Contributions to the Theory of Generalized Differential Equations. II

by

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Introduction

In this paper we continue the study, initiated in [1], of generalized differential equations and their solution families. Our principal concern is an investigation of some implications of Hermes' approximation theorem [2, Theorem 4]. This writer has elsewhere given a paraphrase [3, Theorem 1] of part of the content of Hermes' theorem. In this paper we state a paraphrase (Theorem 1.1) of another part of the content of Hermes' theorem. These two paraphrases are then utilized to obtain two separate improvements of an approximation theorem [1, Theorem 1.3] originally obtained by Zaremba [4, II.8]. One of these improvements leads to a generalization of the fundamental existence theorem [1, Theorem 2.3], [4, II.9] in a form which permits the extension of the rationale of Liapunov's second method to problems of weak stability [5, 6] associated with generalized differential equations. The other improvement of Zaremba's approximation theorem permits a further extension of Zaremba's generalization [4, III.6] of Kneser's classical theorem [7, pp. 15-16] on the connectedness of the attainable set of an ordinary differential equation. Inasmuch as heavy reliance is placed on [1], not only for fundamental results but also for notation and terminology, a familiarity with that paper is assumed.

1. Approximation of Convex Set-Valued Functions

For convenient reference we state first, as Theorem A, the author's paraphrase [3, Theorem 1 and Remark 2] of Hermes' approximation theorem [2, Theorem 4]. For this purpose let $N = I \times W$, where I is a compact interval in E^1 and W is a compact subset of E^n.

THEOREM A. *If R: $N \to \Gamma^n$ is continuous, then there exists a sequence $\{S_k\}$ of continuous functions S_k: $N \to \Gamma^n$ w.th support functions g^k defined by*

$$g^k(t, x, p) \equiv g(S_k(t, x), p),$$

having the following properties:

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(i) *for each k and all* $(t, x) \in N$, $S_k(t, x)$ *is strictly convex and* g^k *is of class* C^2 *on* $N \times (E^n - \{0\})$;

(ii) *for each k and all* $(t, x) \in N$, $\bar{\Delta}(S_{k+1}(t, x), S_k(t, x)) = 0$;

(iii) *for each k and all* $(t, x) \in N$,

$$\bar{\Delta}((R(t, x))^{2^{-(k+1)}}, S_k(t, x)) = \bar{\Delta}(S_k(t, x), (R(t, x))^{2^{-k}}) = 0.$$

Our second paraphrase of Hermes' approximation theorem is

THEOREM 1.1. *If* $R\colon N \to \Gamma^n$ *is continuous, then for each k there exist functions* $\zeta^k\colon N \to E^n$ *and* $\lambda^k\colon N \times (E^n - \{0\}) \to E^n$, *both of class* C^1 *and with* λ^k *satisfying* $\lambda^k(t, x, \alpha\mu) = \lambda^k(t, x, \mu)$, $\alpha > 0$, *such that the sequence* $\{T_k\}$ *of functions* $T_k\colon N \to \Gamma^n$ *defined by*

$$T_k(t, x) = \{f^k(t, x, \mu)\colon \|\mu\| \leqq 1\},$$

where

$$f^k(t, x, \mu) = \zeta^k(t, x)(1 - \|\mu\|) + \|\mu\|\, \lambda^k(t, x, \mu), \mu \neq 0,$$
$$f^k(t, x, 0) = \zeta^k(t, x),$$

has the following properties:

(i) *for each k and all* $(t, x) \in N$, $\bar{\Delta}(T_{k+1}(t, x), T_k(t, x)) = 0$;

(ii) *for each k and all* $(t, x) \in N$,

$$\bar{\Delta}((R(t, x))^{2^{-(k+1)}}, T_k(t, x)) = \bar{\Delta}(T_k(t, x), (R(t, x))^{2^{-k}}) = 0.$$

For the proof of Theorem 1.1 we shall need the following preliminary results.

