentwise, the payoffs of Players 1 and 2. A careful examination of the payoffs reveals that, in this example, the only pure strategy N.E. solution is $s_1 = 3$ and $s_2 = 4$.

In a non-cooperative matrix game, the number of pure strategy N.E. can be as small as zero or as large as the total number of possible outcomes. In any case, it is of fundamental importance to know, at least approximately, how many pure strategy N.E. exist in a particular game. However, this number is not always readily identified, especially in "large" n-person games where n is greater than 2, where the numbers of strategies of the players are very large, and where the payoffs are arranged in a seemingly unstructured manner.

We propose to attack this problem by studying the probability distributions of the number of pure strategy N.E. in "random" n-person games where the payoffs of the various players are modeled as random variables. We focus on n-person games in which each player has a countable number of pure strategies, both because it is often impractical to use mixed strategies, and because many people are uncomfortable using them. Once we adhere only to pure strategies, all that is important is the ordinality of the payoffs; thus, it suffices to study only ordinal games.

In our study of random n-person games, we have found that, under reasonable assumptions, it is difficult to obtain explicit expressions for the probability distribution of the number of N.E. when the numbers of strategies of all n players are finite. In Powers (1986), an explicit solution is given for the case where n = 2.

Earlier research provides some results for limiting cases where the numbers of strategies of certain players go to infinity. Goldberg, Goldman, and Newman (1968) found that the probability that a two-person cardinal game has at least one pure strategy N.E. converges to $1\text{-}e^{-1}$ ($\approx$ 0.6321) as the numbers of strategies of both players go to infinity. Dresher (1970) extended this result to n-person games by showing that the probability that an n-person cardinal game has at least one pure strategy N.E. also converges to $1\text{-}e^{-1}$ as the numbers of strategies of two or more players go to infinity. These researchers modeled the payoffs of their cardinal games as random variables drawn from a continuous distribution, so that for each player, the probability is 1 that all of his payoffs are distinct. This work is thus equivalent to the study of strictly ordinal games, i.e., games where there are no ties among the payoffs of any player, and does not address weakly ordinal games, i.e., games where ties among the payoffs of a player are permitted.

We have found it desirable to identify the complete probability distributions underlying these earlier results. It is also useful to broaden the scope of research by studying the number of N.E. in weakly ordinal games, because individuals often have only a weak preference ordering over all of the possible outcomes of an event. It is intuitively clear that there should be more N.E. in weakly ordinal games, but it is not immediately evident how much of an increase is caused by the weak preferences.

In this paper, we first study the distribution of the number of N.E. in a random n-person strictly ordinal game as the numbers of strategies of one, two, or more players go to infinity. We then extend our results to weakly ordinal games, and study the corresponding limiting distributions. Compared to the results for strictly ordinal games, we find that ties in the payoffs of some players in the weakly ordinal games increase the expected number of N.E. when the numbers of strategies of all players are finite, or if only one player has an infinite number of strategies. However, the distribution of the number of N.E. approaches the same distribution as in the n-person strictly ordinal game when the numbers of strategies of two or more players go to infinity.

2 General Model

We begin by describing our model for a random n-person strictly ordinal game. Consider an n-person strictly ordinal game with Players 1,2,...,n, where Player k has m_k strategies and every player has a strictly ordinal preference over all of the M ($= \Pi\, m_k$) possible outcomes. We assume that: a) for each player, the ordinal payoffs associated with the M possible outcomes are the result of a random drawing of M numbers from 1,2,...,M without replacement (where, without loss of generality, we use the convention that the higher the number associated with an outcome, the more the player prefers it); and b) the ordinal payoffs of the n players are statistically independent of one another. We shall also consider an analogous model for the weakly ordinal game by varying assumption (a) so that for each player, the ordinal payoffs associated with the M possible outcomes are the result of a random drawing of M numbers from 1,2,...,M *with* replacement.

