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Topology optimization of structures composed of one or two materials*

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Abstract Maximization of the integral stiffness of a structure composed of one or two isotropic materials of large stiffness is considered using the homogenization technique. Material is modelled by a second rank composite, and we use the concentrations and orientations of the composite as design variables. Numerical results are presented at the end of the paper.

1 Introduction

During the last decade a number of software systems for shape optimization based on boundary variation have been developed in the field of structural optimization. The efficiency and user-friendliness of many of these systems are so satisfactory that they can be used in industries as CAE (Computer Aided Engineering) tools. The first experiments dealing with boundary variation optimization were carried out in the seventies. Since then, authors such as Esping (1984, 1986), Braibant and Fleury (1984), Haftka and Gandhi (1986), Rasmussen (1990) and Rodrigues (1988) have published results on the subject.

The boundary variation method, however, has some limitations. The result of the optimization process is, e.g. very dependent on the chosen initial design, as the optimized structure is topologically equivalent to this. The user of the optimization program will have to define an initial design of the structure and define how the boundaries of the structure may change. If this is not done appropriately, the gain of the optimization process may be very limited. During boundary variation the user may also have to manually re-define the finite element mesh in order to retain a satisfactory mesh.

Using topology optimization we avoid the abovementioned problems. By this method a prescribed initial topology is not required, and topology optimization can often be applied as a most suitable preprocessor for problems of boundary variation, in which a sensible initial design is essential. Integration of topology and boundary variation optimization was successfully implemented by Bendsøe *et al.* (1990a), Bendsøe and Rodrigues (1990b) and Olhoff *et al.* (1991) by manual definition of the bounds of a boundary variation model based on a topology optimized structure. An automatic interface between the two optimization methods has been developed by Papalambros and Chirehdast (1990) and implemented in the system SAPOP.

This paper deals with topology optimization of plane, linearly elastic structures. Topology optimization is performed as a material distribution problem using a composite material. We apply the method described by Thomsen (1991), which deals with the optimization of plane composite structures consisting of fiber and matrix materials, where the concentration and the orientation of the fibers are used as design variables. Assuming the fiber concentration of such a composite material can take on values from 0 to 100%, and the stiffness of the matrix material is chosen to be very low compared to the stiffness of the fibers, we have in principle defined a topology optimization problem, where there will be "no material" and "material" in domains with 0 and 100% fibers, respectively.

In this paper, the material model is a second rank composite consisting of an isotropic material of large stiffness and a very soft material. The elastic moduli A_{ij} of the composite are obtained by homogenization. On account of the continuous nature of the composite material, the optimization can be carried out with two different fiber concentrations as design variables. The concentration of the material of large stiffness determines whether there will be "material" or "no material" at a given point of a loaded structure. In the following the term "stiff" means that material does not imply "rigid" material. The term "stiff" means material of large but finite stiffness.

The purpose of introducing a composite material is not only to obtain a convenient, continuous material model which can be used to obtain analytical expressions for the elastic moduli. Thus, if the problem had been stated as an integer optimization problem so that either "material" or "no material" could be generated at any point of a design domain, the formulation would *not* have been *correctly proposed*, and the existence of a solution (an optimal design) would not be obvious (Strang and Kohn 1986). The key would then be to reformulate the optimization problem by introducing a family of composites constructed from the basis materials of the original problem. This process is sometimes called *relaxation* and has, e.g. been studied by Murat and Tatar (1985), Lurie and Cherkaev (1986), Kohn and Strang (1986), Thomsen and Olhoff (1990) and Thomsen (1991). Relaxation implies enlarging of the design space, and tends to remove local optima (Kohn 1990). Traditionally, it was thought that one must consider the totality of all possible composites assembled from the set of originally given materials. This approach is called *full relaxation*. Recent investigations, have however, shown that only the set of finite rank laminar composites need to be considered for many optimization problems (Avellaneda 1987; Kohn 1988; Kohn and Lipton 1988). This technique is

*Presented at NATO ASI "Optimization of Large Structural Systems", Berchtesgaden, Germany, Sept. 23 – Oct. 4, 1991 called *partial relaxation* and is performed by introducing some convenient, finite-parameter microstructure. This approach also enlarges the design space but has the disadvantage that the solution may be dependent on the chosen microstructure. In addition, a partial relaxation implies the same difficulty as the discrete material/no material problem, that the problem may have no solution. Studies on bounds on the effective material properties of composite materials composed of two materials have shown that for plane elasticity the stiffest material can be obtained by a layered medium, with layering at two different micro scales (Avellaneda 1987). This means that the existence of solutions is ensured for minimum compliance shape optimization problems if such a material model is used.

