Structural Optimization 2, 213-222 (1990)

Structural Optimization

Design of beams, plates and their elastic foundations for uniform foundation pressure

K. Dems*

Łódź Technical University - J26 PL-90924 Łódź, Poland

R.H. Plaut

Charles E. Via, Jr. Dept. Civil Engineering, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061-0105, USA

Abstract. Beams and circular plates on elastic foundations are considered. In some cases, additional elastic supports are present. The stiffness distribution of the foundation is designed so that the pressure on the foundation is uniform. Sometimes the depth of the beam or plate is also varied, with either a piecewise-constant sandwich or solid cross-section, and a global measure of the deflection is minimized. The total stiffness of the foundation and supports is specified, as well as the volume of the structure. In one type of problem, the edges of the structure are displaced downwards; in the other examples, a downward load is applied. Types of loads include a concentrated central load, a uniform load and a parabolic load. The uniform foundation pressure for the resulting design is often substantially lower than the maximum pressure for a corresponding uniform beam or plate on an elastic foundation with uniform stiffness.

1 Introduction

Optimization of the stiffness distribution of elastic foundations has been treated in several papers. Szelag and Mróz (1978) considered a beam with specified fundamental natural frequency and minimized the total foundation stiffness. Taylor and Bendsøe (1984) displaced a beam downwards at its ends and minimized the maximum foundation pressure. In Dems *et al.* (1987), a measure of the deflection of a beam was minimized, while Plaut (1987) minimized the compliance of beams and plates under uniform loading. The buckling load of a column was maximized by Shin, Haftka and Plaut (1988) and Shin *et al.* (1988).

In Plaut (1989), the cross-sectional area of a sandwich beam and the stiffness distribution of its elastic foundation were optimized simultaneously. The volume of the beam and the total stiffness of the foundation were specified, and the compliance was minimized for uniform loading. In some cases the optimal foundation was concentrated into a single elastic spring, while in others there was a region of uniform foundation with springs at its internal endpoints.

After an initial example involving a beam on elastic supports, beams and circular plates on elastic foundations are considered here. The general objective is the minimization of the maximum pressure transmitted to the foundation. However, the problems to be treated will allow solutions having uniform foundation pressure and this will be stipulated as a constraint. The distribution of foundation stiffness which leads to satisfaction of this constraint is sought. The structures are supported by the foundation and, in most cases, by elastic springs at their edges. The total stiffness of the foundation and springs is specified.

Uniform beams are considered in Section 2. The first example involves three springs designed to take equal forces. In the second example, the beam rests on an elastic foundation and its ends are displaced downwards. Next, a parabolic load is applied to the beam. In the final example, a uniform load is applied to a beam which also has springs at its ends and the stiffness of the springs is chosen such that the total force in the springs is equal to the total force on the foundation.

In Section 3, the depth of the beam is allowed to vary (in a piecewise-constant manner), as well as the stiffness of the foundation. Both sandwich and solid cross-sections

* Visiting Professor, Virginia Polytechnic Institute and State University, August-December 1989

are treated, with given volume. A uniform load and a concentrated central load are considered and springs with specified stiffness act at the ends of the beam. A measure of the beam displacement (the integral of the square of the deflection) is minimized.

In Sections 4 and 5, the previous analysis is extended to the case of axisymmetric circular plates. Uniform plates are analysed in Section 4. The elastic foundation is axisymmetric and there is an elastic support at the edge of the plate. Three examples are treated: downward displacement of the edge, uniform loading and nonuniform loading which varies parabolically with the radius. In Section 5, the depth of the plate is a piecewise-constant function of the radius, with given volume. As for the beams, both sandwich and solid plates are treated, and both uniform and concentrated central loads are applied. The volume displaced by the plate is minimized, with the constraint of uniform foundation pressure. Concluding remarks are presented in Section 6.

2 Uniform beams

Consider a beam with length L, Young's modulus E, moment of inertia I, axial coordinate X with $-L/2 \leq X \leq L/2$, downward deflection $W(X)$ under a downward distributed load $Q(X)$ and an elastic (Winkler) foundation with stiffness (per unit length) $K(X)$. The total stiffness of the foundation and elastic springs is denoted by $\overline{K}_T$. Let I_u be a reference moment of inertia (which will be chosen as I in this section) and let

$$P = K(X)W(X)\,, \quad x = 2X/L\,, \quad D = I/I_u\,, \tag{1}$$

where P is the foundation pressure, which is constrained to be constant. A nondimensional displacement $w(x)$, foundation stiffness $k(x)$ and load $q(x)$ are defined separately in each example and the governing equilibrium equation is

$$Dw''''(x) + k(x)w(x) = q(x)\,. \tag{2}$$

Since $k(x)w(x)$ is constant, (2) can be solved analytically.

2.1 Beam on elastic springs under linearly distributed load

As an initial problem, a beam supported by three elastic springs is considered. Studies involving the optimal locations and/or stiffnesses of flexible supports include Dems *et al.* (1987), Åkesson and Olhoff (1988), Pičuga (1988), Rozvany (1989), and references 1-5 and 8 of Dems *et al.* (1987). Here the objective is equality of the forces in the three springs.

The springs have stiffnesses C_1, C_2 and C_3, with sum $\overline{K}_T$ and are located at $X = -L/2$, L_2, $L/2$, respectively. A linearly distributed load is applied, with $Q(-L/2) = \gamma q_0$ and $Q(L/2) = q_0$. Define

$$w = \frac{16EIW}{q_0 L^4}\,, \quad l_2 = \frac{2L_2}{L}\,, \quad c_j = \frac{L^3 C_j}{8EI}\,, \quad K_T = \frac{L^3 \overline{K}_T}{8EI}\,,$$

$$q(x) = [(1-\gamma)x + 1 + \gamma]/2\,. \tag{3}$$

In terms of these nondimensional quantities, the beam is depicted in Fig. 1a and (2) is applicable with $D = 1$ and $k(x) = 0$.