LEMMA 1.1. *Let* $A \in \Gamma^n$ *be strictly convex and smooth (i.e., each point of the boundary of A lies on a unique support hyperplane); then with* $g_p(A, p)$ *denoting the gradient with respect to p of* $g(A, \cdot)$, *for each* $\beta \in (0, 1)$ *and each* $p \neq 0$ *the point*

$$\zeta(\beta, p) = \beta g_p(A, p) + (1 - \beta) g_p(A, -p)$$

satisfies $\zeta(\beta, p) \in \operatorname{int} A$.

Proof. By [3, Lemma 1] we have

$$g(A, p) = p \circ g_p(A, p), \qquad \|p\| = 1,$$

and the smoothness of A implies that if $p_1 \neq p_2$ then $g_p(A, p_1) \neq g_p(A, p_2)$. For $\|p\| = \|p_0\| = 1$ and $p \neq p_0$, we have

$$p \circ \zeta(\beta, p_0) = \beta p \circ g_p(A, p_0) + (1 - \beta) p \circ g_p(A, -p_0),$$

and then the aforementioned consequence of smoothness together with [3, Lemma 1] implies

$$\circ \zeta(\beta, p_0) < \beta g(A, p) + (1 - \beta) g(A, p) = g(A, p).$$

Moreover

$$p_0 \circ \zeta(\beta, p_0) = \beta g(A, p_0) - (1 - \beta) g(A, -p_0)$$

and since, by virtue of the restrictions on A, $-g(A, -p) < g(A, p)$, $p \neq 0$, we conclude that $p_0 \circ \zeta(\beta, p_0) < g(A, p_0)$. The proof is completed by invoking [1, Lemma 1.4].

COROLLARY 1.1. *Let A be strictly convex and smooth and let $\beta_0 \in (0, 1)$, $p_0 \in E^n - \{0\}$ be fixed; with $\tilde{B} = \{p: \|p\| \leqq 1\}$, the function $\varphi: E^n \to E^n$ defined by*

$$\varphi(p) = \zeta(\beta_0, p_0)(1-\|p\|)+\|p\| g_p(A, p), \qquad p \neq 0,$$
$$\varphi(0) = \zeta(\beta_0, p_0),$$

is a homeomorphism of $\tilde{B}$ onto A.

Proof. That φ is continuous on $E^n - \{0\}$ is a consequence of the continuity of $g_p(A, \cdot)$ on $E^n - \{0\}$ [3, Lemma 1]. Moreover, by the same token, $\|g_p(A, p)\| = \|g_p(A, p/\|p\|)\| \leqq K$, so that

$$\|\varphi(p)-\varphi(0)\| \leqq \|p\| (K+\|\zeta(\beta_0, p_0)\|)$$

when $0 < \|p\| \leqq 1$, implying the continuity of φ at $p = 0$. Now let $\xi \in A$; then the half-line with endpoint at $\zeta(\beta_0, p_0)$ and passing through ξ intersects the boundary of A in a unique point, $x(\xi)$, which lies on a unique support hyperplane to A with outward unit normal $p^* = p^*(x(\xi))$. It follows that $\xi = \varphi(\gamma p^*)$, where

$$\gamma = \gamma(\xi) = \frac{\|\xi - \zeta(\beta_0, p_0)\|}{\|x(\xi) - \zeta(\beta_0, p_0)\|};$$

hence φ is onto and the function inverse to φ is given by

$$\eta(\zeta(\beta_0, p_0)) = 0$$

$$\eta(\xi) = \gamma(\xi)p^*(x(\xi)), \qquad \xi \neq \zeta(\beta_0, p_0).$$

The continuity of η then ensues from that of φ and the compactness of $\tilde{B}$.

Now let us prove Theorem 1.1. Denoting by y a generic point $(t, x) \in N$, we direct the reader's attention to the proof of [3, Theorem 1]. The sets $Q(y_i, \epsilon)$ of that proof satisfy the hypotheses of Corollary 1.1; we denote by $\varphi^\epsilon(y_i, p)$ the mapping of that corollary corresponding to $Q(y_i, \epsilon)$. Then by the argument of [3, Theorem 1], $\varphi^\epsilon(y_i, \cdot)$ is of class C^1 on $E^n - \{0\}$. For the function $Q(\cdot, \epsilon)$ defined by [3, (4)] it follows readily that $Q(y, \epsilon)$ has the representation

$$Q(y, \epsilon) = \{\varphi^\epsilon(y, p): \|p\| \leqq 1\},$$

where $\varphi^\epsilon(y, p)$ is defined as

$$\varphi^\epsilon(y, p) = \sum_{i=1}^{m+1} \alpha_i(y)\varphi^\epsilon(y_i, p).$$

