Structural Optimization 10, 16–32 © Springer-Verlag 1995

Minimum cost design of RC frames using the DCOC method
Part I: columns under uniaxial bending actions

A. Adamu and B.L. Karihaloo

School of Civil & Mining Engineering, The University of Sydney, NSW 2006, Australia

Abstract The paper solves the minimum-cost design problem of RC plane frames. The cost to be minimized includes those of concrete, reinforcing steel and formwork, whereas the design constraints include limits on maximum deflection at a specified node, on bending and shear strengths of beams and on combined axial and bending strength of columns, in accordance with the limit state design (LSD) requirements. The algorithms developed in this work can handle columns under uniaxial bending actions. In the companion paper the numerical procedure is generalized to include columns subjected to biaxial bending. On the basis of discretized continuum-type optimality criteria (DCOC), the design problem is systematically formulated, followed by explicit mathematical derivation of optimality criteria upon which iterative procedures are developed for the solution of design problems when the design variables are the cross-sectional parameters and steel ratios. For practical reasons, the cross-sectional parameters are chosen to be either uniform per member or uniform for several members at a given floor level. The procedure is illustrated on several test examples. It is shown that the DCOC-based methods are particularly efficient for the design of large RC frames.

1 Introduction

The fundamental features of the methods based on COC (Rozvany 1989; Rozvany *et al.* 1990; Rozvany and Zhou 1993a, b) to obtain the minimum-cost design of RC beams subject to strength and deflection constraints, in addition to side constraints, were explained by Adamu *et al.* (1994), and Adamu and Karihaloo (1995) using several test examples of single-span beams. These methods were generalized in their discretized version (Zhou and Rozvany 1992, 1993) to multispan beams with freely varying design variables or uniform cross-section per span by Adamu and Karihaloo (1994a, b). Whenever possible the solutions obtained by COC and/or DCOC methods were compared with those obtained by NLP methods (Kanagasundaram and Karihaloo 1990; Karihaloo 1993).

This paper is devoted to obtaining the minimum-cost design of RC frames. As the design problem involves complex behavioural constraints, it is reformulated to include the constraints on both beams and columns. Optimality criteria are derived and used as a basis for the development of algorithms suitable for solving the optimization problem of these structures.

In the first stage, only beams with uniform cross-sectional parameters per span are considered. However, the steel ratio is allowed to vary freely. The cross-sectional parameters and steel ratio in each column are assumed to be uniform for practical reasons. As indicated by Adamu and Karihaloo (1995), the formwork cost constitutes the major cost of RC construction. In RC multibay and multistorey frames the beam formwork is re-used from floor to floor if the column sizes are kept constant (Ferguson 1979). Hence, it is economical to keep the cross-sectional parameter of the columns in each storey uniform and only vary the amount of steel among them. Further, to facilitate supervision of construction and for economical reasons, it is customary to keep the beam depth and width uniform in a given storey. In the second stage of this study, these observations are exploited in the reformulation of the design problem and derivation of optimality criteria for regular multibay and multistorey frames.

The design constraints include limits on maximum deflection at a prescribed node, bending and shear strengths of beams and uniaxial or biaxial bending strength of columns according to design codes (CEB/FIP 1990; SAA 1988; Warner *et al.* 1988; Ferguson 1979). In Part I of this paper, columns under uniaxial bending actions are considered. However, many columns, especially the corner ones, are subjected to simultaneous moments about both principal axes of the cross-section. In Part II of this paper, columns under biaxial bending actions will be considered. Several examples are solved to demonstrate the versatility of the DCOC-based technique for large RC frames.

2 The design problem

2.1 Problem formulation using an augmented Lagrangian

Consider an RC plane frame consisting of N_b beams and N_c columns. Each beam is further subdivided into N_e elements to account for the variable bending moment, but each column is treated as a single unit and designed for the end moments and axial load.

The beams are assumed to be rectangular in cross-section with width z_{1b}, effective depth z_2^m, and tensile steel ratio z_3^{em} ($e = 1, \ldots, N_e$; $m = 1, \ldots, N_b$); the distance from the centre of tension steel to nearby extreme concrete fibre is d_b'. In this study z_{1b} and d_b' are given, whereas z_2^m and z_3^{em} are design variables. The depth z_2^m is kept constant along the length of a beam member while the steel ratio in each element of the member is permitted to vary. Thus, we have N_e plus one variable per beam member.

In the case of the concrete column section, the steel reinforcement is placed symmetrically with respect to the principal axis of bending (Fig. 1). The width z_{1c}, and the distance from the extreme fibre to the nearby centroid of reinforcement area d'_c are assumed to be given, whereas the effective depth z^c_2 and gross steel ratio z^c_3 [$c = 1, \ldots, N_c$; $A_s = z_{1c}(z^c_2+d'_c)z^c_3$] are design variables. The normal procedure for designing reinforced concrete columns involves end conditions and effective length concepts. To simplify this procedure, the depth z^c_2 and the steel ratio z^c_3 are kept uniform in each column (with no curtailment of reinforcement) giving two unknowns per column.

Thus, an RC frame structure with N_b beams and N_c columns will involve $[(1+N_e)N_b + 2N_c]$ design variables. Note that the superscript m or em indicates that the item in question is related to the beam of span m or element e of span m, respectively, whereas the superscript c alone refers to a column.