This paper deals with maximization of the integral stiffness of a structure with given load and boundary conditions and with a given amount of available material. The integral stiffness of the structure for a given load case is represented by the internal elastic energy. Numerical analysis of the structure is performed by means of the finite element program MODULEF, and the concentration of the stiff material and the axes of orthotropy in each finite element are used as design variables.

Assuming the strain field to be determined at a given design stage, we apply an iterative two-level optimization procedure where:
- the distribution of the amount of available material is determined by using analytical sensitivity analysis and a convex mathematical programming technique, and
- the orientations of the axes of orthotropy are determined using a global optimality criterion method.

The above formulation of topology optimization was recently extended to cover application of two materials with different stiffnesses so that subregions with "material 1", "material 2", "no material" and "mixtures of material 1, material 2 and no material" can be generated in any part of a design domain. This model makes it possible to generate typical sandwich structures.

2 Material model

We now formulate a material model which can describe an arbitrary plane structure consisting of one or two stiff materials. In the literature dealing with topology optimization, several numerical and analytical material models have been used. In all models integer optimization is avoided by using a continuous material model, which can have intermediate values of "material" and "no material", meaning that essentially the optimization can be performed as a sizing problem. Bendsøe and Kikuchi (1988), Suzuki and Kikuchi (1989) and Díaz and Bendsøe (1990) used a numerically determined material model based on a microstructure consisting of an isotropic material with rectangular holes, and they used the orientation and the size of the holes as design variables of the optimization problem.

In this paper we use an analytical model, where two isotropic materials with different stiffnesses can be described by a second rank composite material. The composite is constructed in three micro levels. At the first level we model a composite using "material 1" and "material 2" by turns. The concentrations of the two materials are given in terms of the thicknesses δ_1 and $1-\delta_1$ as shown in Fig. 1a. At the second level we construct a composite composed of the material in Fig. 1a and "very soft material", where the ratio between these is given in terms of δ_2 and $1-\delta_2$, refer to Fig. 1b. At the third level we construct a composite using "material 1" and the composite in Fig. 1b by turns, where the concentrations of these are given in terms of γ and $1-\gamma$, refer to Fig. 1c. The three basis materials used in the material model are isotropic and defined by the stiffness matrices

$$Q^f_{kl} \;:\; \text{material } 1\,, \quad Q^m_{kl} \;:\; \text{material } 2\,,$$

$$Q^-_{kl} \;:\; \text{very soft material}\,.$$

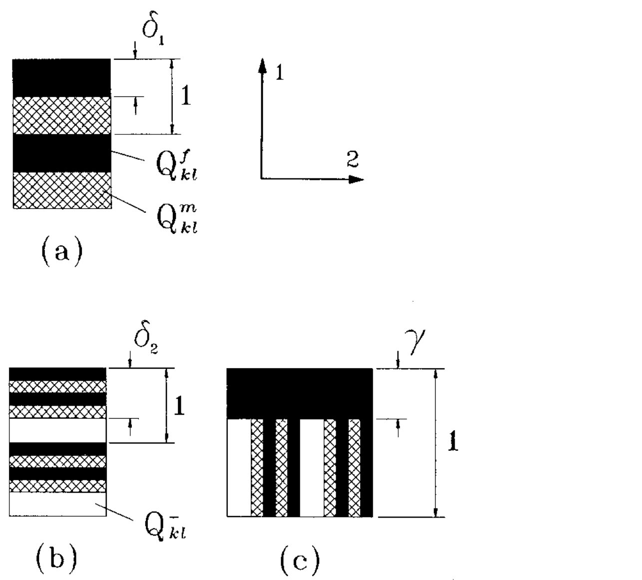


Fig. 1. Construction of composite materials. (a) First level, (b) second level, and (c) third level

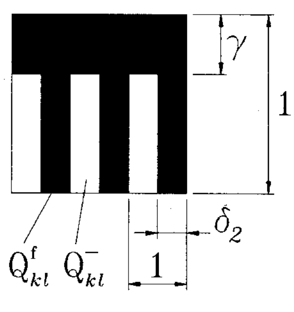


Fig. 2. Material model for optimization with one stiff material

If the design variables of the material model are chosen suitable, we can define topology optimization problems for either one or two stiff materials. If $\delta_1 = 1$, we obtain the composite shown in Fig. 2, which can be used in usual problems of topology optimization, where there may be either "material" or "no material" at a given point of a design domain. This formulation has previously been used by Bendsøe (1989). If all the design variables of the material model in Fig. 1c are allowed to vary between 0 and 1, we obtain a composite, which can describe "material 1", "material 2" and "no material" when suitable values of γ, δ_1 and δ_2 are chosen. We presume that the relatively simple material model in Fig. 1c is general enough to be used in problems of topology optimization. In addition to $\gamma(x)$, $\delta_1(x)$ and $\delta_2(x)$ we apply the material orientation $\theta(x)$ in a point x as a design variable. As mentioned, structures composed of isotropic materials can be generated during the optimization. It is, however, possible that parts of the structure may become anisotropic, as mixtures of the materials may be formed.