The continuity of $\varphi^\epsilon(\cdot, \cdot)$ and $\varphi_p^\epsilon(\cdot, \cdot)$ now follows from Corollary 1.1 and the proof of [3, Theorem 1]. Imitating Hermes' use [2, Theorem 4] of the mollifier technique and invoking [3, Remark 2], we complete the proof of Theorem 1.1.

The next two theorems are the promised improvements of [1, Theorem 1.3]. Only Theorem 1.2 will be proved, the proof of Theorem 1.3 being similar.

THEOREM 1.2. *If $R: N \to \Gamma^n$ is upper semicontinuous, then there exists a sequence $\{Q_m\}$ of continuous functions $Q_m: N \to \Gamma^n$, with support functions ω^m defined by*

$$\omega^m(t, x, p) \equiv g(Q_m(t, x), p),$$

having the following properties:

(i) *for each m and all $(t, x) \in N$, $Q_m(t, x)$ is strictly convex and ω^m is of class C^2 on $N \times (E^n - \{0\})$;*
(ii) *for each m and all $(t, x) \in N$, $\bar{\Delta}(Q_{m+1}(t, x), Q_m(t, x)) = 0$;*
(iii) *for each m and all $(t, x) \in N$, $\bar{\Delta}(R(t, x), Q_m(t, x)) = 0$;*
(iv) *for each $(t, x) \in N$, $\lim_{m\to\infty} \bar{\Delta}(Q_m(t, x), R(t, x)) = 0$.*

THEOREM 1.3. *If $R: N \to \Gamma^n$ is upper semicontinuous, then for each m there exist functions $\zeta^m: N \to E^n$ and $\lambda^m: N \times (E^n - \{0\}) \to E^n$, both of class C^1 and with λ^m satisfying*

$$\lambda^m(t, x, \alpha\mu) = \lambda^m(t, x, \mu), \qquad \alpha > 0,$$

such that the sequence $\{Z_m\}$ of functions $Z_m: N \to \Gamma^n$ defined by

$$Z_m(t, x) = \{f^m(t, x, \mu): \|\mu\| \leqq 1\},$$

where

$$f^m(t, x, \mu) = \zeta^m(t, x)(1 - \|\mu\|) + \|\mu\|\lambda^m(t, x, \mu), \mu \neq 0,$$
$$f^m(t, x, 0) = \zeta^m(t, x),$$

has the following properties:

(i) *for each m and all $(t, x) \in N$, $\bar{\Delta}(Z_{m+1}(t, x), Z_m(t, x)) = 0$;*
(ii) *for each m and all $(t, x) \in N$, $\bar{\Delta}(R(t, x), Z_m(t, x)) = 0$;*
(iii) *for each $(t, x) \in N$, $\lim_{m\to\infty} \bar{\Delta}(Z_m(t, x), R(t, x)) = 0$.*

For the proof of Theorem 1.2, let us call the approximations to $R(t, x)$ corresponding to [1, Theorem 1.3] and to Theorem A approximations of types I and II respectively. Let $\{R_m\}$ be an approximation of type I to R on N and, for each m, let $\{S_k^m\}$ be an approximation of type II to R_m on N. Defining $Q_m = S_m^m$, we see that (i) is an obvious consequence of Theorem A (i). By virtue of [1, Theorem 1.3, (iii)] and Theorem A (iii), (iii) follows from the estimate

$$\bar{\Delta}(R(t, x), Q_m(t, x)) \leqq \bar{\Delta}(R(t, x), R_m(t, x)) + \bar{\Delta}(R_m(t, x), Q_m(t, x)).$$

For the proof of (ii), we find that

$$\begin{aligned}\bar{\Delta}(Q_{m+1}(t, x), Q_m(t, x)) = {} & \bar{\Delta}(Q_{m+1}(t, x), (R_{m+1}(t, x))^{2^{-(m+1)}}) \\ & + \bar{\Delta}((R_{m+1}(t, x))^{2^{-(m+1)}}, (R_m(t, x))^{2^{-(m+1)}}) \\ & + \bar{\Delta}((R_m(t, x))^{2^{-(m+1)}}, Q_m(t, x)).\end{aligned}$$

