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Discrete optimization of geometrically nonlinear truss structures under stability constraints

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Abstract. The paper deals with discrete optimization of elastic trusses with geometrical nonlinear behaviour and constraints on stability. The problem consists of minimizing the weight and determining the optimal member distribution so that the external load does not cause a loss of stability of the structure. Member cross-sections are selected from a catalogue of available sections. Element stresses, elment stability and global structural stability constraints are considered. A controlled enumeration method according to the increasing value of the objective function is applied. Shallow space trusses are numerically analysed.

1 Introduction

The design of truss structures involves the selection of elements from a discrete set of fabricated components. In an optimal design problem, design variables are selected so as to minimize the weight of the structure and satisfy all constraints. Cross-sectional areas of the members are dictated by dominant constraints, depending generally on the geometry and the nature of applied loads. In space structures, element or system stability can become a critical constraint. The paper deals with the discrete optimization of elastic shallow trusses of given layout with geometrical nonlinear behaviour. The problem consists of determining the optimal, minimum weight bar distribution so that external loads do not cause a loss of stability of the structure. Member cross-sections are selected from catalogues of available sections. Element stresses, element stability and global structural stability constraints are considered. Minimum weight design of elastic, geometrically nonlinear trusses with global stability constraints have been examined by Khot (1983), Kamat *et al.* (1984) and Khot and Kamat (1985), where cross-sectional areas were assumed to be continuous.

The simplest technique for coping with discrete variables is to solve the problem by continuous variables methods and then rounding off the continuous solution to the nearest discrete one. This approach often leads to results which are not optimal from the point of view of discrete optimization. The most popular methods applied in discrete optimization of engineering structures can be divided into branch and bound methods, e.g. Cella and Soosaar (1973), Bauer *et al.* (1981); heuristic and approximate methods, e.g. Templeman and Yates (1982); dual formulation methods, e.g. Fleury and Braibant (1982); and controlled enumeration methods, e.g. Fox and Liebman (1980) and Gutkowski *et al.* (1986). In this paper the enumeration method according to the increasing value of the objective function (Greenberg 1971) was used. This leads to the global optimum of the problem described with a linear objective function and arbitrary constraints and catalogues of discrete design variables.

2 Formulation of the problem

The truss structures under consideration are idealized with straight, elastic bar elements of constant cross-sections. The geometry and the partition into k linking groups, i.e. regions in which elements have the same cross-section and material properties, are given. The magnitude of conservative, static external loads $\mathbf{p}$ acting on nodes of the structure is represented by one loading parameter Λ_p. The minimum weight design problem consists of the optimal selection of bar elements from a discrete set-catalogue so that the external load Λ_p does not cause a loss of stability of the structure.

The objective function (the weight of elements) is

$$W(A_i) = \sum_{i=1}^{k} d_i A_i \rho_i \longrightarrow \min , \tag{1}$$

the stress constraint

$$\sigma_{\min} \le \sigma_s^P(A_i, I_i) \le \sigma_{\max}\ , \quad (s = 1, \dots, M)\ , \tag{2}$$

the local stability constraint

$$\sigma_s^P(A_i, I_i) \le \sigma_s^{loc}(A_i, I_i)\ , \quad (s = 1, \dots, M)\ , \tag{3}$$

the global stability constraint

$$\Lambda_{cr}(A_i, I_i) > \Lambda_P\ , \tag{4}$$

and the variable discreteness constraint

$$A_i \in \{\overline{A}_i^1, \overline{A}_i^2, \dots, \overline{A}_i^{r_i}\}\ ,$$

$$I_i \in \{\overline{I}_i^1, \overline{I}_i^2, \dots, \overline{I}_i^{r_i}\}\ , \quad (i = 1, \dots, k)\ , \tag{5}$$

where M is the number of bar elements, k the number of linking groups, A_i the cross-sectional area of bar from the i-th linking group, I_i the moment of inertia corresponding to cross-section A_i, $\{\overline{A}_i^1, \overline{A}_i^2, \dots, \overline{A}_i^{r_i}\}, \{\overline{I}_i^1, \overline{I}_i^2, \dots, \overline{I}_i^{r_i}\}$ catalogues of r_i cross-sectional areas and corresponding moments of inertia in the i-th linking group, r_i the number of available element profiles in the catalogue for the i-th linking group, d_i the sum of the lengths of the i-th linking group elements, ρ_i the mass density, σ_s^P the axial stress of s-th bar corresponding to the external load vector $\mathbf{p}$; $\sigma_{\min}, \sigma_{\max}$ the limiting elastic tensile and compressive stresses in bars, σ_s^{loc} the element axial stress corresponding to elastic buckling, Λ_{cr} the critical load parameter,

$$\mathbf{p} = \Lambda_P \mathbf{p}_o\ , \tag{6}$$

$\mathbf{p}$ the vector of forces externally applied at the nodes and $\mathbf{p}_o$ the reference load vector.