3 Optimization problem

This paper deals with stiffness maximization of linearly elastic structures loaded in plane stress. Structures are analysed using a fixed finite element model with known boundary conditions and in plane loadings. The structure of maximum integral stiffness will be defined as the structure having minimum total elastic strain energy subject to a given loading. The total internal elastic energy U is given by, see e.g. Jones (1975),

$$U=\sum_{i=1}^{n}\left\{\left\{\frac{1}{8}A_{11}\left[(\varepsilon_I+\varepsilon_{II})+(\varepsilon_I-\varepsilon_{II})\cos 2\psi\right]^2+\right.\right.$$
$$+\frac{1}{8}A_{22}\left[(\varepsilon_I+\varepsilon_{II})-(\varepsilon_I-\varepsilon_{II})\cos 2\psi\right]^2+$$
$$+\frac{1}{4}A_{12}\left[(\varepsilon_I+\varepsilon_{II})^2-(\varepsilon_I-\varepsilon_{II})^2\cos^2 2\psi\right]+$$
$$\left.\left.+\frac{1}{2}A_{66}(\varepsilon_I-\varepsilon_{II})^2\sin^2 2\psi\right\}S\right\}_i, \tag{1}$$

where ε_I and ε_{II} are the principal strains, [A] is the matrix of extensional stiffness, ψ is the angle from the direction corresponding to the numerically largest principal strain ε_I to the direction associated with the largest stiffness A_{11} and S is the area of an element. The design variables of the optimization problem are the density and the orientation of "material" in each finite element.

By topology optimization using *one* stiff material, we apply the concentrations γ and δ_2 (refer to Fig. 2) along with the orientation θ of the composite as design variables. We formulate a constraint that enforces the total amount of stiff material to be less than or equal to a given upper bound $\overline{M}$

$$0\le\gamma_i\le 1\quad;\quad\delta_{1i}=1\quad;\quad 0\le\delta_{2i}\le 1\quad;\quad i=1,\dots,n,$$

$$C_1=\sum_{i=1}^{n}\left[\gamma_i+(1-\gamma_i)\delta_{2i}\right]S_i\le\overline{M}. \tag{2}$$

By topology optimization using *two* stiff materials we apply γ, δ_1, δ_2 (refer to Fig. 1c) and θ as design variables, and we enforce the total amounts of "material 1"and "material 2" to be less than or equal to $\overline{M}_1$ and $\overline{M}_2$, respectively,

$$0\le\gamma_i\le 1;\quad 0\le\delta_{1i}\le 1;\quad 0\le\delta_{2i}\le 1;\quad i=1,\dots,n,$$

$$C_2=\sum_{i=1}^{n}\left[\gamma_i+(1-\gamma_i)\delta_{2i}\delta_{1i}\right]S_i\le\overline{M}_1;$$

$$C_3=\sum_{i=1}^{n}\left[(1-\gamma_i)(1-\delta_{1i})\delta_{2i}\right]S_i\le\overline{M}_2. \tag{3}$$

4 Stiffness matrix in terms of design variables

The stiffness matrix of the material shown in Fig. 1c is determined by homogenization in three steps, where stiffnesses are found for each micro level. We follow Bendsøe (1989) in determining the constitutive matrix Q_{kl}^H of a composite

$$Q_{11}^H=\left[M\left(\frac{1}{Q_{11}}\right)\right]^{-1},$$
$$Q_{22}^H=M(Q_{22})-M\left(\frac{Q_{12}^2}{Q_{11}}\right)+\left[M\left(\frac{Q_{12}}{Q_{11}}\right)\right]^2\left[M\left(\frac{1}{Q_{11}}\right)\right]^{-1},$$
$$Q_{12}^H=M\left(\frac{Q_{12}}{Q_{11}}\right)\left[M\left(\frac{1}{Q_{11}}\right)\right]^{-1},$$
$$Q_{66}^H=\left[M\left(\frac{1}{Q_{66}}\right)\right]^{-1}, \tag{4}$$

where $M(f)$ is the average value of a function $f(y)$ in the interval Y,

$$M(f)=\frac{1}{|Y|}\int_Y f(y)\,\mathrm{d}y, \tag{5}$$

and Q_{kl} are the so-called reduced stiffnesses, which for an isotropic material with Young's modulus E, Poisson's ratio ν and plane stress conditions are given by (6) (Jones 1975)

$$Q_{11}=Q_{22}=\frac{E}{1-\nu^2};\;Q_{12}=\frac{\nu E}{1-\nu^2};\;Q_{66}=\frac{E}{2(1+\nu)}. \tag{6}$$