For given total stiffness K_T and load parameter γ, the values of l_2, c_1, c_2, and c_3 which yield equal spring forces can be determined. Results for $K_T = 100$ are presented in Fig. 1b, where l_2 and c_j/K_T are plotted as functions of γ. For $\gamma = 1$, when the load is uniform, $l_2 = 0.5$ and $c_1 = c_3 = 1.70\,c_2$. For this case, if all springs had equal stiffness $c_j = K_T/3$, the force in the central spring would be 50 percent higher than the force in each outer spring. This percentage increases as K_T increases, as does the ratio c_1/c_2 in the optimal solution.

2.2 Beam on foundation subjected to equal end deflections

Suppose that the ends of the beam are displaced down-

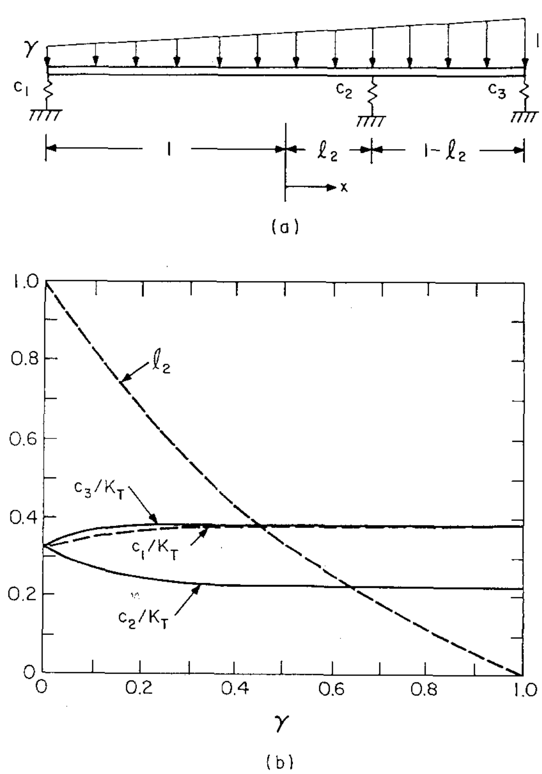

Fig. 1. Beam on elastic springs under linearly distributed load: (a) geometry; (b) spring stiffnesses and internal spring location

wards an amount W_0 and the foundation pressure is to be constant (see Fig. 2a). Define

$$w = \frac{384EIW}{PL^4}, \quad k = \frac{L^4K}{16EI}, \quad K_T = \frac{L^3\overline{K}_T}{8EI},$$
$$w_0 = w(-1) = w(1). \tag{4}$$

Then

$$\int_{-1}^{1} k(x)\,\mathrm{d}x = K_T, \tag{5}$$

and (2) is applicable with $D = 1$, $kw = 24$ and $q = 0$. The optimal foundation stiffness has the form

$$k(x) = 24(w_0 - 5 + 6x^2 - x^4)^{-1}, \tag{6}$$

where $w_0 \geq 5$ is required so that $k(x) \geq 0$ and correspondingly, from (5), K_T must be less than 75.3.

Results based on (6) are shown in Fig. 2b for K_T = 6, 30, 45 and 60. As the total foundation stiffness increases, the concentration of stiffness near the centre of the beam increases. In this example and all subsequent ones, the form of the deflection function w can be inferred from the distribution of k, since kw is constant. Here the deflection is smallest near the centre of the beam and largest near the ends.

A similar problem was treated by Taylor and Bendsøe (1984). They included an upper limit $k_{\max}$ on $k(x)$, and in their numerical example, $k(x) = k_{\max}$ in the central region of the beam.

2.3 Beam on foundation under parabolic load

In this example, let

$$Q(X) = q_1\left(\frac{L^2}{4} - X^2\right), \quad w = \frac{64EIW}{q_1L^6},$$
$$k = \frac{L^4K}{16EI}, \quad K_T = \frac{L^3\overline{K}_T}{8EI} \tag{7}$$

(see Fig. 3a). Then (2) is applicable with $D = 1$, $kw = 2/3$, and $q(x) = 1 - x^2$. The optimal solution has the form

$$k(x) = 240\left(b - 15x^2 + 5x^4 - x^6\right)^{-1}, \tag{8}$$

where b is determined from (5). For $k(x) \geq 0$, it is required that $K_T \leq 130.8$. For K_T = 30, 90 and 120, $k(x)$ is depicted in Fig. 3b. As K_T increases, the available stiffness tends to concentrate near the ends of the beam.

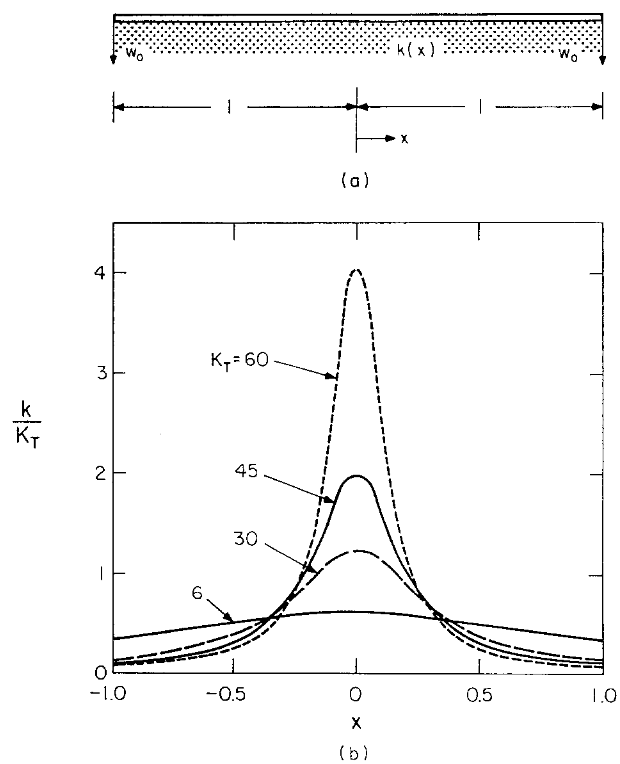

Fig. 2. Beam on foundation with end deflections: (a) geometry; (b) foundation stiffness distribution