The cross-sectional areas A_i and the moments of inertia I_i are design variables. The discrete problem (1)-(5) has been transformed into an integer programming problem by the introduction of new zero-one variables z_j. The number of design variables has increased from k to $\sum_{i=1}^{k} r_i$, but now the enumeration method according to the increasing value of objective function (Greenberg 1971) can be used.

3 Stability constraints

The nonlinear behaviour of the structure may be due to the presence of large deflections or rotations. The shallow truss structures are often characterized by geometric nonlinear behaviour and nonlinear terms in the strain-displacement relations must be included.

An elastic three-dimensional truss structure described by N nodal displacements $q_j (j = 1, \dots, N)$ and one loading parameter Λ is considered. A nonlinear strain displacement relation is obtained by employing the following definition for strain

$$\varepsilon_i = \frac{l_i' - l_i}{l_i}\ , \tag{7}$$

where l_i and l_i' are the undeformed and deformed lengths of the i-th truss. It was assumed that large and finite displacements are allowed and the strain resulting from such a displacement is small enough to permit a linear stress-strain relation. The total potential energy of a structure built up for M truss elements can be expressed as

$$V = \sum_{i=1}^{M} \left(\frac{1}{2} E A_i l_i \varepsilon_i^2\right) - \mathbf{q}^T \mathbf{p} = \Pi(\mathbf{q}) - \Lambda \mathbf{q}^{\mathrm{T}} \mathbf{p}_o\ , \tag{8}$$

where E is Young's modulus and $\mathbf{q} = \{q_i, \dots, q_N\}$ the vector of global displacements of the nodes. A set of N nonlinear equations of equilibrium of the structure is provided by the principle of stationary value of the total potential energy

$$\frac{\partial V}{\partial \mathbf{q}} = \frac{\partial \Pi}{\partial \mathbf{q}} - \Lambda \mathbf{p}_o = \mathbf{0}\ . \tag{9}$$

An accurate location of a critical point (limit or bifurcation point) determined by Λ_{cr} in stability constraint (4) must be calculated. The critical point is characterized by that load level Λ at which (Thompson and Hunt 1984)

$$\det \left| \left[\frac{\partial^2 V}{\partial q_j \partial q_i} \right] \right| = 0\ . \tag{10}$$

The value of $\Lambda_{cr}(\mathbf{p}_{cr} = \Lambda_{cr}\mathbf{p}_o)$ and the corresponding stresses were calculated from the set of $N+1$ nonlinear equations (9) and (10) for the unknown q_j, $(j = 1, \dots, N)$ and Λ. The element stresses σ_s^P in (2) and (3) were obtained from (9) by setting $\Lambda = \Lambda_p$. In both cases the Newton-Raphson iteration technique was used. In the local stability constraint (3) the stresses in the s-th truss element corresponding to its elastic buckling can be expressed in the form

$$\sigma_s^{loc} = \frac{\pi^2 E_s I_s}{A_s l_s^2}\ . \tag{11}$$

4 Enumeration method

The methods of discrete optimization using controlled enumeration consist of the use of such algorithms which allow the optimal solution to be reached by partial enumeration, without checking all feasible variants. The enumeration method according to the increasing value of the objective function (Greenberg 1971) may be applied in the case of a linear objective function with arbitrary constraints and leads to the global optimum. It was first used in the minimum weight optimization of bar structures by Iwanow (1981) and Bauer *et al.* (1981).

In our case the optimization problem may be written

$$W = \sum_{j=1}^{t} c_j z_j = \mathbf{c}^{\mathrm{T}} \mathbf{z} \longrightarrow \min , \tag{12}$$

$$\mathbf{z} \in \mathcal{D} , \tag{13}$$

where $\mathbf{z} = [z_1, \ldots, z_t]$ and $\mathbf{c} = [c_1, \ldots, c_t]$ are vectors of zero-one design variables and constant non-negative coefficients characterizing components of the objective function ($t = \sum_{i=1}^{k} r_i$), and $\mathcal{D} \subset \mathcal{N}^t$ is a set determined by the constraints.