We now consider a composite material composed of two isotropic materials with the stiffnesses Q_{kl}^f and Q_{kl}^m, refer to Fig. 1a. To simplify the calculation both materials are presumed to have the same Poisson's ratio. The elasticity constants Q_{kl}^{H1} of the composite are found by (4)-(6)

$$Q_{11}^{H1}=J_1;\;Q_{22}^{H1}=J_3;\;Q_{12}^{H1}=\nu J_1,\;Q_{66}^{H1}=\frac{1-\nu}{2}J_1, \tag{7}$$

where Q_{11}^{H1} and Q_{22}^{H1} are the stiffnesses corresponding to the orientation of the 1- and 2-axes, refer to Fig. 1, and

$$J_1=\left(\frac{\delta_1}{Q_{11}^f}+\frac{1-\delta_1}{Q_{11}^m}\right)^{-1},\;J_2=\delta_1Q_{11}^f+(1-\delta_1)Q_{11}^m,$$
$$J_3=J_2(1-\nu^2)+\nu^2J_1. \tag{8}$$

The constitutive matrix Q_{kl}^{H3} of the second rank composite shown in Fig. 1c can be determined, by repeating twice the use of (4) and (5)

$$Q_{11}^{H3}=\left(\frac{\gamma}{Q_{11}^f}+\frac{1-\gamma}{Q_{22}^{H2}}\right)^{-1},$$
$$Q_{22}^{H3}=\gamma Q_{11}^f+(1-\gamma)Q_{11}^{H2}-\nu^2\left[\gamma Q_{11}^f+(Q_{11}^{H2})^2\frac{1-\gamma}{Q_{22}^{H2}}-\right.$$
$$\left.-Q_{11}^{H3}\left(\gamma+Q_{11}^{H2}\frac{1-\gamma}{Q_{22}^{H2}}\right)^2\right],\;Q_{12}^{H3}=\nu\left(\gamma+Q_{11}^{H2}\frac{1-\gamma}{Q_{22}^{H2}}\right)Q_{11}^{H3},$$
$$Q_{66}^{H3}=\frac{1-\nu}{2}\left(\frac{\gamma}{Q_{11}^f}+\frac{1-\gamma}{Q_{11}^{H2}}\right)^{-1}, \tag{9}$$

where J_1, J_3, Q_{11}^{H2} and Q_{22}^{H2} are given by (8) and (10).

$$Q_{11}^{H2}=\left(\frac{\delta_2}{J_1}+\frac{1-\delta_2}{Q_{11}^-}\right)^{-1},$$

$$Q_{22}^{H2} = \\ = \delta_2 J_3 + (1-\delta_2) Q_{11}^{-} - \nu^2 \left[\delta_2 J_1 + (1-\delta_2) Q_{11}^{-}\right] + \nu^2 Q_{11}^{H2} . \quad (10)$$

Finally, we determine the matrix of extensional stiffness A_{kl} of a disc with the thickness h (Vinson and Sierakowski 1987)

$$A_{kl} = \int_{-h/2}^{h/2} Q_{kl}^{H3} \, \mathrm{d}z = h Q_{kl}^{H3} . \quad (11)$$

5 Optimization technique

The optimization problem is solved iteratively by a two-level procedure of redesign. The stress/strain field is initially determined by finite element analysis using MODULEF in each loop of redesign, and improved orientations $\theta_i (i = 1, \dots, n)$ of the composite are subsequently determined by means of an optimality criterion at the first level of redesign. At the second level of redesign the material densities δ_{1i}, δ_{2i} and γ_i are improved by a method of analytical sensitivity analysis and mathematical programming.

A notable feature of the present problem is that a usual gradient method may fail in determining the optimal orientation of the composite because local optima normally exist. To circumvent this inherent difficulty, we use results obtained by Pedersen (1989, 1990), who has performed an analytical investigation of the first and second derivatives of the total strain energy with respect to the orientation of the composite. The results of the investigation are summarized in a table shown in the papers by Pedersen (1990) and Thomsen (1991). In an optimization problem where the stiffness of a structure is maximized using the material orthotropy directions as design variables, we may either orient the composite material relative to the principal stress or strain directions (Pedersen *et al.* 1991). Numerical examples, however, show that the best convergence of the optimization problem is obtained, if the composite is rotated relative to the principal stress directions. Coincidence between the largest principal stress and strain directions is always found to be a result of the orientation optimization, and normally these directions will coincide with the material direction associated with the largest stiffness [unless the material has a relatively high shear stiffness, see Pedersen (1990)].