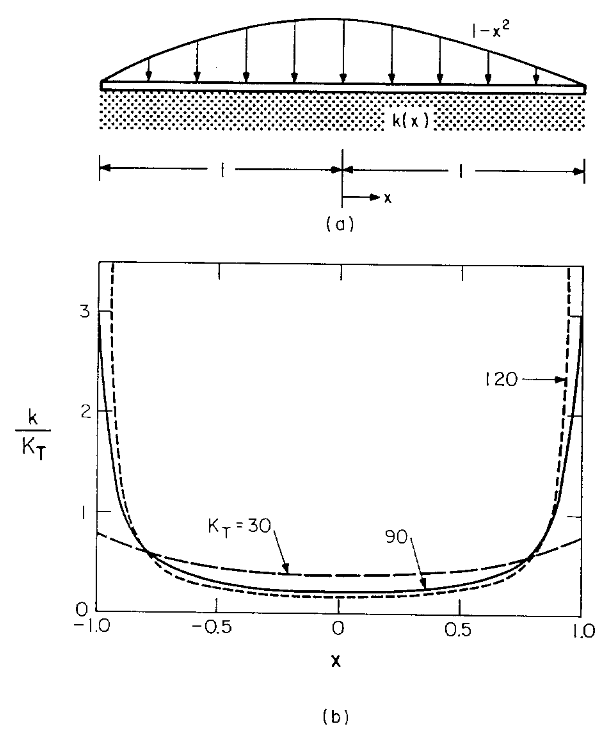

Fig. 3. Beam on foundation under parabolic load: (a) geometry; (b) foundation stiffness distribution

2.4 Beam on foundation and end springs under uniform load

Suppose that the load is uniform and that springs with stiffness C act at the ends of the beam (Fig. 4a). If $C = 0$, the solution would be trivial: $K(X)$ and $W(X)$ would be constant. For nonvanishing C and the constraint of uniform foundation pressure, both $K(X)$ and $W(X)$ vary along the beam. In this example, the value of C is chosen so that the foundation pressure is constant, as always, and in addition the total foundation pressure is equal to the sum of the forces in the two springs [i.e. $KWL = 2CW_0$ where $W_0 = W(L/2) = W(-L/2)$].

Let

$$Q(X) = q_0, \quad w = \frac{16EIW}{q_0 L^4}, \quad k = \frac{L^4 K}{16EI},$$

$$K_T = \frac{L^3 \overline{K}_T}{8EI}, \quad c = \frac{L^3 C}{8EI}. \tag{9}$$

Then (2) is applicable with $D = 1$, $kw = 1 - cw_0$, and $q(x) = 1$, and

$$\int_{-1}^{1} k(x)\,\mathrm{d}x + 2c = K_T, \tag{10}$$

where w_0 denotes the equal nondimensional end deflections of the beam. The optimal stiffness distribution has the form

$$k(x) = (24/w_0)(1 - cw_0)(24 + 5c - 6cx^2 + cx^4)^{-1}, \tag{11}$$

where w_0 is determined from (10). Results are presented in Fig. 4b for $K_T = 24$, 120 and 240. The corresponding values of c are 7.9, 50.9 and 109.2, respectively.

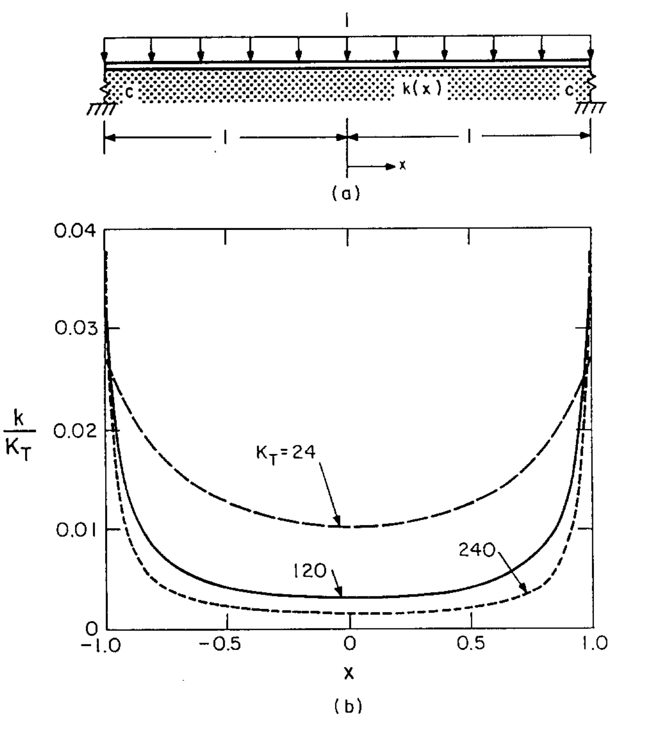

Fig. 4. Beam on foundation and end springs under uniform load: (a) geometry; (b) foundation stiffness distribution

3 Nonuniform beams

In this section, the beam has a rectangular cross-section with variable depth and the volume of the beam is fixed. The depth is assumed to be a piecewise-constant function of X. As in the last example, springs act at the ends of the beam. Here the stiffness C of the springs is specified, along with the total stiffness $\overline{K}_T$. The beam depth variation and the foundation distribution are chosen such that the foundation pressure is uniform and a global measure of the deflection is minimized. Sandwich and solid cross-sections are considered.