We note that assumption (b) is appropriate for games in which the player's evaluations of the outcomes are independent of one another. Both assumptions (a) and (b) are natural and simple suppositions to make when trying to estimate the number of N.E. in an apparently unstructured n-person game. Using our model, we are able to obtain a number of results that are both mathematically interesting from a combinatorial viewpoint, and of value to researchers seeking the number of N.E. in "large" n-person games. We would also like to note that not all of our results require assumptions (a) and (b), and all of them still hold if assumptions (a) is relaxed appropriately, as discussed later.

If we let s_k denote Player k's strategy and $p_k(s_1, s_2,..., s_k,..., s_n)$ denote his ordinal payoff for the outcome $(s_1, s_2,..., s_k,..., s_n)$, then the outcome $(s_1', s_2',..., s_k',..., s_n')$ is a N.E. if and only if $p_k(s_1', s_2',..., s_k',..., s_n') = \mathrm{Max}_{s_k} \{p_k(s_1', s_2',..., s_k,..., s_n')\}$ for all k. In this paper we study the distribution of X, the number of N.E., as the numbers of strategies of different players go to infinity. We present results for both strictly and weakly ordinal games.

3 N-Person Strictly Ordinal Games

In the n-person strictly ordinal game, we first observe that if we fix the strategies of any n-1 players, then there is at most one N.E. associated with that combination of fixed strategies, because all of the players have strictly ordinal preferences. Hence, the maximum number of possible N.E., i.e., Max$\{X\}$, is given by Min $\{ \prod_{k \neq 1} m_k, \prod_{k \neq 2} m_k, \ldots, \prod_{k \neq n} m_k \}$. Applying assumptions (a) and (b) of our model, we note that $E[X]$ = total number of outcomes $\times$ P(a given outcome is a N.E.) $= \prod m_k \times (1/\prod m_k) = 1$.

Now let us consider all of the possible combinations of strategies of players 1, 2 ,..., n-1, and call each of them a "compound strategy." If we match each of the $\prod_{k \neq n} m_k$ compound strategies with the corresponding best response of player n, then we may find that several compound strategies of Players 1, 2, ..., n-1 are associated with the same strategy of Player n. Consider performing this matching when Players 1, 2, ..., n-1 all have finite numbers of strategies, but Player n has an arbitrarily large number of strategies that are all equally likely to be the best response for any compound strategy of Players 1, 2, ..., n-1, and all of the best responses are independent of one another. Intuitively, the probability should approach 1 that each compound strategy of Players 1, 2,...,n-1 is associated with a distinct strategy of Player n, because it is unlikely that among the very large number of strategies of Player n, the finite number ($\prod_{k \neq n} m_k$) of best responses do not come from distinct strategies.

Given that all of the N.E. of a game must come from the outcomes that already have the best responses of Player n, it follows that the probability that any one of these outcomes is a N.E. is equal to the probability that Players 1, 2 ,..., n-1's ordinal payoffs are all greatest in that outcome, i.e., $1/\prod_{k \neq n} m_k$. Thus, if we let the number of strategies of Player n go to infinity while keeping the numbers of strategies of the other players finite, then we identify the existence of a N.E. as a Bernoulli event with probability $1/\prod_{k \neq n} m_k$, and quickly see that the limiting distribution of the number of N.E. is a binomial random variable with parameters $\prod_{k \neq n} m_k$ and $1/\prod_{k \neq n} m_k$. Formal proof of the above result can be found in Powers (1986).

Of course, we would also like to know the distribution of the number of N.E. as the numbers of strategies of two or more players go to infinity simultaneously. This problem would be easy if we knew that this limiting distribution exists (see, for example, Gelbaum and Olmsted (1964)). If this were the case, then we could first let m_n go to infinity, and then let $\prod_{k \neq n} m_k$ go to infinity. Using the result that we just noted for the case when only Player n has an infinite number of strategies, we would conclude that the distribution of the number of N.E. when $\prod_{k \neq n} m_k$ goes to infinity must be the Poisson distribution with mean 1 (by the DeMoivre-Laplace limit theorem). Unfortunately, we do not know that this limit exists, and so we can-not take advantage of the earlier result. However, we can apply a useful result of Chen (1975) to show that the limiting distribution is indeed Poisson with mean 1.