The second stage in the loop of redesign consists in determining an improved distribution of the amount of material, i.e. to obtain improved values of the design variables δ_{1i}, δ_{2i} and γ_i $(i = 1, \dots, n)$. We apply a dual method of mathematical programming using mixed variables as developed by Svanberg (1987) and implemented in the computer code MMA (Method of Moving Asymptotes). To this end we need the sensitivities of the objective function and the constraints with respect to the above-mentioned design variables.

Results of Pedersen (1990, 1991) show that by means of Clapeyron's theorem and the principle of virtual displacements for structures with design independent loads, the gradient of the total strain energy u can be determined from the gradient of the specific strain energy u_i of a given element, whose strain field is considered to be fixed

$$\frac{\mathrm{d}U}{\mathrm{d}a_i} = -\frac{\partial u_i}{\partial a_i} S_i , \quad i = 1, \dots, n . \quad (12)$$

Here a_i denotes any of the design variables δ_{1i}, δ_{2i} or γ_i $(i = 1, \dots, n)$. Thus, the sensitivities of the total strain energy U with respect to δ_{1i}, δ_{2i} and γ_i can be determined by (1) and (12), assuming the strain field to be fixed, and restricting variation to the stiffness matrix $[\mathbf{A}]$. For the i-th element of the discretized geometry we obtain the following expression for sensitivities with respect to the design variables a_i:

$$U_{,a_i} = -\left\{\left\{\frac{1}{8} A'_{11} \left[(\varepsilon_I + \varepsilon_{II}) + (\varepsilon_I - \varepsilon_{II}) \cos 2\psi\right]^2 + \right.\right. \\ + \frac{1}{8} A'_{22} \left[(\varepsilon_I + \varepsilon_{II}) - (\varepsilon_I - \varepsilon_{II}) \cos 2\psi\right]^2 + \\ + \frac{1}{4} A'_{12} \left[(\varepsilon_I + \varepsilon_{II})^2 - (\varepsilon_I - \varepsilon_{II})^2 \cos^2 2\psi\right] + \\ \left.\left. + \frac{1}{2} A'_{66} (\varepsilon_I - \varepsilon_{II})^2 \sin^2 2\psi \right\} S \right\}_i , \\ i = 1, \dots, n . \quad (13)$$

Here A'_{kl} is a shortened notation for the derivatives $\mathrm{d}A_{kl}/\mathrm{d}a_i$ of a component of the stiffness matrix $[\mathbf{A}]$. These sensitivities have been analytically derived by Thomsen (1992), and results can be found in that paper. Sensitivities of the constraints in (2) and (3) are readily derived analytically, and thus we have all necessary sensitivity information required for the optimization at the second level of redesign.

6 Examples of optimization using one material of large stiffness

Initially, we consider examples of topology optimization where we only use *one* material of large stiffness. The optimization method has been tested for a number of structures with various design domains and boundary conditions, and we have chosen examples where the results can be compared with analytical solutions.

The optimization is performed iteratively by choosing the *orientation* and *concentration* in each finite element which *separately* maximize the stiffness of the structure. We apply an initial geometry consisting of an anisotropic material as shown in Fig. 2 with orientations $\theta_i = 0$ and concentrations $\gamma_i = \delta_{2i}$ $(i = 1, \dots, n)$.

Let us now consider a Michell truss, which is an analytical solution to an optimization problem. A Michell truss is obtained by volume minimization of a "truss-universe", which has prescribed upper bounds on allowable tension and compression stresses, see Hemp (1973). Figure 3 shows such an example, where the force 2P is carried by the truss structure in the design domain ABCD. The same topology was confirmed by discretized truss optimization (Rozvany and Zhou 1991, p. 62; Zhou and Rozvany 1991, p. 321). By the topology optimization we presume that the optimal structure is symmetrical about the centreline in Fig. 3, and using this we reduce the design domain as shown in Fig. 4.

