On the Foundations of Game Theory: The Case of Non-Archimedean Utilities

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Abstract: Contrary to what appears to have become an accepted part of the folklore of game theory, a finite two-person zero-sum game with non-Archimedean utilities may have no equilibrium-point solution, and either one or both players may have no "minimax" strategy. Even when both players have "minimax" strategies, such a game may lack an equilibrium point.

1. Introduction

In concluding his exposition of HAUSNER's multidimensional (lexicographic, non-Archimedean) expected utility theory [HAUSNER], THRALL [p. 186] says that "This discussion illustrates the fact that non-Archimedean utilities are perfectly satisfactory for game theory". This viewpoint has been perpetuated in a number of later writings, examples being [AUMANN, p. 453], [FERGUSON, pp. 20–21] and [LUCE and RAIFFA, p. 27], and appears to have become an accepted fact in the folklore of game theory.

The purpose of this note is to show that, even in the simplest case of finite two-person zero-sum games, the actual state of affairs with non-Archimedean utilities is vastly different from the wellknown results [LUCE and RAIFFA, NASH, VON NEUMANN-MORGENSTERN] under Archimedean or VON NEUMANN-MORGENSTERN utilities. In particular, a non-Archimedean finite two-person zero-sum game may have no stable solution (equilibrium point), either one or both players may have no "minimax" strategy, and even when both players have "minimax" strategies the game may have no equilibrium point.

The fourth section proves these assertions. The next section outlines the non-Archimedean utility theory. Section 3 presents some definitions involved in the present investigation and discusses an example to show what can happen under non-Archimedean utilities. The paper concludes with a brief discussion of the results.

2. Lexicographic Expected Utility

Throughout, we assume that there are two players with n and m pure strategies respectively. $K = \{(i,j): i = 1, \ldots, n \text{ and } j = 1, \ldots, m\}$ is the set of nm pure-strategy

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pairs. We shall let X be the set of probability vectors $x = (x_1, \dots, x_n)$ that characterize player 1's mixed strategies: $x_i \geq 0$ for all i, and $\Sigma x_i = 1$. Similarly, Y is the set of probability vectors $y = (y_1, \dots, y_m)$ that characterize player 2's mixed strategies.

With $\succsim$ the preference-or-indifference relation for player 1 on the set of probability distributions on K, we assume that $\succsim$ is a weak order (transitive and connected, or complete). Strict preference ($\succ$) and indifference ($\sim$) are defined in the usual way: $p \succ q$ iff not $(q \succsim p)$, $p \sim q$ iff $p \succsim q$ and $q \succsim p$.

We assume further that the independence axioms (including "strong" versions) of expected-utility theory hold for $\succsim$, but do not assume that the VON NEUMANN-MORGENSTERN Archimedean axiom holds. (In our terms, the latter says that if $p \succ q$ and $q \succ r$ then there are α, β strictly between 0 and 1 for which $\alpha p + (1-\alpha) r \succ q$ and $q \succ \beta p + (1-\beta) r$.) The structure of expected-utility under these axioms is discussed by CHIPMAN [1971], FISHBURN [1971 b], HAUSNER, and THRALL. In particular, using an observation in FISHBURN [1971 a, p. 575], it follows from the weak order and independence axioms and from the finiteness of K that there exist a finite number of real-valued functions $u_1, \dots, u_N$ on K such that, with

$$u(x,y) = (\Sigma_{i,j} u_1(i,j) x_i y_j, \dots, \Sigma_{i,j} u_N(i,j) x_i y_j), \tag{1}$$

$$(x,y) \succsim (x',y') \quad \text{iff} \quad u(x,y) \geq^L u(x',y') \tag{2}$$

for all (x,y) and (x',y') in $X \times Y$, where $\geq^L$ is the usual lexicographic order on N-dimensional Euclidean space: $(a_1, \dots, a_N) \geq^L (b_1, \dots, b_N)$ iff $a_k = b_k$ for all k or $a_k > b_k$ for the smallest k for which $a_k \neq b_k$.

