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A Note on Special Thue Systems With a Single Defining Relation[1]

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Abstract. It is shown that a Thue system of the form $T_1 = \{(w, e)\}$ is Church-Rosser if and only if there is a Thue system T_2 that is Church-Rosser and is equivalent to T_1.

1. Introduction

Thue systems are combinatorial rewriting systems often studied in computability theory. In the past decade Thue systems have been used to specify context-free languages in terms of unions of congruence classes; for this, the Church-Rosser property and its variations play an important role (see [2], [3], [5]). The Church-Rosser property has been investigated for abstract reduction or replacement systems, term-rewriting systems, tree-manipulating systems, etc. (see [6, 10, 12, 13, 14]), and in each case is very useful.

If a Thue system T_1 is not Church-Rosser, then it may be the case that there is a Thue system T_2 that is Church-Rosser and is equivalent to T_1 in the sense that T_2 generates the same Thue congruence as T_1. Ó'Dúnlaing [13] has shown that it is undecidable for a finite Thue system T_1, whether there is a (finite or infinite) Thue system T_2 that is Church-Rosser and equivalent to T_1. Jantzen [7] investigated the specific Thue system $\{(abbaab, e)\}$ which is not Church-Rosser, and showed that there is no (finite or infinite) Church-Rosser Thue system that is equivalent to $\{(abbaab, e)\}$.

In this note it is shown that a one-relator special Thue system $T_1 = \{(w, e)\}$ is Church-Rosser if and only if there is a (finite or infinite) Thue system T_2 that is Church-Rosser and is equivalent to T_1. This result contrasts with that of Ó'Dúnlaing and also reveals the basis for Jantzen's result.

For an introduction to the literature on Thue systems and replacement systems, see [2, 3, 5, 6, 12, 13, 14].

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2. Thue Systems

If Σ is a finite alphabet, then Σ^* is the free monoid with identity e generated by Σ. If $w \in \Sigma^*$, then the *length* of w is denoted by $|w|$: $|e| = 0$, $|a| = 1$ for $a \in \Sigma$, and $|wa| = |w| + 1$ for $w \in \Sigma^*$, $a \in \Sigma$.

A *Thue system* T on a finite alphabet Σ is a subset of $\Sigma^* \times \Sigma^*$. The *Thue congruence generated by* T is the reflexive, transitive closure $\overset{*}{\leftrightarrow}$ of the relation defined as follows: if $(u, v) \in T$ or $(v, u) \in T$, then for every $x, y \in \Sigma^*$, $xuy \leftrightarrow xvy$. The *congruence class* of $z \in \Sigma^*$ (mod T) is $[z] = \{w \in \Sigma^* | w \overset{*}{\leftrightarrow} z\}$. The *monoid presented by* T has as elements the congruence classes of Σ^* (mod T), and as multiplication $[x] \circ [y] = [xy]$, so that $[e]$ is the monoid identity. Every finitely generated monoid is presented by some Thue system. Thue systems T_1 and T_2 are *equivalent* if they define the same congruence, i.e., for all x, y, $x \overset{*}{\leftrightarrow} y$ (mod T_1) if and only if $x \overset{*}{\leftrightarrow} y$ (mod T_2). Thus, equivalent Thue systems present the same monoid.

For a Thue system T, write $x \to y$ if $x \leftrightarrow y$ and $|x| > |y|$, write $x \vdash\!\dashv y$ if $x \leftrightarrow y$ and $|x| = |y|$, and write $x \mapsto y$ if $x \to y$ or $x \vdash\!\dashv y$. A string x is *irreducible* if there is no y such that $x \to y$ and is *minimal* if $x \overset{*}{\leftrightarrow} y$ implies $|x| \leqslant |y|$.

Let T be a Thue system.

(a) T is *Church-Rosser* if $x \overset{*}{\leftrightarrow} y$ implies that for some z, $x \overset{*}{\to} z$ and $y \overset{*}{\to} z$.

(b) T is *confluent* if $w \overset{*}{\to} x$ and $w \overset{*}{\to} y$ implies that for some z, $x \overset{*}{\to} z$ and $y \overset{*}{\to} z$.

(c) T is *preperfect* if $x \overset{*}{\leftrightarrow} y$ implies that for some z, $x \overset{*}{\mapsto} z$ and $y \overset{*}{\mapsto} z$.

A Thue system on alphabet Σ is *special* if $T \subseteq \Sigma^* \times \{e\}$.

A Thue system with no length-preserving relations is confluent if and only if it is Church-Rosser (a simple proof is in [5]), so that a special Thue system is confluent if and only if it is Church-Rosser. If a Thue system is Church-Rosser, then each congruence class has a unique irreducible element and a string is irreducible if and only if it is minimal [5, 6, 10].

3. Results

A string w is *primitive* if there is no string x and integer $k > 1$ such that $w = x^k$; otherwise, w is *imprimitive*. In either case, the shortest x such that $w = x^k$ is the *root* of w, denoted $\rho(w)$. If for some u, v with $0 < |u| < |w|$, $uw = wv$, then w has *overlap*.