Figure 5 shows results of topology optimization where 4-node elements have been used. The available amount of material is set to be 40% of the design domain in volume. From Fig. 5 it appears that the structure is very similar to the Michell truss in Fig. 3. We see that only very few elements

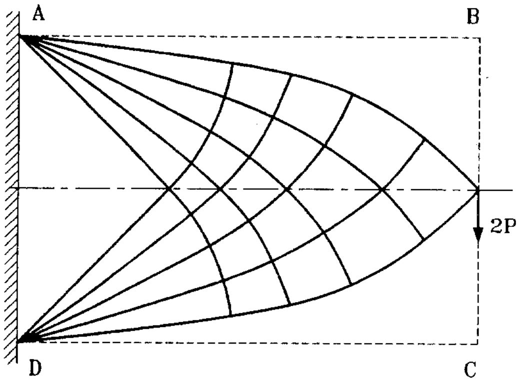


Fig. 3. Michell truss. Analytical solution for truss optimization problem in the design domain ABCD

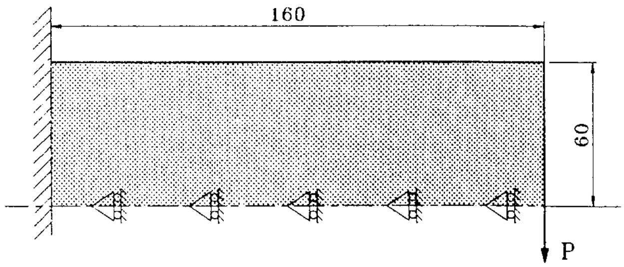


Fig. 4. Design domain, load and boundary conditions using symmetry

in the optimized structure remain anisotropic (hatched elements), which means that the design domain has been separated into sub-domains consisting of very soft material (holes) and isotropic stiff material, respectively.

Similar results were obtained by Rozvany *et al.* (1992), who used a solid isotropic microstructure with penalty (SIMP) for intermediate densities. The fact that in Fig. 5 herein most non-empty elements contain isotropic material supports the validity of the SIMP approach for this problem.

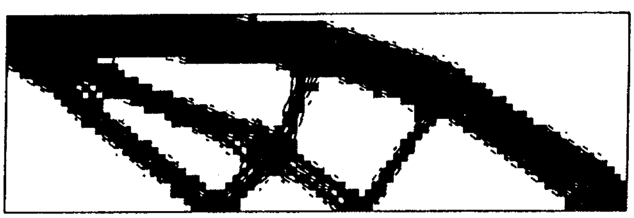


Fig. 5. Optimal topology. Volume = 40% ($n = 28 \times 90$)

Figure 6 illustrates two other examples of Michell trusses which are simply supported and fixed against displacements in two points, respectively, and where the design domains are the upper half planes. By topology optimization of the structures with the load and boundary conditions shown in Fig. 6, we utilize the symmetry about the vertical centrelines and thus only analyse the left half of the structures assuming the right edges to be simply supported, refer to Fig. 7. Available amounts of material of the structures in Figs. 7a and b are set to be 30% and 25%, respectively, and we obtain the optimal topologies in Fig. 8.

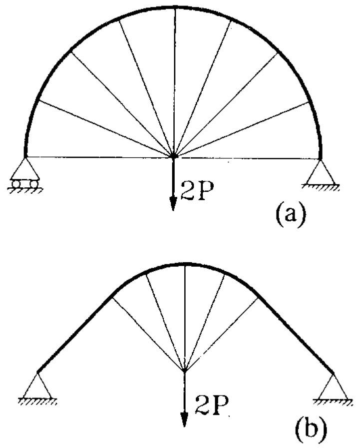


Fig. 6. Michell trusses

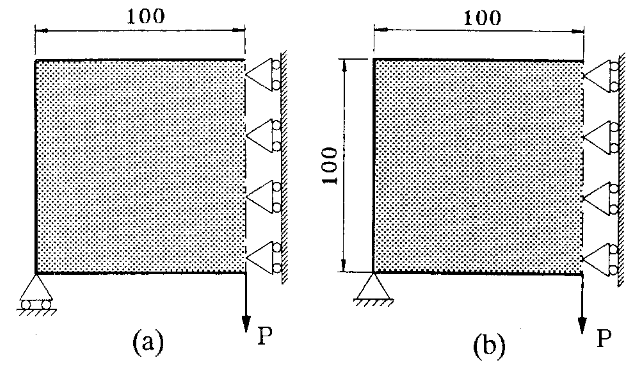
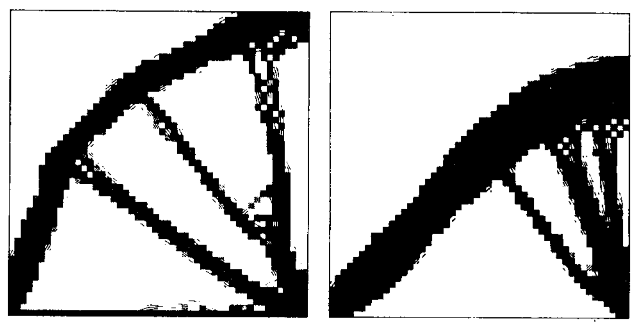


Fig. 7. Design domain, load and boundary conditions using symmetry



(a) Volume=30% (b) Volume=25%

Fig. 8. Optimal topologies ($n = 50 \times 50$)

Michell trusses all follow two geometrical conditions (Hemp 1973, pp 70-101):

- if a pair of tension and compression members meet at a point, they must be orthogonal (see e.g. the intersession, where the force is acting in Fig. 3);
- if two tension (compression) members and one compression (tension) member meet at a point, then the compression (tension) member must be orthogonal to the other two [see e.g. the upper (lowest) row of members in Fig. 3

and the intersection points along the curved lines in Figs. 6a and b].

