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Restricted One-Counter Machines with Undecidable Universe Problems*

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Abstract. We show the undecidability of the universe problem for two restricted classes of nondeterministic one-counter machines. These classes are among the simplest known for which the universe problem can be shown unsolvable.

1. Introduction

In this paper, we introduce two very simple subclasses of *one*-counter machines with restricted nondeterminism and show that the "universe" problem (whether an automaton accepts *all* its input strings) for each subclass is unsolvable.

The equivalence problem for deterministic finite-turn pushdown machines and for deterministic one-counter machines is decidable [9, 10]. Equivalence is also decidable for deterministic two-way multicounter machines whose inputs and counters are reversal-bounded [6]. On the other hand, even the (simpler) universe problem for the class C of nondeterministic *one*-counter machines whose counters make at most *one* reversal is unsolvable [1, 3]. This last result clearly illustrates the power of nondeterminism even in simple computing machines. As another example, it is well known that relational equivalence is decidable for deterministic generalized sequential machines (gsm's). However, in a recent paper [7], it is shown that relational equivalence is undecidable for ε-*free* nondeterministic gsm's whose input or output alphabet is restricted to *one* letter. (The unsolvability without the alphabet restriction has been shown earlier in [4].) A related problem, the universe problem for relations defined by

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multitape finite automata making at most one nondeterministic move, is also undecidable [2]. Here we consider another instance of the following general problem: Suppose a decision problem is undecidable for a given class of nondeterministic machines but decidable for the deterministic subclass. Find a subclass consisting of machines with "restricted" nondeterminism for which the problem remains unsolvable. We feel that a study of this kind will give us a better understanding of the true nature of nondeterminism as it relates to decision questions. For a related research effort, one that concerns hierarchies of computations based on the number of nondeterministic steps, see [8].

In this short note we exhibit two very simple subclasses C_1 and C_2 of C whose universe problems are unsolvable. The only nondeterminism involved in the operations of the machines comprising $C_1(C_2)$ consists essentially of deciding when to reverse the counter (when to start using the counter for the first time). We believe that C_1 and C_2 are among the simplest known classes of machines containing the finite-state acceptors for which the universe problem can be shown undecidable.

C_1 consists of counter machines M satisfying the following conditions: (1) M has no ε-moves (ε is the null string), (2) M can make at most one counter-reversal, (3) The only nondeterminism involved in the operation of M is in deciding when to enter the counter-*decreasing* (= counter-reversal) mode. (This implies that once the counter-decreasing mode is entered, M's moves are deterministic.) Formally, a machine in C_1 can be specified by a 5-tuple $M=\langle K,\Sigma,\delta,q_0,F\rangle$, where K, Σ, and $F\subseteq K$ are finite nonempty sets of states, input alphabet and accepting states, respectively. $K=K_1\cup K_2$ with $K_1\cap K_2=\varnothing$, where K_1 consists of the states used in the counter-increasing mode and K_2 are the states used in the counter-decreasing mode. The start state q_0 is in K_1 and δ is a mapping from $K\times\Sigma\times\{0,1\}$ into the subsets of $K\times\{-1,0,+1\}$ satisfying the following conditions (the third parameter in the domain is 0 or 1 depending on whether the counter has zero or nonzero value, respectively):

(a) For each (q,a,b) in $K_1\times\Sigma\times\{0,1\}$, $\delta(q,a,b)$ is empty or of the form $\{(p_1,d_1)\}$ or $\{(p_2,d_2),(p_3,d_3)\}$, where $d_i\geqslant 0, p_1$ is in $K_1\cup K_2, p_2$ is in K_1, and p_3 is in K_2. Thus, the only nondeterminism possible is in changing from increasing mode to decreasing mode.

(b) For each (q,a,b) in $K_2\times\Sigma\times\{0,1\}, \delta(q,a,b)$ is empty or of the form $\{(p,d)\}$, where p is in K_2 and $d\leqslant 0$. This means that once the counter-decreasing mode is entered, the machine remains deterministic in this mode. Note that since M has no ε-moves, ε is accepted by M if and only if q_0 is in F.

The second class C_2 consists of counter machines M satisfying (1) and (2) above, and (3′) the only nondeterminism involved in the operation of M is in deciding when to enter the counter-*using* mode, i.e., when to start using the counter for the first time. (Again, it is understood that once the counter-using mode is entered, M's moves are deterministic.) Thus, a machine in C_2 can be specified by a 5-tuple $M=\langle K,\Sigma,\delta,q_0,F\rangle$, where K, Σ, and F are as above, $K=K_1\cup K_2\cup K_3$ with $K_i\cap K_j=\varnothing(i\neq j)$ and q_0 in K_1. δ is a mapping from $K\times\Sigma\times\{0,1\}$ into the subsets of $K\times\{-1,0,+1\}$ satisfying the following conditions:

(c) For each (q,a) in $K_1\times\Sigma, \delta(q,a,1)$ is empty, and $\delta(q,a,0)$ is either empty or of the form $\{(p_1,0)\}$ or $\{(p_2,0),(p_3,0)\}$, where p_1 is in $K_1\cup K_2, p_2$ is in K_1, and p_3 is in K_2.