3.1 Beam with sandwich cross-section

Suppose the beam has a width B, a core of constant height $2A$ and face sheets with equal thickness T_j for $X_{j-1} < X < X_j$, $j = 1, \ldots, n$, with $X_0 = 0$ and $X_n = L/2$. The beam is symmetric about $X = 0$ and the face-sheet thickness of a uniform beam with the same volume is denoted T_u. The stiffness of the core is neglected, and $I = 2A^2BT_j$ for $X_{j-1} < X < X_j$. For the reference beam, $I_u = 2A^2BT_u$.

Define x and D as in (1) and denote

$$t_j = \frac{T_j}{T_u}, \quad x_j = \frac{2X_j}{L}, \quad k = \frac{L^4 K}{16EI_u}, \quad K_T = \frac{L^3 \overline{K}_T}{8EI_u},$$

$$c = \frac{L^3 C}{8EI_u}, \quad w = \frac{8EI_u W}{Q_0 L^3}, \quad w_0 = w(-1) = w(1), \tag{12}$$

when a concentrated load Q_0 is applied at $X = 0$ (Fig. 5). If the beam is subjected to a uniform load q_0, the quantity Q_0 in (12) is replaced by $q_0 L/2$. Assume that constraint (10) on total stiffness of the foundation and springs is satisfied. The volume constraint is given by

$$\sum_{j=1}^{n} (x_j - x_{j-1}) t_j = 1, \tag{13}$$

where $x_0 = 0$ and $x_n = 1$. In the equilibrium equation (2), $D = t_j$ for $x_{j-1} < x < x_j$, $q(x) = \delta(x)$ for the concentrated load (with $kw = 0.5 - cw_0$), and $q(x) = 1$ for the uniform load (with $kw = 1 - cw_0$). The equation is solved analytically in each region with constant depth and continuity conditions on deflection, slope, bending moment and shear force are then applied, along with boundary conditions.

The objective functional

$$G = \int_{-1}^{1} w^2(x)\, \mathrm{d}x \tag{14}$$

is minimized with the use of a modified steepest descent method. In the numerical results, the beam is divided into 40 segments of equal length (i.e. $x_j = j/n$, $n = 20$), $c = 20$ and $K_T = 60$. The optimal thickness variations of the face sheets are depicted in Fig. 6 and the corresponding optimal distributions of foundation stiffness are shown in Fig. 7. Solid lines are associated with the central concentrated load and dashed lines are for the uniform load. The thickness tends to zero at the ends and a maximum at the centre, where it is larger for the concentrated load, while the foundation stiffness is lowest at the centre.

If the beam were uniform ($t_j = 1$) and the foundation stiffness were constant ($k = 10$), the maximum foundation pressure would be 72 percent higher than the pressure for the optimal solution in the case of a concentrated load and 30 percent higher for the uniform load case. If the beam were uniform and the foundation were designed for uniform pressure, as in Section 2, the value of G in (14) would be 77 percent higher than the value for the optimal nonuniform beam for the concentrated load and 28 percent higher for the uniform load.

The optimal thickness variations in Fig. 6 can be approximated by simple analytical functions $t(x)$. For the concentrated load, if one considers a symmetric beam with

$$t(x) = 2(1 - x) \quad \text{for} \quad 0 \le x \le 1\,, \tag{15}$$

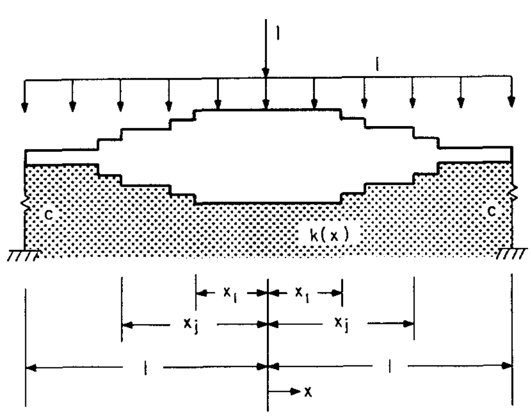

Fig. 5. Geometry of nonuniform beam on foundation and end springs

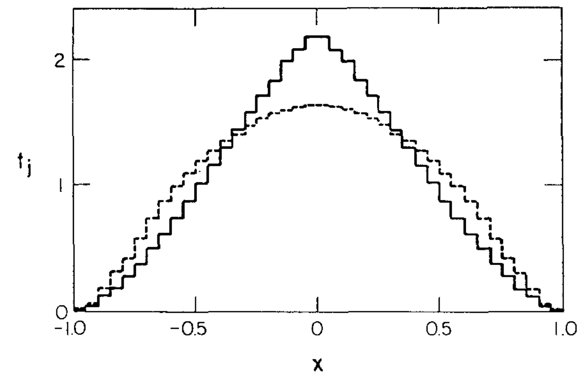

Fig. 6. Optimal face-sheet thickness variation for sandwich beam with $c = 20$ and $K_T = 60$ under concentrated load (——) and uniform load (- - - -)

then the symmetric stiffness distribution providing uniform foundation pressure is given by

$$k(x) = (0.263 - 0.338x^2 + 0.0834x^3)^{-1}\,, \tag{16}$$

and the corresponding value of G is 8.8 percent higher than that for the design in Fig. 6. For the uniform load, the thickness variation