The first and third terms of the right member of this inequality are zero by virtue of Theorem A (iii); the second term is zero by virtue of [1, Theorem 1.3 (ii)]. Finally, from Theorem A (iii) we obtain

$$\bar{\Delta}(Q_m(t, x), R(t, x)) \leqq 2^{-m} + \bar{\Delta}(R_m(t, x), R(t, x)),$$

from which (iv) follows by virtue of [1, Theorem 1.3 (iv)].

2. A New Existence Theorem for Generalized Differential Equations

The central result of this section is a generalization of the fundamental existence theorem [1, Theorem 2.3], [4, II.9] for the Cauchy problem

$$\dot{x} \in R(t, x), \; x(t_0) = x_0. \tag{1}$$

We shall need the class $\mathscr{V}(D)$ of functions $V\colon D \to E^1$, where D is an open subset of $E^1 \times E^n$, which are of class C^1 on D and for which the gradient $V_x(t, x)$ with respect to x does not vanish on D.

THEOREM 2.1. *Let D be an open subset of $E^1 \times E^n$ containing (t_0, x_0) and let $R\colon D \to \Gamma^n$ be upper semicontinuous; then for each $V \in \mathscr{V}(D)$ there exists a solution φ of* (1) *satisfying*

$$V'(t, \varphi(t)) \equiv \frac{d}{dt} V(t, \varphi(t)) = V_t(t, \varphi(t)) - g(R(t, \varphi(t)), \; -V_x(t, \varphi(t)))$$

almost everywhere on its maximal interval of existence. Moreover, all solutions of (1) *may be continued to the boundary of D.*

Proof. Let the set $N = I \times W$ of the preceding section be chosen in such a way that t_0 is the midpoint of I, W is a neighborhood of x_0 and $N \subset D$. On N we construct an approximation to R of the type given in Theorem 1.2; we condense our notation by defining

$$\nu^m(t, x, p) = -\omega^m(t, x, -p),$$

$$\nu(t, x, p) = -g(R(t, x), -p).$$

The functions $k^m\colon N \to E^n$ defined by

$$k^m(t, x) = \nu_p^m(t, x, V_x(t, x)), \qquad m = 1, 2, 3, \cdots,$$

are continuous on N. As in the proof of [2, Theorem 5], we may infer that the function $\zeta\colon N \to E^1$ defined by

$$\zeta(t, x) = \max \{\|\sigma\|: \sigma \in Q_1(t, x)\}$$

has a maximum M on N. It follows that

$$\|k^m(t, x)\| \leqq M, \qquad (t, x) \in N, \qquad m = 1, 2, 3, \cdots,$$

so that there is an interval $[t_1, t_2]$ containing t_0 in its interior and independent of m on which all solutions of the differential equations

$$\dot{x} = k^m(t, x), \quad x(t_0) = x_0, \quad m = 1, 2, 3, \cdots, \tag{2}$$

exist. Since $k^m(t, x) \in Q_m(t, x)$ on N, an argument like that for [1, Theorem 2.3] shows that every sequence $\{x^m\}$ of solutions of (2) on $[t_1, t_2]$ has a subsequence which converges uniformly on that interval to a solution φ of (1).

The function $\nu(\cdot, \varphi(\cdot), V_x(\cdot, \varphi(\cdot)))$ is Lebesgue summable on $[t_1, t_2]$ since, by [1, Corollary 1.2], it is lower semicontinuous—hence measurable—on $[t_1, t_2]$ and by [1, Lemma 1.6] it is bounded on $[t_1, t_2]$. Let us define $V^*\colon D \to E^1$ by

$$V^*(t, x) = V_t(t, x) + \nu(t, x, V_x(t, x)); \tag{3}$$

it is a consequence of the definition of ν and the fact that φ is a solution of (1) that