Because (1) is a vector bilinear form with the noted constraints on x and y, $u(x,y)$ can be lexicographically maximized or minimized over one variable with the other held fixed. For example, $\min_y u(x,y)$ is determined as follows. Let $v_k(x,j) = \Sigma_i u_k(i,j) x_i$, so that

$$u(x,y) = (\Sigma_j v_1(x,j) y_j, \dots, \Sigma_j v_N(x,j) y_j).$$

Proceeding on the basis of $\geq^L$, determine all y in Y that minimize $\Sigma_j v_1(x,j) y_j$. The set Y_1 of minimizing y will be the set of all vectors in Y with zero components for the j for which there is a j' with $v_1(x,j') < v_1(x,j)$. Y_1 is therefore a nonempty polytope in Y. If Y_1 has only one element (there is a unique smallest $v_1(x,j)$), this element is the overall lexicographic minimizer. If Y_1 has more than one element, we then determine the set of y in Y_1 that minimize $\Sigma_j v_2(x,j) y_j$. Call this set Y_2. It is a nonempty polytope in Y_1. If Y_2 has only one element, it is the overall lexicographic minimizer. If Y_2 has more than one element, continue in the obvious way.

Thus, $\min_y u(x,y)$ and $\max_x u(x,y)$ exist in all cases. However, as we shall note in the next section, there may be no $x \in X$ that lexicographically maximizes $\min_y u(x,y)$. Section 4 presents an example where neither $\max_x \min_y u(x,y)$ nor

$\min_y \max_x u(x, y)$ exists. This is in sharp contrast to the Archimedean theory, for which $N = 1$ in (1), where these double extrema always exist.

3. Definitions and Example

For simplicity, we shall assume that our finite two-person game is zero-sum. This means that player 2's preference-or-indifference order is the dual of $\succsim$, which permits the vector utilities of player 2 to be taken as the negatives of the vector utilities of player 1.

Under the zero-sum condition, we are most interested in maximin strategies for player 1 and minimax strategies for player 2 (with respect to player 1's utilities). Formally, x is a maximin strategy for player 1 if and only if $\min_y u(x, y) \geq^L \min_y u(x', y)$ for all x' in X. And y is a minimax strategy for player 2 if and only if $\max_x u(x, y') \geq^L \max_x u(x, y)$ for all y' in Y.

We shall say that (x, y) is an equilibrium point if and only if

$$(x, y') \succsim (x, y) \succsim (x', y) \quad \text{for all} \quad (x', y') \quad \text{in} \quad X \times Y.$$

As in the usual zero-sum theory, neither player can improve his position by departing from an equilibrium strategy as long as his opponent plays an equilibrium strategy.

To illustrate the problems that can arise with non-Archimedean utilities, let $(n, m) = (2, 2)$ with the following two-dimensional utilities on K for player 1:

		Player 2	
		p	$1 - p$
Player 1	a	(1,0)	(0,0)
	$1 - a$	(0,0)	(0,1)

We shall let (a, p) denote the mixed-strategy pair in which $x = (a, 1 - a)$ and $y = (p, 1 - p)$. Then

$$u(a, p) = (ap, (1 - a)(1 - p))$$

and (a, p) is an equilibrium point if and only if

$$(aq, (1 - a)(1 - q)) \geq^L (ap, (1 - a)(1 - p)) \geq^L (bp, (1 - b)(1 - p))$$

for all b and q in $[0, 1]$. If $p > 0$, the right-hand $\geq^L$ demands $a = 1$; but if $a = 1$, the left-hand $\geq^L$ demands $p = 0$. And if $p = 0$, the right-hand $\geq^L$ demands $a = 0$, whereas if $a = 0$ then the left-hand $\geq^L$ requires $p = 1$.

Therefore this game has no equilibrium point.