$$t(x) = 1.5(1 - x^2) \tag{17}$$

leads to

$$k(x) = (0.179 - 0.156x^2)^{-1} \tag{18}$$

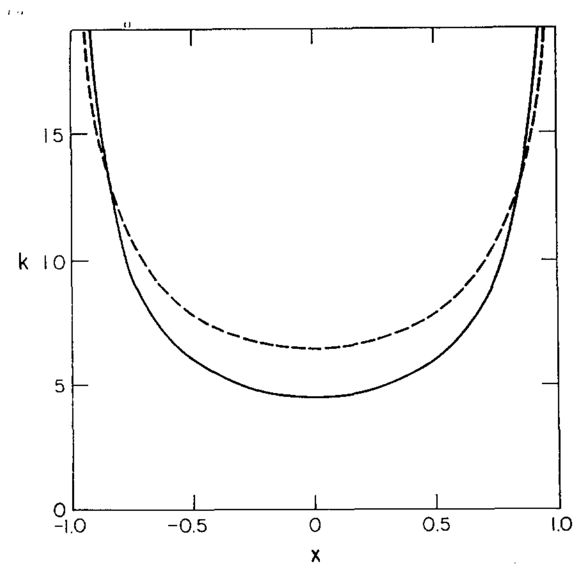

Fig. 7. Optimal foundation stiffness distribution for sandwich beam with $c = 20$ and $K_T = 60$ under concentrated load (——) and uniform load (- - - -)

for uniform pressure and a value of G that is 3.5 percent higher than that for the optimal piecewise-constant design shown in Fig. 6.

3.2 Beam with solid cross-section

Suppose the beam is homogeneous with a rectangular cross-section having width B, depth H_j and moment of inertia $I = BH_j^3/12$ for $X_{j-1} < X < X_j$, $j = 1, \dots, n$ with $H_j = H_u$ and $I = I_u$ for the reference uniform beam. Using the nondimensional quantities in (1) and (12) and defining $h_j = H_j/H_u$, the constraint of constant volume is given by (13) with t_j replaced by h_j and the foundation stiffness constraint is still given by (10). The equilibrium equation is given by (2) with $D = h_j^3$ for $x_{j-1} < x < x_j$ and the same quantities $q(x)$ and kw as for the sandwich cross-section.

Again, the functional G in (14) is minimized for the case of 40 equal-length segments, $c = 20$ and $K_T = 60$. The optimal depth variations for a concentrated load (solid lines) and a uniform load (dashed lines) are depicted in Fig. 8. The corresponding foundation stiffness distributions are very similar to those in Fig. 7. If the beam were uniform ($h_j = 1$) and the foundation stiffness were constant ($k = 10$), the maximum foundation pressure would be 91 percent higher than the pressure for the optimal solution in the case of a concentrated load and 41 percent higher for the uniform load case. If the beam were uniform and the foundation were designed for uniform pressure, as in Section 2, the value of G in (14) would be 128 percent higher in the case of a concentrated load and 49 percent higher in the case of a uniform load. These improvements for the solid beam are greater than those for the sandwich beam.

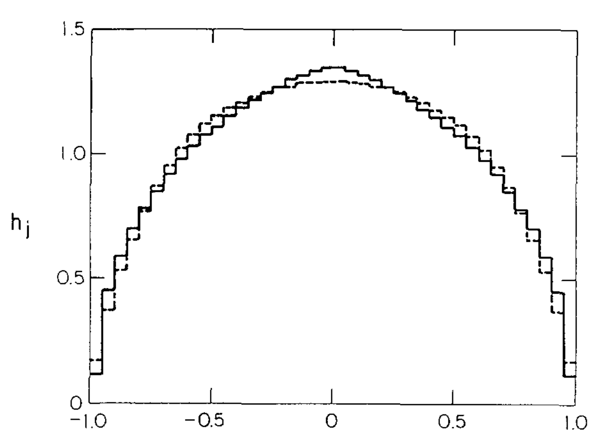

Fig. 8. Optimal depth variation for solid beam with $c = 20$ and $K_T = 60$ under concentrated load (——) and uniform load (- - - -)

4 Uniform circular plates

Consider a circular plate with radius R, Young's modulus E, Poisson's ratio ν, flexural rigidity $\overline{D}$, radial coordinate $\overline{r}$, downward deflection $W(\overline{r})$ under a downward axisymmetric load $Q(\overline{r})$, an axisymmetric elastic foundation with stiffness (per unit area) $K(\overline{r})$ and an elastic support at the edge $\overline{r} = R$ with stiffness (per unit length) C. The total stiffness of the foundation and elastic support is denoted $\overline{K}_T$ and D_u is the flexural rigidity of a reference plate (which will equal $\overline{D}$ in this section).

Let

$$P = K(\overline{r})W(\overline{r}), \quad r = \frac{\overline{r}}{R}, \quad D = \frac{\overline{D}}{D_u},$$

$$w = \frac{D_u W}{\beta}, \quad k = \frac{R^4 K}{D_u}, \quad K_T = \frac{R^2 \overline{K}_T}{2\pi D_u},$$

$$c = \frac{R^3 C}{D_u}, \quad w_0 = w(1), \tag{19}$$

where P is the constant foundation pressure and β depends on $Q(\overline{r})$. In nondimensional terms,

$$k(r)w(r) = 1 - 2cw_0 \tag{20}$$

(except for the case in Section 4.1), the constraint on total stiffness becomes

$$\int_0^1 k(r) r \, \mathrm{d}r + c = K_T\,, \tag{21}$$

and the equilibrium equation is given by

$$D \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{ r \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w}{\mathrm{d}r}\right)\right]\right\} + kw = q(r)\,, \tag{22}$$

where $q(r)$ is defined in each example. Since kw is constant, (22) can be solved analytically.

In the following examples with uniform plates, the spring constant C will be chosen to minimize the functional J defined by

$$J = F_s^5 + F_f^5, \quad F_s = 2\pi RCW_0, \quad F_f = \pi R^2 P\,, \tag{23}$$

where $W_0 = W(R)$. Here F_s is the total force in the elastic support and F_f is the total force on the foundation. Minimization of J tends to equalize these two forces.