Consider n individuals, Players 1, 2, ..., n, where Player k has m_k strategies, and each player has strictly ordinal preferences over the M ($= \Pi m_k$) outcomes of the game. Without loss of generality, we assume that $m_1 \leq m_2 \leq \ldots \leq m_n$. Let s = $(s_1, s_2, \ldots, s_n)$ and p_k(s) denote Player k's ordinal payoff for the outcome s. We assume that, for each $k = 1, 2, \ldots, n$, the p_k(s) are all independently obtained by a random drawing of M numbers from 1 to M without replacement, and we define the following indicator functions:

$$\mathbf{I}_k(s_1, s_2, \ldots, s_k', \ldots, s_n) = 1 \quad \text{if } p_k(s_1, s_2, \ldots, s_k', \ldots, s_n) = \underset{s_k}{\text{Max}} \{p_k(s_1, s_2, \ldots, s_k, \ldots, s_n)\}$$

$$= 0 \quad \text{otherwise, and}$$

$$\mathbf{I}(\mathrm{s}) = \Pi\, \mathbf{I}_k(\mathrm{s}).$$

With this notation, we present the following result.

Theorem 1: In an n-person strictly ordinal game, the probability distribution of the number of N.E. approaches the Poisson distribution with mean 1 as the numbers of strategies of two or more players approach infinity.

Proof: Let X denote the number of N.E. Then, by definition, $X = \Sigma\, \mathbf{I}(\mathrm{s})$, where the **I**(s) are Bernoulli random variables with mean $1/M$. Clearly, **I**(s) and **I**(s′) are dependent if less than two of the elements of s and s′ are different, and independent otherwise. For any s, let $B(\mathrm{s})$ = {s′ : less than two elements of s′ and s are different}. Furthermore, let $b_1 = \sum_{\mathrm{s}} \sum_{\mathrm{s}' \in B(\mathrm{s})} E[\mathbf{I}(\mathrm{s})]E[\mathbf{I}(\mathrm{s}')]$, and $b_2 = \sum_{\mathrm{s}} \sum_{\mathrm{s} \neq \mathrm{s}' \in B(\mathrm{s})} E[\mathbf{I}(\mathrm{s})\mathbf{I}(\mathrm{s}')]$.
Then $b_1 = M[\Sigma(m_k\text{-}1)+1] \times (1/M^2)$ and $b_2 = M[\Sigma(m_k\text{-}1)] \times 0$.
Since $\lim_{m_{n-1}, m_n \to \infty} b_1 = \lim_{m_{n-1}, m_n \to \infty} b_2 = 0$, it follows from Chen (1975) that, as m_{n-1} and m_n go to infinity, $X \xrightarrow{\text{dist.}} P$, where $P \sim$ Poisson (1). Because this result holds for all $m_1, m_2, \ldots, m_{n-2}$, it must also hold as any or all of them approach infinity. ■

We observe that for the Poisson distribution with mean 1, $P(X > 0) = 0.6321$. Thus, a N.E. is not so rare when the numbers of strategies of two or more players go to infinity.

4 N-Person Weakly Ordinal Games

We now proceed to describe our model for the n-person weakly ordinal game. We use the same notation as that in the n-person strictly ordinal game, but consider n players who may have weakly ordinal preferences over all of the possible outcomes. Again, without loss of generality, we assume that $m_1 \leq m_2 \leq \ldots \leq m_n$, and let $(s_1, s_2, \ldots, s_n)$ denote, componentwise, the strategies chosen by Players 1, 2, ..., n. We also assume that, independent of the payoffs of the other players, the payoffs of each Player k are obtained by a random drawing of $M(=\Pi m_k)$ numbers *with* replacement.