Continuing with the example, we look for a maximin strategy for player 1. For the first step

$$\min_p u(a,p) = \begin{matrix} (0,1-a) & \text{if} \quad a > 0 \\ (0,0) & \text{if} \quad a = 0\,. \end{matrix}$$

Since the first components are uniformly zero, only the second components are involved in maximizing over a. However, because of the discontinuity at $a = 0$, there is no value of a in $[0,1]$ that maximizes $\min_p u(a,p)$. Hence player 1 has no maximin strategy.

On the other hand, player 2 has a minimax strategy, since

$$\max_a u(a,p) = \begin{matrix} (p,0) & \text{if} \quad p > 0 \\ (0,1) & \text{if} \quad p = 0 \end{matrix}$$

and $p = 0$ minimizes $\max_a u(a,p)$. Hence $p = 0$ is player 2's unique minimax strategy.

However, due to the lack of an equilibrium point, we can still find ourselves going in circles, as in the pure strategy cycles of Archimedean zero-sum games with no pure-strategy equilibrium. If player 2 plays his minimax strategy, player 1 can do best with $a = 0$, obtaining a utility of $(0,1)$. But if player 1 takes $a = 0$, player 2's best strategy is $p = 1$. And so forth.

4. Theorems

We shall now summarize some general results for the non-Archimedean theory. Throughout this section, all games are assumed to be finite two-person zero-sum games, with utilities for player 1 satisfying (1) and (2). The first theorem shows that some aspects of the Archimedean theory carry over to the non-Archimedean case.

Theorem 1:

If (x,y) is an equilibrium point then x is a maximin strategy for player 1 and y is a minimax strategy for player 2. Moreover, all equilibrium points are interchangable and equivalent: that is, if (x,y) and (x^*,y^*) are equilibrium points then (x,y^*) and (x^*,y) are equilibrium points and $(x,y) \sim (x^*,y^*)$.

The following theorems present the results that distinguish the non-Archimedean case from the Archimedean case.

Theorem 2:

There are games where player 1 has a maximin strategy, player 2 has a minimax strategy, and there is no equilibrium point.

Theorem 3:

There are games in which player 1 has a maximin strategy and player 2 has no minimax strategy; there are games in which player 1 has no maximin strategy and player 2 has a minimax strategy; there are games in which player 1 has no maximin strategy and player 2 has no minimax strategy.

Theorem 1 is proved in the usual fashion. For the first part, let (x,y) be an equilibrium point, with $u(x,y') \geq^L u(x,y) \geq^L u(x',y)$ for all x' in X and y' in Y. Then $\min_{y'} u(x,y') = u(x,y)$. If x were not a maximin strategy for player 1, then there would be an x' in X such that $\min_{y'} u(x',y') >^L \min_{y'} u(x,y') = u(x,y)$. In particular, $u(x',y) >^L u(x,y)$, which contradicts $u(x,y) \geq^L u(x',y)$. Hence x is a maximin strategy for player 1. The proof that y is a minimax strategy for player 2 is similar.

Secondly, if both (x,y) and (x^*,y^*) are equilibrium points then $u(x,y') \geq^L u(x,y) \geq^L u(x',y)$ and $u(x^*,y') \geq^L u(x^*,y^*) \geq^L u(x',y^*)$ for all x' in X and y' in Y. In particular, $u(x,y^*) \geq^L u(x,y) \geq^L u(x^*,y) \geq^L u(x^*,y^*) \geq^L u(x,y^*)$, so that $(x,y^*) \sim (x,y) \sim (x^*,y) \sim (x^*,y^*)$ by (2). It follows from weak order for $\succsim$ that $(x,y') \succsim (x,y^*) \succsim (x',y^*)$ and $(x^*,y') \succsim (x^*,y) \succsim (x',y)$ for all x' in X and y' in Y, so that (x,y^*) and (x^*,y) are equilibrium points.