4.1 Plate subjected to edge deflection

Suppose that the edge of the plate is displaced downwards an amount W_0, as shown in Fig. 9a in nondimensional terms. If $\beta = R^4 P$ in (19), then (22) is applicable with $D = 1$, $kw = 1$ and $q = 0$. The optimal foundation stiffness is

$$k(r) = 64(64w_0 - 3 + 4r^2 - r^4)^{-1}, \quad (24)$$

where w_0 is determined from constraint (21). It is required that $w_0 > 3/64$ to assure that $k(r)$ is positive. Results are presented in Fig. 9b for $K_T = 1$, 10 and 100. For $K_T = 100$ the value of $k(0)/K_T$ is 62.5. As for beams, the stiffness tends to concentrate near the centre of the plate as K_T increases.

4.2 Plate under uniform load

Suppose that $Q(\bar{r}) = q_0$ (Fig. 10a) and $\beta = q_0 R^4$ in (19). Then $D = 1$ and $q = 1$ in (22) and the optimal solution has the form

$$k(r) = 32(1 - 2cw_0)w_0^{-1}(32 + 3c - 4cr^2 + cr^4)^{-1}, \quad (25)$$

where once again w_0 is determined from (21). This stiffness distribution is plotted in Fig. 10b for $K_T = 1, 10$ and 100. As K_T increases, the stiffness tends to concentrate near the edge of the plate.

4.3 Plate under parabolic load

In this example, $Q(\bar{r}) = q_1(R^2 - \bar{r}^2)$, as shown in Fig. 11a, $\beta = q_1 R^6$ in (19), and in (22), $D = 1$ and $q = 1 - r^2$. The optimal foundation stiffness distribution is depicted in Fig. 11b for $K_T = 1$, 10 and 25. If K_T is too large, a solution with $k \geq 0$, $0 \leq r < 1$ does not exist.

5 Nonuniform circular plates

In this section, the depth of the plate is assumed to be a piecewise-constant function of $\bar{r}$ and the volume of the plate is fixed. The stiffness C of the elastic support at the edge is specified, along with the total stiffness $\overline{K}_T$. In the optimal solution, the foundation pressure is uniform and the displaced volume under given loading is minimized.

5.1 Plate with sandwich cross-section

Suppose the plate has a core of constant depth and face sheets with equal thickness T_j for $\bar{r}_{j-1} < \bar{r} < \bar{r}_j$, $j = 1, \ldots, n$ with $\bar{r}_0 = 0$ and $\bar{r}_n = R$. Denote the corresponding flexural rigidity by $\overline{D}_j$ and let T_u and D_u be values for a reference uniform plate with the same volume. In addition to the nondimensional quantities in (19), define

$$t_j = \frac{T_j}{T_u}, \quad r_j = \frac{\bar{r}_j}{R}. \quad (26)$$

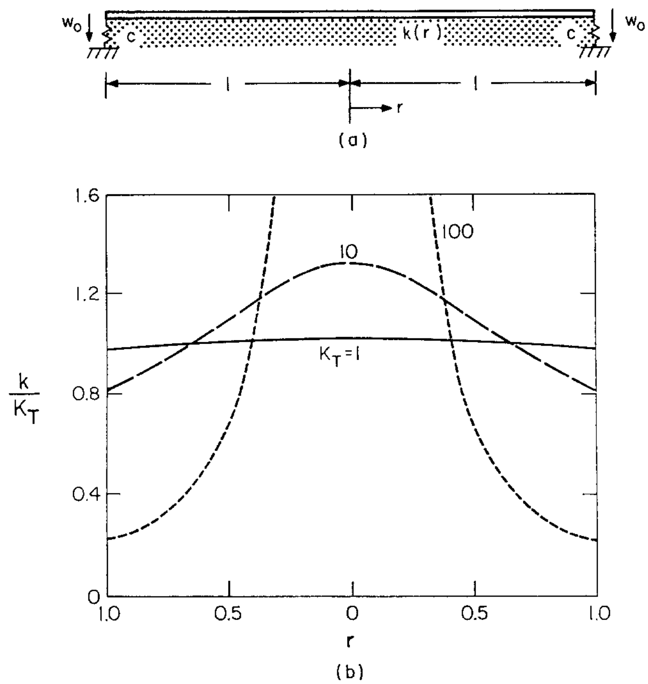

Fig. 9. Circular plate with edge deflection: (a) geometry; (b) foundation stiffness distribution

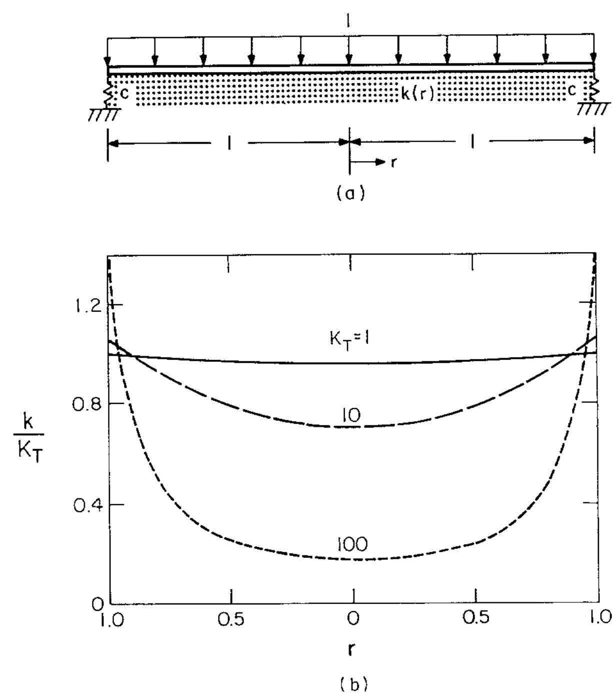

Fig. 10. Circular plate under uniform load: (a) geometry; (b) foundation stiffness distribution

Then (20)-(22) are applicable with $D = t_j$ for $r_{j-1} < r < r_j$ If a concentrated load Q_0 is applied at the centre of the plate, $\beta = Q_0 R^2/\pi$ in (19) and $q = \delta(r)$ in (22), while for a uniform load q_0, $\beta = q_0 R^4$ and $q = 1$. In nondimensional terms, a cross-section along a diameter of the plate would look like Fig. 5 with x, x_1 and x_j replaced by r, r_1 and r_j, respectively.