(d) For each (q,a,b) in $K_2 \times \Sigma \times \{0,1\}, \delta(q,a,b)$ is empty or of the form $\{(p,d)\}$, where p is in $K_2 \cup K_3$ and $d \geqslant 0$.

(e) For each (q,a,b) in $K_3 \times \Sigma \times \{0,1\}, \delta(q,a,b)$ is empty or of the form $\{(p,d)\}$, where p is in K_3 and $d \leqslant 0$.

2. Undecidability of the Universe Problem for C_1

The proof of the undecidability of the universe problem for C_1 involves the construction for each single-tape Turing machine Z of a counter machine M_1 in C_1 over some input alphabet Σ. M_1 has the property that the language it accepts is equal to Σ^* if and only if Z does not halt on blank tape.

Theorem 1. *The universe problem for the class C_1 is undecidable.*

Proof. Let Z be a single-tape Turing machine (TM) [5] with state set K and tape alphabet $\Gamma = \{0,1,b\}$ (b for blank). Assume that $K \cap \Gamma = \varnothing$, that the start state q_0 is not a halting state, and that Z does not write blanks. A configuration of Z can be represented by a string of the form xqy, where q is in K and either x is in $b^+(0+1)^*$ and y is in $(0+1)^*b^+$ or x is in b^+ and y is in $b(0+1)^+b^+$. (Note that any string representing a configuration of Z must begin and end with at least one blank.) xqy represents the configuration in which Z is in state q scanning the leftmost symbol of y. If α represents a configuration of Z, then for any $i,j \geqslant 0, b^i\alpha b^j$ represents the same configuration. For any $i,j \geqslant 1, b^i q_0 b^j$ represents the initial configuration of Z on blank tape. If α represents a configuration of Z, we shall simply say α is a configuration of Z. Clearly, Z halts on blank tape if and only if there exist integers $s \geqslant 4$ and $t \geqslant 2$ and configurations $\alpha_1, \ldots, \alpha_t$ such that $|\alpha_1|$(= length of α_1) $= |\alpha_2| = \ldots = |\alpha_t| = s$ and the sequence $\langle \alpha_1, \ldots, \alpha_t \rangle$ is a halting computation of Z on blank tape. ($t \geqslant 2$ since q_0 is not a halting state, and $s \geqslant 4$ since Z does not write blanks.) Note that for any $1 \leqslant i < t$, configuration α_{i+1} differs from α_i in at most 3 positions.

Let # be a new symbol and let $\Sigma = K \cup \{0,1,b,\#\}$. For any string x and $0 \leqslant m < n \leqslant |x|$, denote by $[x,m,n]$ that portion of x starting with the $(m+1)$-st symbol and ending with the n-th symbol. Define a predicate $P_1(x)$ on Σ^* as follows. For x in $\Sigma^*, P_1(x)$ is true if and only if there exist integers $s \geqslant 4$ and $t \geqslant 2$ such that (see Fig. 1):

(1) The length of x is $2^t s$.

(2) For $i = 1,2,\ldots,t$,

$$\alpha_i = \left[x, (2^i - 1)s, 2^i s\right]$$

is a configuration of Z, and all other symbols in x are the new symbol #.

(3) The sequence $\langle \alpha_1, \ldots, \alpha_t \rangle$ is a halting computation of Z on blank tape. Note that if $P_1(x)$ is true, then for any $1 \leqslant r \leqslant s$,

$$(2^{i+1} - 1)s + r = ((2^i - 1)s + r) + 2^i s.$$

Let $L_1 = \{x | x \text{ in } \Sigma^*, P_1(x) \text{ is false}\}$. Then $L_1 = \Sigma^*$ if and only if Z does not halt on blank tape. We will describe a machine M_1 in C_1 accepting L_1. The theorem

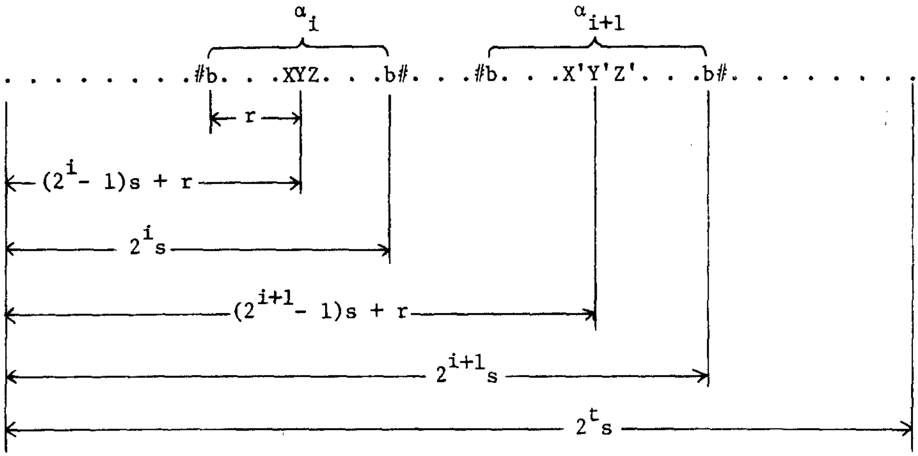

Fig. 1. Format of x when $P_1(x)$ is true

would then follow from the unsolvability of the halting problem for Turing machines.