Nivat [11] has shown that it is decidable whether a finite Thue system is confluent. (Also, see [4].) For Thue systems T with no length-preserving relations, Nivat's algorithm amounts to testing for the following: for every pair of (not necessarily distinct) relations with $|u_1| > |v_1|$ and $|u_2| > |v_2|$, and $(u_1, v_1) \in T$ or $(v_1, u_1) \in T$, and $(u_2, v_2) \in T$ or $(v_2, u_2) \in T$, (i) if there exist x, y such that $u_1 x = y u_2$ and $|x| < |u_2|$, then there exists z such that $v_1 x \overset{*}{\to} z$ and $y v_2 \overset{*}{\to} z$, and (ii) if there exist x, y such that $u_1 = x u_2 y$, then there exists z such that $v_1 \overset{*}{\to} z$ and $x v_2 y \overset{*}{\to} z$. Conditions (i) and (ii) are referred to as the "Nivat criteria."

Now consider a Thue system $T = \{(w, e)\}$. There are four (mutually exclusive) possibilities for the structure of w:

Case 1. w is primitive and has no overlap.

Case 2. w is imprimitive and $\rho(w)$ has no overlap.

Case 3. w is primitive and has overlap.

Case 4. w is imprimitive and $\rho(w)$ has overlap.

In Cases 1 and 2 it follows from Nivat's criteria that $T=\{(w,e)\}$ is confluent since (i) is vacuous in Case 1 and trivial in Case 2, and (ii) is vacuous in both cases.

Consider Case 3. If w is primitive and w has overlap, then there exist strings, x, y and integer $k>0$ such that $w=(xy)^k x$ and $xy \neq yx$ [9]. Thus, $xyw=wyx$, $|xy|=|yx|$, and $xy \overset{*}{\leftrightarrow} xyw = wyx \overset{*}{\leftrightarrow} yx$. Thus, xy and yx are congruent $(\mathrm{mod}\, T)$ and are irreducible (since $|xy|=|yx|<|w|$) but unequal (in Σ^*), so that $[xy]$ has two irreducible elements. Hence, T is not Church-Rosser. Second, since $T=\{(w,e)\}$, if u and v are strings such that $u \overset{*}{\leftrightarrow} v$, then the remainder of $|u|$ upon division by $|w|$ equals the remainder of $|v|$ upon division by $|w|$. Thus $|xy|=|yx|<|w|$ implies that xy, yx are minimal with respect to the congruence $\overset{*}{\leftrightarrow}$, that is, for any finite or infinite Thue system generating the congruence T, the strings xy and yx are both irreducible. Since $xy \neq yx$, this means that no Church-Rosser system generates this congruence.

Consider Case 4. Note that there exist strings x, y and integers t, k such that $w=\rho(w)^t$, $t>1$, $\rho(w)=(xy)^k x$, $k \geqslant 1$, and $xy \neq yx$. Now $((xy)^k x)^{t-1}(xy)^k w = ((xy)^k x)^{t-1}(xy)^k((xy)^k x)^t = ((xy)^k x)^t (yx)^k ((xy)^k x)^{t-1} = w(yx)^k((xy)^k x)^{t-1}$ so $w \to e$ implies $((xy)^k x)^{t-1}(xy)^k \overset{*}{\leftrightarrow} (yx)^k((xy)^k x)^{t-1}$. Let $u=((xy)^k x)^{t-1}(xy)^k$ and $v=(yx)^k((xy)^k x)^{t-1}$ so that $|u|=|v|<|w|$, $u \overset{*}{\leftrightarrow} v$, and $u \neq v$ (since $xy \neq yx$). Thus, just as in Case 3, u and v are distinct strings that are congruent and minimal with respect to the congruence generated by T. Hence, neither T nor any other Thue system equivalent to T is Church-Rosser.

Thus, we have the result.

Theorem. *Let $T=\{(w,e)\}$. There is a (finite or infinite) Church-Rosser Thue system equivalent to T if and only if T is Church-Rosser.*

One might ask about the computational difficulty of determining for a string w which of cases 1–4 holds. Avenhaus and Madlener [1] have noted that the pattern-matching algorithm of Knuth, Morris, and Pratt [8] can be used to decide in linear time which of the four cases holds.

One cannot obtain the analogous result for preperfect systems. To see this, let $T_1=\{(aba,e)\}$ and $T_2=\{(aba,e),\ (ab,ba)\}$ where $\Sigma=\{a,b\}$. In T_1, $ab \overset{*}{\leftrightarrow} ababa \overset{*}{\leftrightarrow} ba$ so that T_2 is equivalent to T_1. The analysis of Case 3 above shows that the special system T_1 is not Church-Rosser and, since T_1 has no length-preserving rules, not preperfect. Since $(ab,ba) \in T_2$, a and b commute by means of length-preserving rules. Since for every $w \in \{a,b\}^*$ there exist unique $p \geq 0$ and $q \geq 0$ such that $p+q=|w|$ and $w \overset{*}{\vdash\!\dashv} a^p b^q$, and $aab \vdash\!\dashv aba \to e$, congruence classes of the congruence generated by T_2 are $[ab^n]$ and $[b^n]$ for every $n \geq 0$. Thus, T_2 is preperfect.

For any Thue system T_1 the preperfect Thue system $T_2=\{(u,v) \| |u| \geqslant |v| \text{ and } u \overset{*}{\leftrightarrow} v\ (\mathrm{mod}\, T_1)\}$ is equivalent to T_1, so that every Thue *congruence* is preperfect.

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