The volume constraint is given by

$$\sum_{j=1}^{n} (r_j^2 - r_{j-1}^2) t_j = 1 . \tag{27}$$

As for nonuniform beams, the equilibrium equation can be solved analytically. The objective functional to be minimized here is chosen to be

$$G = \int_0^1 w(r) r \, dr . \tag{28}$$

For the case of a uniform load, G is proportional to the compliance (i.e. the work done by the load). The foundation design function $k(r)$ and plate design parameters t_j must satisfy constraints (21) and (27).

In the numerical results, $r_j = j/n$, $n = 10$, $\nu = 0.3$, $c = 20$ and $K_T = 40$. Figure 12 depicts the optimal thickness variations of the face sheets and Fig. 13 illustrates the corresponding foundation stiffness distributions. In comparison with the results in Fig. 6 for a sandwich beam, the thickness of the face sheets for the plate is not concentrated as much near the centre. If the plate thickness were uniform ($t_j = 1$) and the foundation were designed for uniform pressure, as in Section 4, the value of G in (28) would be 35 percent higher in the case of the concentrated load and 5 percent higher for the uniform load.

5.2 Plate with solid cross-section

Suppose the plate is homogeneous and has depth H_j for

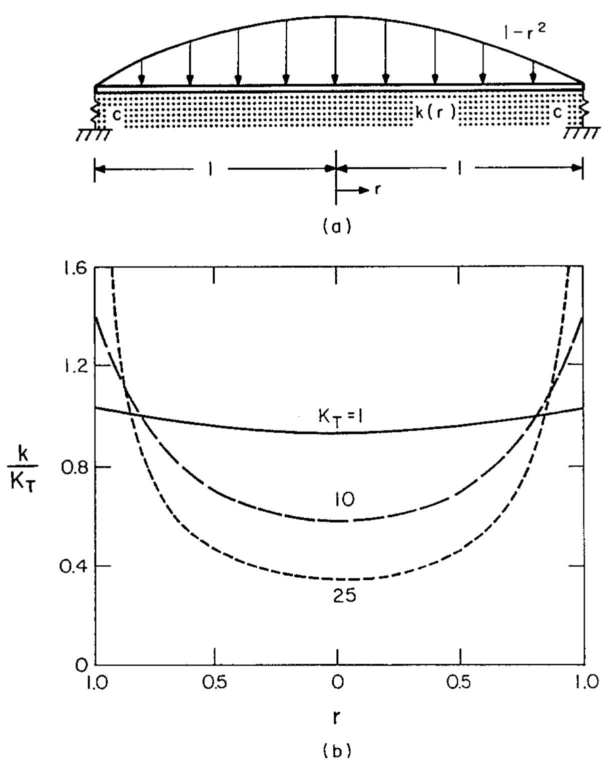

Fig. 11. Circular plate under parabolic load: (a) geometry; (b) foundation stiffness distribution

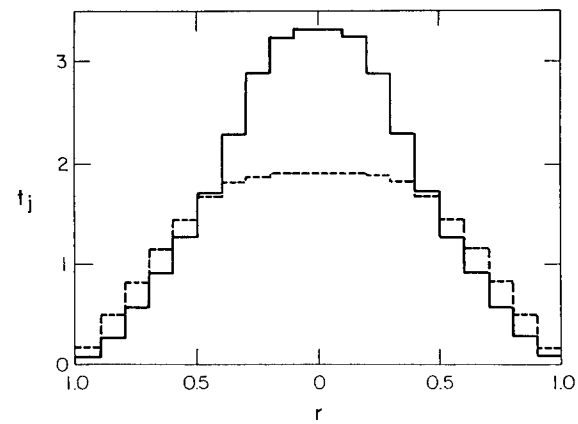

Fig. 12. Optimal face-sheet thickness variation for sandwich circular plate with $c = 20$, $K_T = 40$ and $\nu = 0.3$ under concentrated load (——) and uniform load (- - - -)

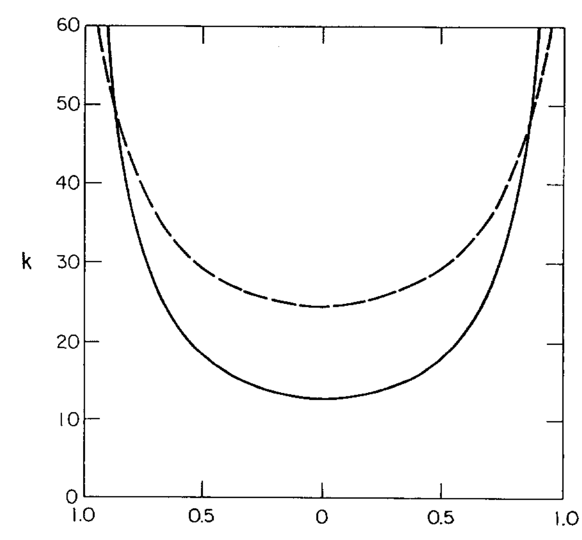

Fig. 13. Optimal foundation stiffness distribution for sandwich circular plate with $c = 20$, $K_T = 40$ and $\nu = 0.3$ under concentrated load (——) and uniform load (- - - -)

$\bar{r}_{j-1} < \bar{r} < \bar{r}_j$, with $H_j = H_u$ and $\overline{D}_j = D_u$ for the reference uniform plate. Using the nondimensional quantities in (19) and defining $h_j = H_j/H_u$, the volume constraint is given by (27) with t_j replaced by h_j, and the equilibrium equation is given by (22) with $D = h_j^3$ for $r_{j-1} < r < r_j$. Again, (20) and (21) are satisfied and G in (28) is minimized.