$$V'(t, \varphi(t)) \geqq V^*(t, \varphi(t))$$

almost everywhere on $[t_1, t_2]$. Let us suppose that the set P defined by

$$P = \{t \in [t_1, t_2]: V'(t, \varphi(t)) > V^*(t, \varphi(t))\}$$

has positive Lebesgue measure; we shall show that this assumption leads to a contradiction. Defining the positive number ξ by

$$\xi = 2^{-1} \int_P [V'(\tau, \varphi(\tau)) - V^*(\tau, \varphi(\tau))]d\tau,$$

we find that

$$V(t_2, \varphi(t_2)) - V(t_1, \varphi(t_1)) = \int_{t_1}^{t_2} V^*(\tau, \varphi(\tau))d\tau + 2\xi.$$

Now letting $\{x^m\}$ denote a sequence of solutions of (2) converging uniformly on $[t_1, t_2]$ to φ, we obtain from the last formula, (2) and the definition of k^m, the formula

$$(4) \qquad [V(t_2, \varphi(t_2)) - V(t_2, x^m(t_2))] - [V(t_1, \varphi(t_1)) - V(t_1, x^m(t_1))]$$
$$= \int_{t_1}^{t_2} [V^*(\tau, \varphi(\tau)) - V^*(\tau, x^m(\tau))]d\tau + 2\xi, \ m = 1, 2, 3, \cdots.$$

But now we may find $L = L(\xi) > 0$ such that if $m > L$, then

$$|V(t, \varphi(t)) - V(t, x^m(t))| < \xi/2, \ t \in [t_1, t_2]$$

and

$$\int_{t_1}^{t_2} [V^*(\tau, \varphi(\tau)) - V^*(\tau, x^m(\tau))]d\tau > -\xi;$$

together with (4), these estimates produce the absurdity $\xi < \xi$. Thus P must be of measure zero, so that

$$(5) \qquad V'(t, \varphi(t)) = V^*(t, \varphi(t))$$

almost everywhere on $[t_1, t_2]$.

The proof of Theorem 2.1 is completed by observing that if φ is a solution of (1) on an interval (τ_1, τ_2) and $(\tau_1, \varphi(\tau_1+0))$ $[(\tau_2, \varphi(\tau_2-0))] \in D$, then the foregoing construction may be repeated to continue φ as a solution of (1) satisfying (5) to the left of τ_1 [right of τ_2]. The final step is then an application of [8, Theorem 4].

Turning our attention now to questions of weak stability for (1), we note that the definitions given below are in some respects more general than those given by Roxin [5, 6]. Throughout the remainder of this section we shall assume that $R: E^1 \times E^n \to \Gamma^n$ is upper semicontinuous and that $\pi: E^1 \to \Omega^n$ is a given continuous function.

Definition 2.1. We say that (1) is *attracted to* π if and only if the following conditions are satisfied: for each $\epsilon > 0$ and $t_0 \in E^1$, there exists $\mu = \mu(\epsilon, t_0) > 0$ such that for each x_0 satisfying $0 < \alpha(x_0, \pi(t_0)) < \mu$, there exists a solution φ of (1) satisfying $0 < \alpha(\varphi(t), \pi(t)) < \epsilon$ on its right maximal interval of existence $[t_0, T)$ and, provided $T < \infty$, $\lim_{t \to T} \alpha(\varphi(t), \pi(t)) = 0$.

Definition 2.2. We say that (1) is *quasi-asymptotically attracted to* π if and only if the following conditions are satisfied: for each $t_0 \in E^1$, there exists $\sigma = \sigma(t_0) > 0$ such that for each x_0 satisfying $0 < \alpha(x_0, \pi(t_0)) < \sigma$, there exists a solution φ of (1) satisfying $0 < \alpha(\varphi(t), \pi(t))$ on its right maximal interval of existence $[t_0, T)$ and $\lim_{t \to T} \alpha(\varphi(t), \pi(t)) = 0$.

Definition 2.3. We say that (1) is *asymptotically attracted to* π if and only if it is both attracted and quasi-asymptotically attracted to π.