The idea of the enumeration method consists of forming a sequence of design variable vectors $\mathbf{z}$

$$\{\mathbf{z}^0, \mathbf{z}^1, \mathbf{z}^2, \ldots\} , \tag{14}$$

so that the referred sequence of objective functions

$$\{W^0, W^1, W^2, \ldots\} ,\ W^i = W(\mathbf{z}^i), \qquad (i = 0, 1, \ldots) \tag{15}$$

was non-decreasing. If $n_{\min}$ is the smallest natural $n \in \mathcal{N}$, for which the condition (13) is satisfied, then the solution of the optimization problem (12) and (13) is

$$\mathbf{z}^{\min} = \mathbf{z}^{n_{\min}} , \quad W^{\min} = W\left(\mathbf{z}^{n_{\min}}\right) . \tag{16}$$

The problem of finding the minimum of the objective function (12) is transformed to forming the sequences (14), (15) and to checking whether the subsequent vector $\mathbf{z}^i$ belongs to the admissible region. The first point encountered belonging to this region is the solution of the problem. The detailed algorithm is presented by Greenberg (1971).

5 Numerical examples

Numerical examples deal with the discrete optimization of shallow trusses. The material is characterized by $E = 7000 \times 10^7$ N/m^2, $-\sigma_{\min} = \sigma_{\max} = 20 \times 10^7$ N/m^2, $\rho = 27500$ N/m^3. Constraints (2)-(5) were considered, bars of circular sections were chosen and cross-sectional areas A are given in optimization results (corresponding moments of inertia $I = A^2/4\pi$). Computations were performed on an IBM PC/XT.

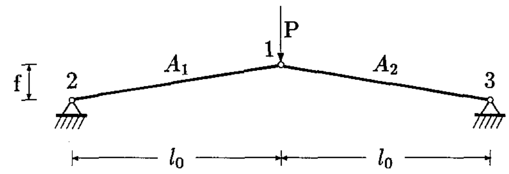

Fig. 1. Two-bar truss

5.1 Two-bar truss

The shallow symmetric two-bar truss shown in Fig. 1 ($l_0 = 2.00$ m, $f = 0.07$ m and $P = 1.5$ kN) was considered.

The cross-sections of bars A_1 and A_2 were chosen from the catalogue

$$A_i[\mathrm{m}^2 \times 10^{-4}] \in \{12.57; 15.9; 19.63; 23.76; 28.27; 33.18; 38.48\} , \qquad (i = 1, 2) .$$

The results of discrete optimization for two variants: (a) $A_1 = A_2$ and (b) A_1, A_2 optional, are given in Table 1.

Table 1. Optimal results for the two-bar truss

	A_1 [m$^2 \times 10^{-4}$]	A_2 [m$^2 \times 10^{-4}$]	P_{cr} [kN]	W [N]
a	23.76	23.76	1.728	261.5
b	19.63	23.76	1.563	238.8
	23.76	19.63	1.563	238.8

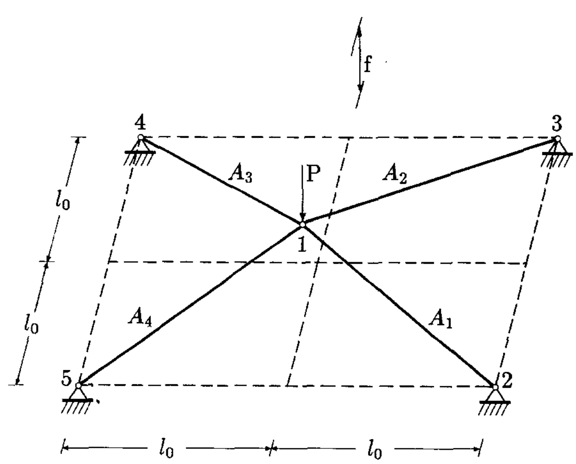

Fig. 2. Four-bar symmetrical truss

Table 2. Optimal results for four-bar symmetrical truss

	a	b	c	d
A_1 $[m^2 \times 10^{-4}]$	38.48	33.18	23.76	23.76
A_2 $[m^2 \times 10^{-4}]$	38.48	33.18	44.18	44.18
A_3 $[m^2 \times 10^{-4}]$	38.48	38.48	23.76	23.76
A_4 $[m^2 \times 10^{-4}]$	38.48	38.48	44.18	44.18
P_{cr} [kN]	3.138	2.907	2.775	2.775
W [N]	1197.583	1115.109	1057.222	1057.222

The discrete optimal solution is characterized by elements of different cross-sectional areas. Discrete optimization of a structure of symmetrical geometry and loading does not always lead to the symmetry of elements. The continuous variable optimization of such a structure is connected with elements of equal areas, see e.g. Khot (1983) and Khot and Kamat (1985). The two bar symmetrical truss of different element areas is characterized by the asymmetrical loss of stability mode.