This model of a random n-person weakly ordinal game is a natural extension of our model of a random n-person strictly ordinal game. The drawing with replacement simulates the possibility of ties among a player's payoffs. Let Z be the number of N.E. in this weakly ordinal game. We know that, given the strategies of the rest of the players, there is possibly more than one "maximal outcome" (best response) for each player. Since N.E. can be formed from these maximal outcomes, we anticipate a greater number of N.E. in this game than in the strictly ordinal game, and this is confirmed by the following lemma.

Lemma 1: In an n-person weakly ordinal game, the expected number of N.E. is greater than or equal to 1.

Proof: For $k = 1, 2, \ldots, n$, fix the strategies of all but Player k, and consider the one or more maximal outcomes of Player k. Identify one of these outcomes as the "designated" maximal outcome and the rest, if any, as "undesignated" maximal outcomes. Then define, for all $k = 1, 2, \ldots, n$,

$$\mathbf{I}_k(\mathbf{s}) = \begin{cases} 1 & \text{if } \mathbf{s} \text{ gives Player } k \text{ a designated maximal outcome} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{J}_k(\mathbf{s}) = \begin{cases} 1 & \text{if } \mathbf{s} \text{ gives Player } k \text{ an undesignated maximal outcome} \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$\mathbf{I}(\mathbf{s}) = \Pi\, \mathbf{I}_k(\mathbf{s})$$

$$\mathbf{J}(\mathbf{s}) = \Pi\, [\mathbf{I}_k(\mathbf{s}) + \mathbf{J}_k(\mathbf{s})] - \Pi\, [\mathbf{I}_k(\mathbf{s})].$$

We recognize that
$\Sigma\, \mathbf{I}(\mathbf{s}) = X$, a random variable with the same distribution as the number of N.E. in the strictly ordinal game, and we know that $E[X] = 1$ (from section 3). Let us also define $\Sigma\, \mathbf{J}(\mathbf{s}) = Y$. Since Y is a non-negative random variable, it follows that $E[Y] \geq 0$. We can see that $Z = X + Y$, and so $E[Z] = E[X] + E[Y] \geq 1$.

■

Consider an n-person weakly ordinal game in which the number of strategies of Player n goes to infinity while the numbers of strategies of the other players remain finite. Given any compound strategy chosen by Players 1, 2, ..., k-1, k+1, ..., n, the probability that all of the m_k ordinal payoffs are distinct for Player $k (k = 1, 2, ..., n\text{-}1)$ goes to 1, because Player k's finite m_k ordinal payoffs are drawn with replacement from an arbitrarily large collection of numbers: $1, 2, ..., M$. Thus, in the limit, there is only one maximal outcome for Player k when the strategies of the rest of the players are fixed, and this is similar to the strictly ordinal n-person game. However, this similarity exists only for Players k, where $k \neq n$. For Player n, there is, with positive probability, more than one maximal outcome in the limit when the strategies of the rest of the players are fixed, because, in this case, an infinite number of ordinal payoffs, m_n, is drawn from M numbers, where M is just a finite constant times m_n.

This result changes the nature of the problem dramatically, and leads to a different limiting distribution from the corresponding strictly ordinal case. In Powers (1986), it is shown that $\varphi_Z(t)$, the moment generating function associated with this limiting distribution, is

$$\{[1/(1\text{-}e^{-1/M'})]\cdot[e^{(e^t-1)/M'^2} - e^{-1/M'}]\}^{M'}, \text{ where } M' = m_1 \times m_2 \times ... \times m_{n-1}.$$

Although it is not possible to extract the probability distribution function from this moment generating function, we can easily find the first moment, $E[Z] = \varphi_Z'(0) = 1/[M'(1\text{-}e^{-1/M'})]$, as well as higher moments.

If we let more than one player have an infinite number of strategies (or equivalently, let M' go to infinity), then $E[Z]$ approaches 1. (Note that this does not prove that $E[Z]$ approaches 1 as the numbers of strategies of two or more players go to infinity *simultaneously*.) This suggests that no extra N.E. are contributed by the undesignated maximal outcomes. Although this may seem odd at first glance, it is actually quite apparent.