Remark 2.1. The pattern established by these three definitions makes it clear how one may proceed by analogy with Liapunov stability theory to give definitions of uniform attractedness, global asymptotic attractedness, etc. It is not our purpose here to give an exhaustive treatment of this generalization of weak stability theory, but rather to show how Theorem 2.1 provides the means to develop the analogue of Liapunov's second method for this theory. Thus we shall be content with obtaining sufficient conditions for attractedness and asymptotic attractedness.

For our next group of definitions we shall need the following sets:

$$\Sigma_\pi = \{(t, x) \in E^1 \times E^n \colon \alpha(x, \pi(t)) = 0\};$$

$$N_\pi^\delta = \{(t, x) \in E^1 \times E^n \colon 0 < \alpha(x, \pi(t)) < \delta\}.$$

That N_π^δ is an open subset of $E^1 \times E^n$ is an easy consequence of the continuity of $\alpha(\cdot, \pi(\cdot))$; Σ_π is clearly closed by the same token.

Definition 2.4. A function $V\colon E^1 \times E^n \to E^1$ is *positive definite* (*with respect to* π) if and only if it satisfies the following conditions:

(i) $V(t, x) = 0$ if and only if $(t, x) \in \Sigma_\pi$;

(ii) there exists a continuous, strictly increasing function ρ on E^1 satisfying $\varphi(0) = 0$ and

$$\rho(\alpha(x, \pi(t))) \leqq V(t, x)$$

on $\Sigma_\pi^\sim$, the complement of Σ_π.

Definition 2.5. A function $V\colon E^1 \times E^n \to E^1$ is *decrescent* (*with respect to* π) if and only if there exists a continuous, strictly increasing function λ on E^1 satisfying $\lambda(0) = 0$ and

$$|V(t, x)| \leqq \lambda(\alpha(x, \pi(t)))$$

on $\Sigma_\pi^\sim$.

Definition 2.6. A continuous function $V\colon E^1 \times E^n \to E^1$ will be called a *gauge function* if and only if V is positive definite with respect to π and $V \in \mathscr{V}(\Sigma_\pi^\sim)$.

We may now state our main results on attractedness.

THEOREM 2.2. *If there is a gauge function V and a $\delta > 0$ for which the function V^* defined by* (3) *satisfies $V^*(t, x) \leqq 0$ on N_π^δ, then* (1) *is attracted to* π.

THEOREM 2.3. *If there is a decrescent gauge function V and a $\delta > 0$ for which $V^*(t, x) \leqq -\psi(\alpha(x, \pi(t)))$ on N_π^δ, where the function ψ on E^1 is continuous, non-negative and strictly increasing, then* (1) *is asymptotically attracted to* π.

For the proof of Theorem 2.2, we observe that $V \in \mathscr{V}(N_\pi^\delta)$, so that by (5) and the hypothesis for V^* we have

$$V'(t, \varphi(t)) \leqq 0 \qquad \text{a.e. on } [t_0, T), \tag{6}$$

where φ is the solution of (1) whose existence is ensured by Theorem 2.1. We assert that there exists $\sigma = \sigma(t_0) \in (0, \delta)$ such that if $0 < \alpha(x_0, \pi(t_0)) < \sigma$, then $V(t_0, x_0) < \rho(\delta)$. Indeed, since by [1, Lemma 1.5] $(\pi(t_0))^\delta \in \Omega^n$, the uniform continuity of $V(t_0, \cdot)$ on $(\pi(t_0))^\delta$ implies the existence of $\sigma(t_0)$ such that $|V(t_0, x_0) - V(t_0, x_1)| < \rho(\delta)$ when $\|x_0 - x_1\| < \sigma$ and $x_0, x_1 \in (\pi(t_0))^\delta$. In particular this is true if x_1 is a point of $\pi(t_0)$ nearest x_0, from which the assertion follows. Now if $0 < \alpha(x_0, \pi(t_0)) < \sigma(t_0)$ and $\alpha(\varphi(T-0), \pi(T)) = \delta$, we obtain from (6) the absurdity

$$\rho(\delta) = \rho(\alpha(\varphi(T-0), \pi(T))) \leqq V(T, \varphi(T-0)) \leqq V(t_0, x_0) < \rho(\delta).$$

Thus when $0 < \alpha(x_0, \pi(t_0)) < \sigma(t_0)$, the only modes in which $(t, \varphi(t))$ can tend to the boundary of N_π^δ as $t \to T-0$ are (i) $\alpha(\varphi(T-0), \pi(T)) = 0$ or (ii) $T = \infty$. The remainder of the proof of Theorem 2.2 and the proof of Theorem 2.3 follow by means of standard Liapunov arguments applied to φ (*vide* [9, pp. 14–15]).