5.2 Four-bar symmetrical truss

The four-bar symmetrical truss in Fig. 2 ($l_0 = 2.00$ m, $f = 0.07$ m) was optimized for design load $P = 2.75$ kN.

The optimal results for different variants of partition into linking groups: (a) $A_1 = A_2 = A_3 = A_4$; (b) $A_1 = A_2, A_3 = A_4$; (c) $A_1 = A_3, A_2 = A_4$; (d) optional A_i, $(i = 1,2,3,4)$; and cross-sectional areas from the catalogue

$$A_i[m^2 \times 10^{-4}] \in \{19.63; 23.76; 28.27; 33.18; 38.48; 44.18; 50.27\}, \qquad (i = 1,2,3,4)$$

are given in Table 2.

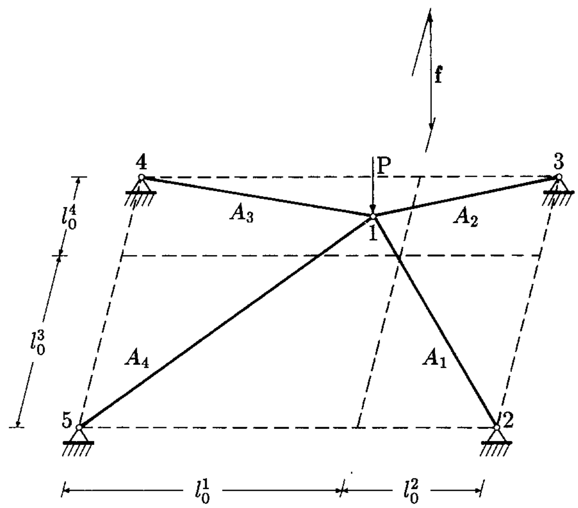

Fig. 3. Four-bar unsymmetrical truss

The discrete optimal solution of this symmetrically constructed and loaded truss, as in the first example 5.1 is not connected with the selection of equal areas elements.

5.3 Four-bar asymmetrical truss

The optimization results of the four-bar asymmetrical truss shown in Fig. 3 ($l_0^1 = 2.50$ m, $l_0^2 = 1.50$ m, $l_0^3 = 2.25$ m, $l_0^4 = 1.75$ m, $f = 0.07$ m) for $P = 3.5$ kN and the catalogue and variants of linking groups as in Example 5.2 are given in Table 3. The symbol d describes the optimal solution for arbitrary A_i $(i = 1,2,3,4)$. The results d' are connected with the next solution satisfying all constraints in the variant d of partition into linking groups, following the optimal one in the sequence of increasing values of the objective function. Note that solutions d and d' correspond to close objective functions, but they are characterized by a quite different distribution of elements.

5.4 Dome structure

The 30 member three-dimensional dome structure shown in Fig. 4 was optimized for a concentrated load P applied in the vertically downward direction at node 1. Nodes 8-19 are fixed. The coordinates of the node points in one quarter are given in Table 4. The coordinates of the other node points can be found by symmetry.

The structure was partitioned into four linking groups. Region 1 includes bars 1-6, region 2 includes bars 7-12, region 3 includes 13-18, and region 4 includes bars 19-30. The optimization results for two design loads $P = 15$ kN and $P = 25$ kN and for the catalogue

$$A_i[m^2 \times 10^{-4}] \in \{12.57; 15.9; 19.63; 23.76; 28.27; 33.18;$$

Table 3. Optimal results for four-bar unsymmetrical truss

	a	b	c	d	d'
A_1 $[m^2 \times 10^{-4}]$	44.18	44.18	38.48	44.18	38.48
A_2 $[m^2 \times 10^{-4}]$	44.18	44.18	44.18	44.18	50.27
A_3 $[m^2 \times 10^{-4}]$	44.18	38.48	38.48	38.48	33.18
A_4 $[m^2 \times 10^{-4}]$	44.18	38.48	44.18	33.18	38.48
P_{cr} [kN]	3.867	3.569	3.626	3.512	3.519
W [N]	1388.396	1309.848	1298.147	1238.783	1239.543

$38.48; 44.18; 50.27\}, \qquad (i = 1,2,3,4)$

are given in Table 5.