Considering the optimal topologies, we see that they all comply with these conditions. If we had used larger amounts of available material, the topologies would have had frame character and would have differed from Michell trusses, which have joint connections in the intersection points of the truss members.

By topology optimization we do not take stability of compression loaded truss members into consideration. To obtain a practically useful truss structure we could apply the optimized topologies as initial designs in a traditional truss optimization program. Truss optimization has been undertaken by Pedersen (1972), who minimized the total weight of a truss structure considering cross-sectional areas at the bar and joint coordinates as design variables. Pedersen considers truss optimization for several simultaneous load cases taking stress constraints into consideration for bars in tension and stability constraints for bars in compression in the elastic (Euler) and plastic domains. A corresponding method is applied by Olhoff *et al.* (1991), who minimize the weight of a truss structure with fixed compliance using a toplogy optimized structure as an initial design for the truss optimization program SCOTS.

7 Examples of optimization using two materials of large stiffnesses

Finally we consider examples of topology optimization, where the material is modelled by one very soft material with elastic moduli Q_{kl}^{-} and *two* stiff materials with the elastic moduli Q_{kl}^{f} and Q_{kl}^{m}, respectively. We consider optimization of the cantilever beam in Fig. 9, and once more use symmetry.

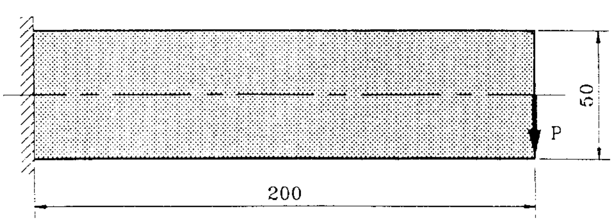


Fig. 9. Cantilever beam

Initially, we show two examples where the stiffness ratios between "material 1" and "material 2" are set to be 10 and 75, respectively,

$$\text{(a)} \quad Q_{kl}^{f} = 10 Q_{kl}^{m}, \qquad \text{(b)} \quad Q_{kl}^{f} = 75 Q_{kl}^{m}. \tag{14}$$

The available amounts of "material 1" and "material 2" are set to be 20% and 65%, respectively, of the design domain volume. During the optimization we choose the *orientations* θ_i and *concentrations* γ_i, δ_{1i} and δ_{2i}, which maximize the stiffness of the structure.

Figure 10 shows the optimal topologies of the two examples where the structures have been discretized into 12 × 96 4-node elements. The hatching densities of an element in two perpendicular directions are proportional to the elastic moduli A_{11} and A_{22}, and the orientation of the hatching indicates the corresponding directions. The optimized structures mainly consist of orthotropic material. Only very few elements along the upper left edges consist of isotropic "material 1", and isotropic "material 2" elements have not been generated. The stiffer material along the upper edge carries the large normal stresses, whereas the shear stresses are carried by a softer orthotropic material, the stiffnesses of which are almost equal in the two principal material directions. Elements at the right, upper and the left, lower corner have small strain energy densities due to the applied load and boundary conditions, and no material is distributed in these elements.

For the examples in Figs. 10a and b, we have chosen the material orientations θ_i such as to *maximize* the stiffness of the structure. We now show two examples where we choose the material orientations that *minimize* the stiffness of the structure, while we still choose the material concentrations δ_{1i}, δ_{2i} and γ_i that *maximize* the stiffness. When this is done, anisotropic material becomes very unfavourable, because the material stiffnesses in the directions of the principal stresses become very low relative to the material consumption. This leads to generation of topologies that only consist of "holes" and isotropic "material 1" and "material 2". Figures 11a and b show two examples, where we have used the same elastic moduli Q_{kl}^{f} and Q_{kl}^{m} and amounts of available material as in Fig. 10, but where the orientations are chosen such as to minimize the stiffness. It should be noted that the orientation of the material has no influence on the stiffness when all material has become isotropic. In this figure the distributions of isotropic "material 1" and "material 2 " are illustrated by black and hatched domains, respectively, whereas white elements illustrate void. It appears that stiff "material 1" is distributed along the upper edge of the structure in order to carry the largest normal stresses, and that shear stresses are carried by softer "material 2" like in ordinary sandwich structures composed of two "skins" and one "core". For the structure in Fig 11b it has been advantageous to use an amount of the stiff "material 1" as "shear force reinforcement", because the stiffness ratio $Q_{kl}^{f}/Q_{kl}^{m} = 75$ of this example is large, and since there is a relatively large amount of "material 1" available. Note that the topologies in Figs. 11a and b represent *local* optima, and they store 3% and 26% more elastic energy than the topologies in Figs. 10a and b, respectively. The purpose of the examples in Fig. 11 has been to demonstrate how to obtain a topology only consisting of isotropic materials, which is preferable from the point of view of manufacture.