Remark 2.2. The sufficiency of the condition of [10, Theorem 1] may be seen to be a consequence of Theorem 2.1. It is noteworthy that the class of gauge functions examined in [10] may be useful in connection with the theorems on attractedness which we have proved above.

3. Generalization of the Kneser-Zaremba Theorem

In his extension [4, III.6] of Kneser's theorem [7, pp. 15–16], Zaremba proved, in effect, that if $G_0 \in \Omega^n$ is connected then $\mathscr{A}(t, t_0, G_0)$ is connected for each $t \in I$. In Theorem 3.1 below, we find the even stronger conclusion that $\bar{H}(t_0, G_0)$ is a connected subset of $\mathscr{C}^n(I)$; from Theorem 3.1, the Zaremba-Kneser theorem follows as a corollary. Setting $H = H_1 \cup H_2$ in the example following Corollary 1.3 of [1] discloses the fact that for an element $H \in \mathscr{H}^n(I)$, $G(t; H)$ may be connected even though H is not. This fact provides still more cogent evidence for the value of making the solution family $\bar{H}(t_0, G_0)$ of a generalized differential equation the fundamental object of analysis.

Before stating Theorem 3.1, it will prove desirable to list some preliminary results concerning elements of Ω^n and $\mathscr{H}^n(I)$ which are also connected subsets of E^n and $\mathscr{C}^n(I)$ respectively.

LEMMA 3.1. *Let $H \in \mathscr{H}^n(I)$ be connected; then $G(t; H)$ is connected for each $t \in I$.*

Proof. We shall prove the contrapositive; hence, suppose that for some $t_1 \in I$, $G(t_1; H)$ is not connected. Then there exist disjoint $A, B \in \Omega^n$ such that $A \cup B = G(t_1; H)$. Let H_A be the largest subset of H for which $G(t_1; H_A) = A$, with a similar definition for H_B. It is evident that $H = H_A \cup H_B$ and that H_A and H_B are nonvoid and disjoint. Let $h \in H$ be a cluster point of H_A; then $h(t_1)$, as a cluster point of A, is in A. Thus $h \in H_A$, and we conclude that $H_A, H_B \in \mathscr{H}^n(I)$. Consequently, H is not connected.

LEMMA 3.2. *Let the sequence* $\{H_m\} \subset \mathscr{H}^n(I)$ *satisfy* $\bar{\sigma}(H_{m+1}, H_m) = 0$ *for all m and suppose that each* H_m *is connected; then the set* H^* *defined by* $H^* = \cap H_m$ *satisfies:* (i) $H^* \in \mathscr{H}^n(I)$; (ii) H^* *is connected;* (iii) $\lim_{m\to\infty} \bar{\sigma}(H_m, H^*) = 0$.

Proof. Immediate from [1, Theorem 1.6] and [11, p. 163, F(b)].

LEMMA 3.3. *If* $G \in \Omega^n$ *is connected, then for each* $\eta > 0$, G^η *is arcwise connected.*

Proof. The set $G_1^\eta = \{x \in E^n: \alpha(x, G) < \eta\}$ has the following easily obtained representation:

$$G_1^\eta = \bigcup \{B(x, \eta): x \in G\},$$

where $B(x, \eta)$ is the open ball of radius η and center x. Thus G_1^η is open and since $B(x, \eta)$ is connected, it is a consequence of [12, p. 83, Problem 9] that G_1^η is connected. This in turn implies, by virtue of [12, Theorem 2–8V], that G_1^η is arcwise connected. Since any point ξ satisfying $\alpha(\xi, G) = \eta$ can be joined to a point $y \in G$ nearest ξ by a line segment all of whose points (except ξ) lie in G_1^η, the lemma follows.