6 Conclusions

The optimal discrete solution of the problem under consideration must not be evaluated by 'rounding off' known continuous variable solutions. Taking discrete greater sizes near the solution in continuous variables may lead to a solution which is not optimal from the discrete optimization point of view and taking smaller sizes may cause a violation of the constraints. The discrete optimal design can be obtained by discrete programming methods (see Pyrz 1990). The continuous variable optimization of structures of symmetrical geometry and loading usually leads to the symmetrical distribution of elements. The discrete minimum weight designs of such problems is not always symmetrical. Discrete optimization results depend on the partition of the structure into linking groups and on the catalogues of available sections. The large number of linking groups usually leads to the structure of less weight. The most time consuming part of the computer solution of the problem under consideration is the evaluation of nonlinear critical loads. In practical design, the most effective numerical algorithms of nonlinear stability analysis or approximate, sufficiently exact methods must be applied. The enumeration method according to the increasing value of the objective function can be a useful tool for

Table 4. Coordinates of the node points of the dome structures

node point	x $[m \times 10^{-2}]$	y $[m \times 10^{-2}]$	z $[m \times 10^{-2}]$
1	0.0	0.0	65.041
3	300.0	0.0	48.923
4	150.0	259.8075	48.923
11	600.0	0.0	0.0
12	450.0	259.8075	16.404
13	300.0	519.615	0.0
14	0.0	519.615	16.404

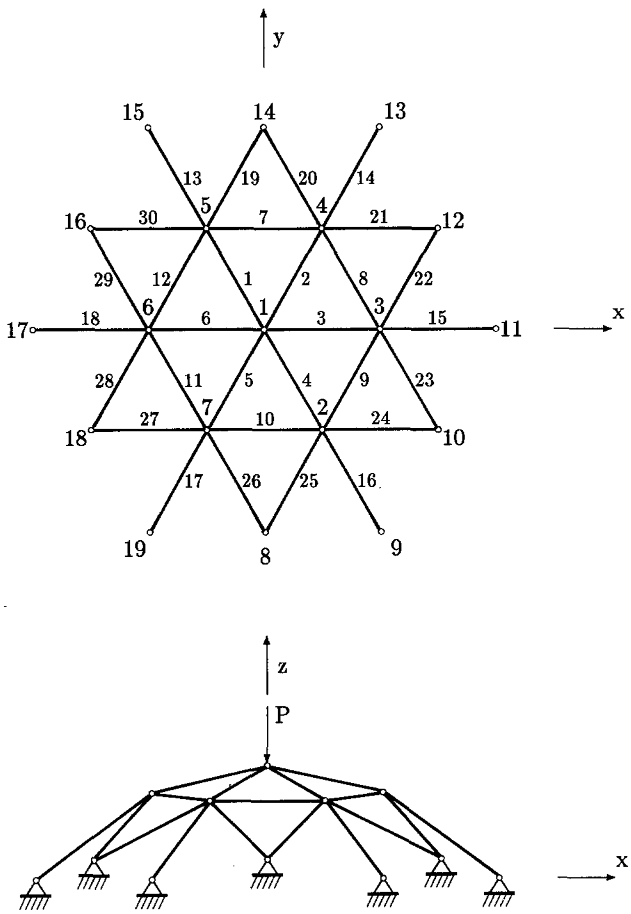

Fig. 4. Dome structure

discrete optimization problems with linear objective functions and arbitrary constraints and catalogues of available design variables. This guarantees the global optimum of the problem. The effectiveness of the method is limited by the number of design variables and the number of standard sections in catalogues. It can be increased if the solution of the continuous variable optimization problem is known. The continuous variable optimal solution of the minimization problem is the lower bound of discrete optimal results. There is no need to check the constraints if discrete design variables correspond to objective functions less than the known continuous optimization solution. In the author's opinion, the most effective algorithms of discrete optimization of the problem under consideration with a large number of design variables can be obtained by the skillful connection of discrete and continuous variable optimization methods.

The conclusions formulated above deal with properties of discrete optimal solutions observed on the basis of the presented numerical examples and need not occur in all discrete optimization problems of the considered class of structures. They can, however, be important in the case of practical discrete optimization of engineering structures.

Table 5. Optimal results for the dome structure

P[kN]	$A_1[\mathrm{m}^2 \times 10^{-4}]$	$A_2[\mathrm{m}^2 \times 10^{-4}]$	$A_3[\mathrm{m}^2 \times 10^{-4}]$	$A_4[\mathrm{m}^2 \times 10^{-4}]$	P_{cr}[kN]	W [N]
15.0	38.48	15.9	19.63	12.57	16.33	6177.587
25.0	44.18	33.18	19.63	12.57	26.486	7315.504

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