For $k = 1, 2, ..., n$, consider Player k's m_k ordinal payoffs given a particular compound strategy chosen by the rest of the players. Since these m_k ordinal payoffs are drawn from M numbers with replacement, and M approaches infinity at a faster rate than m_k, it follows that the probability that Player k's m_k ordinal payoffs are all distinct (i.e., that there is only one maximal outcome) goes to 1. We note that this is true for *all* players, and that this structure is similar to the structure of the n-person strictly ordinal game. Hence, we would expect that as the numbers of strategies of two or more players go to infinity, the probability distribution of the number of N.E. in an n-person weakly ordinal game also approaches the Poisson distribution with mean 1. The following lemma is needed to establish this result formally in Theorem 2.

Lemma 2: In an n-person weakly ordinal game, the expected number of N.E. approaches 1 as the numbers of strategies of two or more players go to infinity.

Proof: $E[Z] = \sum_{\mathbf{s}} E[\mathbf{I}(\mathbf{s})+\mathbf{J}(\mathbf{s})] = \sum_{\mathbf{s}} \prod_{k} P(\mathbf{I}_k(\mathbf{s})+\mathbf{J}_k(\mathbf{s}) = 1)$

$$= (\Pi m_k)\, \Pi \{(\Pi m_k)^{-m_k} [(\Pi m_k)^{m_k-1} + (\Pi m_k-1)^{m_k-1} + (\Pi m_k-2)^{m_k-1} + ... + (\Pi m_k - (\Pi m_k-1))^{m_k-1}]\}.$$

Let r be the number of players with an infinite number of strategies, where $2 \leq r \leq n$, and let $m = m_{n-(r-1)} = ... = m_{n-1} = m_n$. Let v be the index for Players $1, 2, ..., n\text{-}r$, and let $N = \Pi m_v$, so that $M = Nm^r$. Then $E[Z]$

$= N\ \Pi\{(Nm^r)^{-m_v}\ [(Nm^r)^{m_v-1} + (Nm^r-1)^{m_v-1} + (Nm^r-2)^{m_v-1} + \ldots + (Nm^r - (Nm^r-1))^{m_v-1}]\} \cdot m^r\{(Nm^r)^{-m}\ [(Nm^r)^{m-1} + (Nm^r-1)^{m-1} + (Nm^r-2)^{m-1} + \ldots + (Nm^r-(Nm^r-1))^{m-1}]\}^r$

$$= N \prod_v [(Nm^r)^{-1} \cdot \sum_{i=0}^{Nm^r-1} (1-i/(Nm^r))^{m_v-1}] \cdot$$

$$[(1/(Nm^{r-1})) \sum_{i=0}^{Nm^r-1} (1-i/(Nm^r))^{m-1}]^r, \text{ and}$$

$$\lim_{m\to\infty} E[Z]$$

$$= N[\prod_v \int_0^1 (1-u)^{m_v-1}\, du] \cdot \lim_{m\to\infty} [(1/Nm^{r-1})) \sum_{i=0}^{Nm^r-1} (1-i/(Nm^r))^{m-1}]^r$$

(where u is just a dummy variable)

$$= N(1/N) \cdot \lim_{m\to\infty} [(1/(Nm^{r-1})) \sum_{i=0}^{Nm^r-1} (1-i/(Nm^r))^{m-1}]^r.$$

Now, $[(1/(Nm^{r-1})) \sum_{i=0}^{Nm^r-1} (1-i/(Nm^r))^{m-1}]^r$

$$\leq [(1/(Nm^{r-1})) \sum_{i=0}^{Nm^r-1} (e^{-i/(Nm^r)})^{m-1}]^r$$

$$\leq [(1/(Nm^{r-1})) \sum_{i=0}^{\infty} (e^{-i/(Nm^r)})^{m-1}]^r$$

$$= [(1/(Nm^{r-1})) \cdot 1/(1-e^{-(m-1)/(Nm^r)})]^r$$

$$= [(1/N) \cdot m^{-(r-1)}/(1-e^{-(m-1)/(Nm^r)})]^r.$$

Using L'Hôpital's rule, we find that $\lim_{m\to\infty} [m^{-(r-1)}/(1-e^{-(m-1)/(Nm^r)})] = N$.