To prove Theorem 2, consider the $(n,m) = (2,2)$ game where player 1's two-dimensional utilities are as follows:

		Player 2	
		p	$1-p$
Player 1	a	$(1,1)$	$(0,1)$
	$1-a$	$(0,0)$	$(1,0)$

Using the notation as in the preceding example,

$$u(a,p) = (ap + (1-a)(1-p), a)$$

so that (a,p) is an equilibrium point if and only if

$$(aq + (1-a)(1-q), a) \geq^L (ap + (1-a)(1-p), a) \geq^L (bp + (1-b)(1-p), b)$$

for all b and q in $[0,1]$. The right-hand $\geq^L$ requires

$$\begin{aligned} a = 1 &\quad \text{if} \quad p \geq 1/2 \\ a = 0 &\quad \text{if} \quad p < 1/2 . \end{aligned}$$

But the left-hand $\geq^L$ requires $p = 0$ if $a = 1$, and $p = 1$ if $a = 0$. Hence there is no equilibrium point.

However, player 1 has a maximin strategy since

$$\min_p u(a,p) = \begin{array}{lll} (a,a) & \text{if} & a < 1/2 \\ (1/2,1/2) & \text{if} & a = 1/2 \\ (1-a,a) & \text{if} & a > 1/2 \end{array}$$

and $a = 1/2$ maximizes this; and player 2 has a minimax strategy since

$$\max_a u(a,p) = \begin{matrix} (1-p,0) & \text{if} & p < 1/2 \\ (1/2,1) & \text{if} & p = 1/2 \\ (p,1) & \text{if} & p > 1/2 \end{matrix}$$

and $p = 1/2$ minimizes this. Hence $(a,p) = (1/2, 1/2)$ is a joint "minimax" strategy pair. But it is not an equilibrium, as just proved, and if one player sticks to his "minimax" strategy then the other can gain by departing from his "minimax".

The example of the preceding section suffices for the first two parts of Theorem 3. For the final assertion we use an $(n,m) = (2,3)$ game with the following three-dimensional utilities:

	p	q	r
a	$(1,0,0)$	$(0,0,0)$	$(0,1,1)$
$1-a$	$(0,0,0)$	$(0,1,1)$	$(0,1,0)$

With $x = (a, 1-a)$ and $y = (p,q,r)$,

$$u(x,y) = (ap, r + (1-a)q, ar + (1-a)q).$$

For player 1,

$$\min_y u(x,y) = \begin{matrix} (0,1-a,1-a) & \text{if} & a > 0 \\ (0,0,0) & \text{if} & a = 0 \end{matrix}$$

and there is no a which maximizes $\min_y u(x,y)$. For player 2,

$$\max_x u(x,y) = \begin{matrix} (p,r,r) & \text{if} & p > 0 \\ (0,1,q) & \text{if} & p = 0 \text{ and } q > 0 \\ (0,1,1) & \text{if} & p = q = 0 \end{matrix}$$

and there is no $y = (p,q,r)$ that minimizes this. (To minimize over y we consider the cases where $p = 0$, but there is no minimizing y for these cases.)

Hence player 1 has no maximin strategy and player 2 has no minimax strategy.

5. Discussion

Some years ago, Thrall presented an argument that convinced many people (myself included) that non-Archimedean utilities could give the same types of results in game theory that obtain with Archimedean or unidimensional utilities. However, as shown in this paper, we have been under an illusion: without the Archimedean assumption, very little remains of the standard solution theory. This is true even when all the non-Archimedean axioms of traditional expected-utility theory are adopted, however unpalatable some of these may be to some people.

The perseverance of this illusion may be due to a widespread uncritical acceptance of the Archimedean axiom. Indeed, a "so what" reaction to this paper may be forthcoming from persons who find nothing amiss with that axiom. On the other hand, examples in the literature (e.g., CHIPMAN [p. 221], FISHBURN [1970, p. 110], and THRALL) suggest to others that the Archimedean assumption may be inapplicable in some situations.

The impression remains that game theory without the Archimedean axiom is rather barren.

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