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(a)
(b)

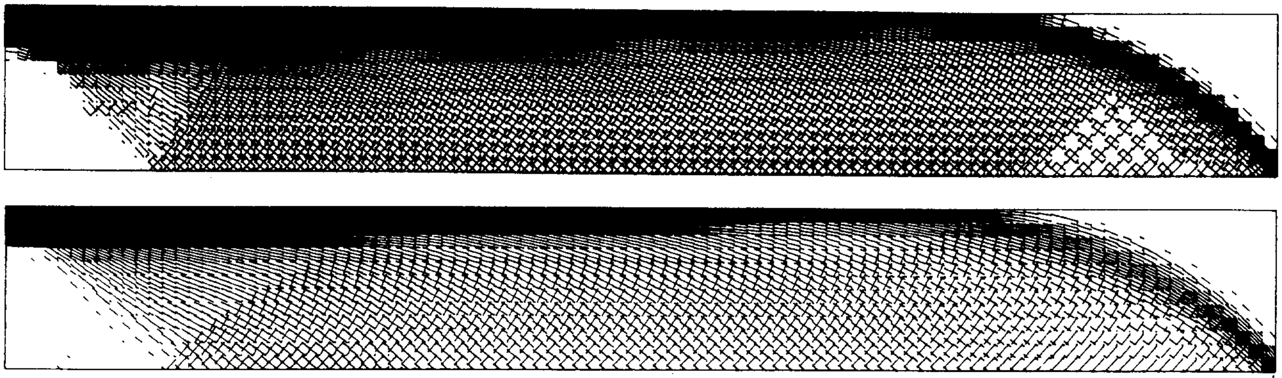

Fig. 10. Optimal topologies determined by using the orientations θ_i that maximize the stiffness. Hatching density is proportional to the stiffness. (a) $Q^f_{kl} = 10Q^m_{kl}$, (b) $Q^f_{kl} = 75Q^m_{kl}$ ($n = 12 \times 96$)

(a)
(b)

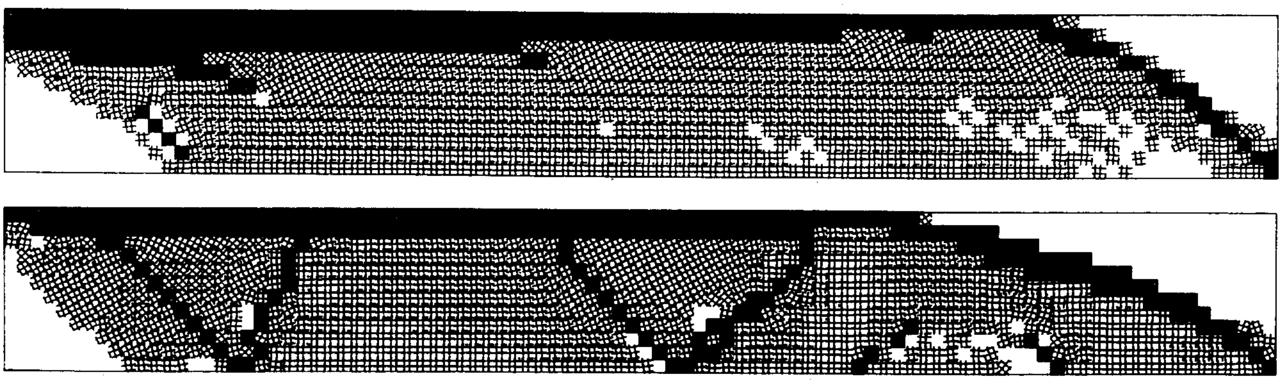

Fig. 11. Optimal topologies determined by using the orientations θ_i that minimize the stiffness. Black and hatched domains illustrate isotropic "material 1" and "material 2", respectively. (a) $Q^f_{kl} = 10Q^m_{kl}$, (b) $Q^f_{kl} = 75Q^m_{kl}$ ($n = 12 \times 96$)

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