We only sketch the operation of M_1 leaving most of the details to the reader. Given an input string x, M_1 moves right on the input, incrementing the counter for each right move. At some point, M_1 guesses that it has reached a position r within some configuration α_i. This position is represented by Y in Fig. 1. Then, without changing the counter, M_1 moves right until it reaches the # to the right of α_i. When M_1 reaches the #, it moves right, decrementing the counter for each right move, until the counter becomes zero. (Note that if $P_1(x)$ is true then at the time the counter becomes zero, M_1 must be on the same position r within α_{i+1}. This position is represented by Y' in Fig. 1.) M_1 then checks whether Y' and its neighbors X' and Z' are appropriate for α_{i+1} to be a valid successor of α_i. (We assume that X, Y, Z have been recorded in the finite control.) If they are *not* appropriate, or if α_1 and α_t are *not* seen (in passing) to be correct initial and halting configurations, respectively, then M_1 accepts the input; otherwise, M_1 does not accept the input in this particular computation. □

3. Undecidability of the Universe Problem for C_2

The proof of the unsolvability of the universe problem for the class C_2 follows the technique above.

Theorem 2. *The universe problem for the class C_2 is undecidable.*

Proof. Let Z and Σ be as in the proof of Theorem 1. We assume that Z can only halt after making an even positive number of moves. Define a predicate $P_2(x)$ on Σ^* as follows. For x in $\Sigma^*, P_2(x)$ is true if and only if there exist integers $s \geqslant 4$ and $t \geqslant 1$ such that:

(1) x is of the form $\# \alpha_1 \# \alpha_2 \ldots \# \alpha_{2t}$.

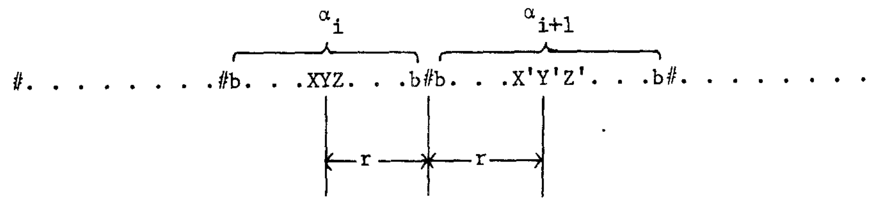

Fig. 2. Format of x when $P_2(x)$ is true

(2) $|\alpha_1| = |\alpha_2| = \ldots = |\alpha_{2t}| = s$.

(3) The sequence $\langle \alpha_1, \alpha_2^r, \ldots, \alpha_{2t-1}, \alpha_{2t}^r \rangle$ is a halting computation of Z on blank tape, where $\alpha_i^r =$ reverse of string α_i.

Let $L_2 = \{x | x \text{ in } \Sigma^*, P_2(x) \text{ is false}\}$. Then $L_2 = \Sigma^*$ if and only if Z does not halt on blank tape. We can now construct a machine M_2 in C_2 accepting L_2. Given an input string x, M_2 moves right on the input without using the counter. At some point, M_2 guesses that it has reached a symbol within some string α_i and decides to enter the counter-using mode. This symbol is represented by Y in Fig. 2. M_2 then moves right, incrementing the counter for each right move, until it reaches the # to the right of α_i. When M_2 reaches the #, it moves right, decrementing the counter for each right move, until the counter becomes zero. Then M_2 checks whether X', Y', Z' are appropriate if α_{i+1}^r is a valid successor of α_i. (We assume that X, Y, Z have been recorded in the finite control.) If they are *not* appropriate, or if α_1 and α_{2t}^r are *not* seen (in passing) to be correct initial and halting configurations, respectively, then M_2 accepts the input; otherwise, M_2 does not accept the input in this particular computation. □

4. Conclusion

The undecidability of the universe problem for the class C of nondeterministic one-counter machines whose counters make at most one reversal is well known [1,3]. We have strengthened this result by exhibiting two very simple subclasses C_1 and C_2 of C for which the universe problem remains unsolvable. It is obvious from the definitions of C_1 and C_2 that they are properly included in C. Let L, L_1 and L_2 be the classes of languages accepted by machines in C, C_1 and C_2, respectively. We believe that $L_1 - L_2 \neq \varnothing, L_2 - L_1 \neq \varnothing$ and $L - (L_1 \cup L_2) \neq \varnothing$. Candidate languages for $L_1 - L_2$, $L_2 - L_1$ and $L - (L_1 \cup L_2)$ are:

$A = \{a^i b^j c^k | i,j,k \geqslant 1, i = j \text{ or } i = k\}$,

$B =$ reverse (A) and

$C = A \cup B$.

We shall consider these and other questions (e.g., closure properties, characterizations) in a future paper.

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