Thus, $\lim_{m\to\infty} E[Z] \leq 1$. Together with Lemma 1, we see that $\lim_{m\to\infty} E[Z] = 1$.

■

Theorem 2: In an n-person weakly ordinal game, the probability distribution of the number of N.E. approaches the Poisson distribution with mean 1 as the numbers of strategies of two or more players go to infinity.

Proof: Let m be defined as in the proof of Lemma 2. Then the following are true:

$\lim_{m \to \infty} E[X] = 1$ (from the discussion in section 3) and

$\lim_{m \to \infty} E[Z] = 1$ (from Lemma 2).

Since $E[Z] = E[X] + E[Y]$, we conclude that $\lim_{m \to \infty} E[Y] = 0$.

Using Markov's inequality and the fact that $Y \geq 0$, we see that for all $\epsilon > 0$, $\lim_{m \to \infty} P(|Y - 0| \geq \epsilon) \leq \lim_{m \to \infty} E[Y]/\epsilon$,
which implies that $\lim_{m \to \infty} P(|Y - 0| \geq \epsilon) = 0$
and $\lim_{m \to \infty} P(|Y - 0| > \epsilon) = 0$.

It then follows that

$Y \xrightarrow{\text{prob.}} 0$,
and so $Y \xrightarrow{\text{dist.}} 0$.

We know from Theorem 1 in section 3 that $X \xrightarrow{\text{dist.}} P$, where $P \sim$ Poisson (1). Since $Z = X + Y$, $X \xrightarrow{\text{dist.}} P$, and $Y \xrightarrow{\text{dist.}} 0$, we conclude by standard convergence theory (see, for example, Chung (1974)) that $Z \xrightarrow{\text{dist.}} P$.

■

5 Relaxation of Assumptions

We recall that two simplifying assumptions were made in our models for both the strictly and weakly ordinal games. They are in essence: a) the complete symmetry of the ordinal payoffs across outcomes for each player, and b) the independence of different player's ordinal payoffs.

In our analysis of the number of N.E. in n-person strictly ordinal games, we took advantage only of the symmetry across outcomes for each Player k, given any compound strategy chosen by all of the other players. Therefore, Theorem 1 still holds if the ordinal payoffs of Player k, given each compound strategy, are drawn randomly without replacement from a sample of m_k or more distinct integers.

In the case of n-person weakly ordinal games, if the ordinal payoffs of Player k, given any compound strategy chosen by the other players, are drawn randomly with replacement from a sample of $[\text{Max}\{m_k\}]^a$ distinct integers, where a is greater than 1, then Theorem 2 still holds. In all cases, a relaxation of assumption (a) is possible in that we need for each player only symmetry among the ordinal payoffs of his outcomes for a given compound strategy chosen by the other players, and the number of possible rankings of the outcomes by the players to go to infinity at the proper rate.

6 Conclusion

In this paper, we have considered the number of N.E. in random n-person non-cooperative games in matrix form. Because we have chosen to study only pure strategy N.E., it sufficed to conduct our study in terms of ordinal games. Under two very simple assumptions, we obtained interesting results concerning the number of N.E. in strictly and weakly ordinal games. Specifically, we found limiting probability distributions of the number of N.E. as the numbers of strategies of some players go to infinity. Under our assumptions, the results indicate that pure strategy N.E. are not at all uncommon in the class of n-person games, and that we should not expect many more N.E. in weakly ordinal games than in strictly ordinal games if two or more players have very large numbers of strategies.

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