Using the same loading conditions and parameters as for the sandwich cross-section, the optimal depth variations of the solid plate are presented in Fig. 14. The corresponding foundations are similar to those in Fig. 13. If the plate were uniform ($h_j = 1$) and the foundation were designed for uniform pressure, the value of G in (28) would be 67 percent higher for the concentrated load case and 11 percent higher for the uniform load.

6 Concluding remarks

Symmetric beams and axisymmetric circular plates on elastic foundations of the Winkler type have been treated. If the stiffness of the foundation can be distributed in a nonuniform manner, various objectives can be optimized, such as vibration frequencies, buckling loads and compliance. Attention here has been focused on the foundation pressure, which also was considered in an example by Taylor and Bendsøe (1984). Often the maximum pressure which would be transmitted to a uniform foundation can be reduced substantially by an optimal distribution of the foundation stiffness.

If K denotes the foundation stiffness ($K > 0$) and W is the downward deflection of the beam or plate, then KW is the pressure. With the exception of an initial example which only involved spring supports, the foundations here were designed so that the pressure was uniform, i.e. KW was constant and the minimum pressure was equal to the maximum pressure. This meant that W could not be zero or negative. Rigid supports with zero deflection could not be included, as well as beams or plates which would deflect upward in some regions and cause a negative pressure (tension) or lift off the foundation (if it only acts under compression). In such cases, the constraint of uniform pressure could be replaced by the objective of minimizing the maximum pressure. This topic is left for future research, as well as consideration of other foundation models.

In some of the examples, end springs were present in addition to the elastic foundation, with the total stiffness specified. For uniform beams and plates, the stiffnesses of these elastic supports were chosen to equalize the total forces acting on the foundation and on these end supports (Sections 2.4 and 4.1-4.3); for nonuniform beams and plates, the spring stiffnesses were fixed (Sections 3 and 5).

An additional objective can be considered if both the foundation stiffness and the structural material can be distributed optimally. In Sections 3 and 5, global measures of the deflection were minimized, subject to the constraint of uniform foundation pressure. If the depth of the beam or plate were allowed to vary continuously with position, the resulting optimality conditions could not be solved analytically. Therefore, piecewise-constant functions were employed. The lengths of the uniform-depth segments were specified, although they also could have been optimized. Sandwich and solid cross-sections were analysed, with concentrated central loads and uniform loads. The percentage decrease in the deflection measure was found to be greater for a concentrated load than a uniform load, and greater for a solid cross-section than a sandwich cross-section.

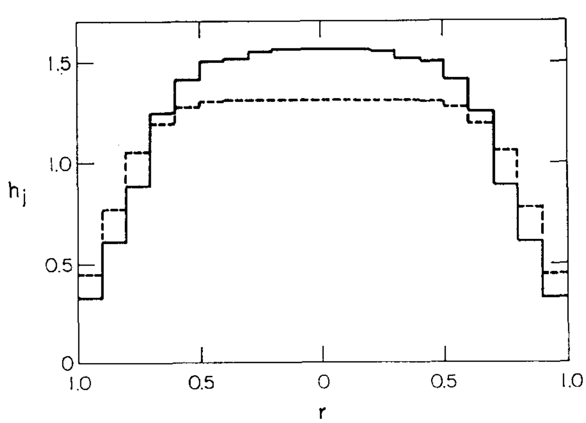

Fig. 14. Optimal depth variation for solid circular plate with $c = 20$, $K_T = 40$ and $\nu = 0.3$ under concentrated load (——) and uniform load (- - - -)

References

Åkesson, B.; Olhoff, N. 1988: Minimum stiffness of optimally located supports for maximum value of beam eigenfrequencies. *J. Sound Vib.* **120**, 457-463

Dems, K.; Plaut, R.H.; Banach, A.S.; Johnson, L.W. 1987: Optimization of elastic foundation for minimum beam deflection. *Int. J. Solids Struct.* **23**, 1551-1562

Pičuga, A. 1988: Optimality conditions for multiple loaded structures – integrating control and finite element method. In: Rozvany, G.I.N.; Karihaloo, B.L. (eds.) *Structural optimization.* Dordrecht: Kluwer

Plaut, R.H. 1987: Optimal beam and plate foundations for minimum compliance. *J. Appl. Mech.* **54**, 255-257

Plaut, R.H. 1989: Simultaneous optimization of beams and their elastic foundations for minimum compliance. *J. Appl. Mech.* **56**, 629-632

Rozvany, G.I N. 1989: *Structural design via optimality criteria: the Prager approach.* Dordrecht: Kluwer

Shin, Y.S.; Haftka, R.T.; Plaut, R.H. 1988: Simultaneous analysis and design for eigenvalue maximization. *AIAA J.* **26**, 738-744

Shin, Y.S.; Haftka, R.T.; Watson, L.T.; Plaut, R.H. 1988: Tracing structural optima as a function of available resources by a homotopy method. *Comp. Meth. Appl. Mech. Engrg.* **70**, 151-164

Szelag, D.; Mróz, Z. 1978: Optimal design of elastic beams with unspecified support conditions. *Zeit. ang. Math. Mech.* **58**, 501-510

Taylor, J.E.; Bendsøe, M.P. 1984: An interpretation for min-max structural design problems including a method for relaxing constraints. *Int. J. Solids Struct.* **20**, 301-314

Received Jan. 16, 1990