Now let us assume that $R: I \times E^n \to \Gamma^n$ is upper semicontinuous and satisfies either (i) or (ii) of [1, Theorem 2.4]. By virtue of [1, Theorem 2.5], if $(t_0, G_0) \in I \times \Omega^n$ then $\bar{H}(t_0, G_0) \in \mathscr{H}^n(I)$, where $\bar{H}(t_0, G_0)$ is the restriction to I of the solution family of the generalized differential equation

$$\dot{x} \in R(t, x), \qquad x(t_0) \in G_0. \tag{7}$$

THEOREM 3.1. *Under the foregoing assumptions, if* $G_0 \in \Omega^n$ *is connected then* $\bar{H}(t_0, G_0)$ *is connected.*

Proof. Consider the problem

$$\dot{x} \in (R(t, x))^1, \qquad x(t_0) \in G_0^1; \tag{8}$$

we denote by $\bar{J}(t_0, G_0^1)$ the restriction to I of the solution family of (8). By [1, Lemma 2.2], $\bar{J}(t_0, G_0^1) \in \mathscr{H}^n(I)$ and then $F(\bar{J}(t_0, G_0^1)) \in \Psi^n(I)$. Setting $N = F(\bar{J}(t_0, G_0^1))$, we may construct on N an approximation to R of the type described in Theorem 1.3. We denote by $\bar{H}_m(t_0, G)$ the restriction to I of the solution family of

$$\dot{x} \in Z_m(t, x), \qquad x(t_0) \in G, \qquad m = 1, 2, 3, \cdots. \tag{9}$$

Inasmuch as (*vide* the proof of [1, Theorem 2.3])

$$\bar{H}(t_0, G_0) = \bigcap \bar{H}_m(t_0, G_0^{m^{-1}}),$$

by virtue of Lemma 3.2, it will suffice to show that for each m, $\bar{H}_m(t_0, G_0^{m^{-1}})$ is connected. To this end, we utilize the proof recently devised by Hermes [13, p. 261] for a closely related problem. By Theorem 1.3 and the Filippov Lemma [1, Lemma 1.12], to each solution φ of (9) with $G = G_0^{m^{-1}}$ there corresponds a function $u: I \to E^n$ having Lebesgue measurable components and satisfying $\|u(t)\| \leq 1$ on I such that φ is a solution on I of the ordinary differential equation

$$\dot{x} = f^m(t, x, u(t)), \qquad x(t_0) = \varphi(t_0).$$

Now let $\varphi^0 \neq \varphi^1$ be two solutions of (9) with $G = G_0^{m-1}$ and let u^0, u^1 be the corresponding functions whose existence is ensured by the Filippov Lemma. Since by Lemma 3.3, G_0^{m-1} is arcwise connected, there exists a homeomorphism c on $[0, 1]$ into G_0^{m-1} such that $c(i) = \varphi^i(t_0)$, $i = 0, 1$. Defining $u(\cdot, \beta)$ on I by

$$u(t, \beta) = \beta u^1(t) + (1-\beta)u^0(t),$$

we consider the differential equation

$$\dot{x} = f^m(t, x, u(t, \beta)), \qquad x(t_0) = c(\beta). \tag{10}$$

From the form of f^m given in Theorem 1.3, we deduce readily that f^m is uniformly Lipschitzian on the compact set $N \times \tilde{B}$; hence for each $\beta \in [0, 1]$, (10) has a unique solution whose value at t we denote by $\varphi(t, \beta)$. Thus the function $\varphi(\cdot, \cdot)$ is continuous on $I \times [0, 1]$. Inasmuch as $\varphi(\cdot, \beta)$ is also a solution of (9) with $\varphi(\cdot, i) = \varphi^i$, $i = 0, 1$, for each $\beta \in [0, 1]$, we have demonstrated that to each pair of elements of $\bar{H}_m(t_0, G_0^{m-1})$ there corresponds a connected subset of $\bar{H}_m(t_0, G_0^{m-1})$ containing the pair. By [12, Lemma 2–8VI] we may conclude that $\bar{H}_m(t_0, G_0^{m-1})$ is connected, and the proof is complete.

As indicated earlier, by virtue of Lemma 3.1 the Kneser-Zaremba theorem is an immediate corollary to Theorem 3.1.

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