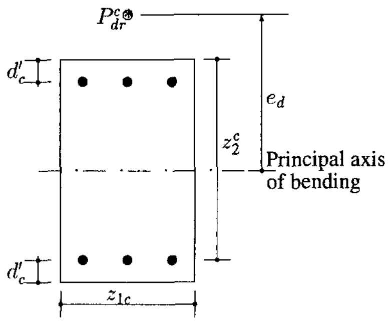

Fig. 1. Column section with symmetric steel reinforcement

The axial stiffness is given by:
for a beam $EA^{em} = EA^m = k_b z_{1b} z^m_2 = \overline{\ell}_{2b} z^m_2$,
for a column $EA^c = k_c z_{1c} z^c_2 = \overline{\ell}_{2c} z^c_2$,
in which k_b and k_c are constants that account for d'_b and d'_c. In most cases they range between 1.05 to 1.10, and $\overline{\ell}_{2b} = E_c z_{1b} k_b$, $\overline{\ell}_{2c} = E_c z_{1c} k_c$.

The flexural rigidity EI is:
for a beam $EI^{em} = EI^m = 0.045 E_c z_{1b}(z^m_2)^3 = \ell_{2b}(z^m_2)^3$, and
for a column $EI^c = 0.045 E_c z_{1c}(z^c_2)^3 = \ell_{2c}(z^c_2)^3$.

The objective function is the cost of construction which includes the costs of concrete, reinforcing steel and formwork, and is

$$\phi = \sum_{m=1}^{N_b}\sum_{e=1}^{N_e} \psi^{em}_b(z^m_2, z^{em}_3) + \sum_{c=1}^{N_c} \psi^c_c(z^c_2, z^c_3)\,, \tag{1}$$

where

$$\psi^{em}_b(z^m_2, z^{em}_3) = [z^m_2(z^{em}_3 m_{2b} + m_{1b}) + \overline{c}_{1b}]L^{em}\,,$$

$$\psi_c(z^c_2, z^c_3) = [z^c_2(z^c_3 m_{2c} + m_{1c}) + z^c_3 m_{2c} d'_c + \overline{c}_{2c}]L^e\,,$$

with

$$m_{1b} = (z_{1b} + 2c_{fc})c_c\,,\ m_{2b} = z_{1b}(c_{sc} - 1)c_c\,,$$

$$\overline{c}_{1b} = [z_{1b}d'_b + (z_{1b} + 2d'_b)c_{fc}]c_c\,,\ m_{1c} = (z_{1c} + 2c_{fc})c_c\,,$$

$$m_{2c} = z_{1c}(c_{sc} - 1)c_c\,,\ \overline{c}_{2c} = [z_{1c}d'_c + 2c_{fc}(z_{1c} + d'_c)]c_c\,.$$

As the flexibility relationships are used to express the behavioural constraints, the statically determinate and stable frame element chosen is the one with joint B [right-end (see Fig. A.1 of Appendix A)] built-in.

The displacement at a specified node using the notation described in Appendix A is computed using

$$u_m = \sum_{m=1}^{N_b}\sum_{e=1}^{N_e}\left[\{\overline{\mathbf{F}}^{em}_{fA}\}^T[\mathbf{f}^{em}_A]\{\mathbf{F}^{em}_{fA}\} + \{\overline{\mathbf{F}}^{em}_{fA}\}^T\{\hat{\mathbf{u}}^{em}_{fA}\}\right] + \sum_{c=1}^{N_c}\left[\{\overline{\mathbf{F}}^c_{fA}\}^T[\mathbf{f}^c_A]\{\mathbf{F}^c_{fA}\} + \{\overline{\mathbf{F}}^c_{fA}\}^T\{\hat{\mathbf{u}}^c_{fA}\}\right]\,, \tag{2}$$

which is identical to

$$u_m = \sum_{m=1}^{N_b}\sum_{e=1}^{N_e}\left(\frac{Q^{em}_1}{A^m E} + \frac{Q^{em}_2}{EI^m}\right) + \sum_{c=1}^{N_c}\left(\frac{Q^c_1}{A^c E} + \frac{Q^c_2}{EI^c}\right)\,, \tag{3}$$

where

$$Q^i_1 = P^i_A \overline{P}^i_A L^i - \frac{1}{2}q^i(L^i)^2\overline{P}^i_A\,,$$

$$Q^i_2 = \frac{V^i_A(L^i)^2}{6}\left(2\overline{V}^i_A L^i - 3\overline{M}^i_A\right) + \frac{M^i_A L^i}{2}\left(2\overline{M}^i_A - \overline{V}^i_A L^i\right) + \frac{p^i(L^i)^3}{24}\left(4\overline{M}^i_A - 3\overline{V}^i_A L^i\right)\,.$$

Here, the superscript i becomes em for beam elements and c when the unit is a column. If there is no distributed load within the element or the unit considered, p^i or q^i or both are set equal to zero. Moreover, if there are other types of load, their effect can be included through the so-called equivalent load system.

The strength constraints for a beam element are as follows:
(a) the flexural strength constraint

$$q^{em}_1 = \frac{|M^{em}_{\max}|}{z_{1b}(z^m_2)^2 z^{em}_3(c_2 - c_3 z^{em}_3)} - 1.0 \le 0\,, \tag{4}$$

(b) the shear strength constraint against web crushing

$$q^{em}_2 = \frac{|V^{em}_{\max}|}{\ell_4 z^m_2} - 1.0 \le 0\,, \tag{5}$$

$c_2 = 0.87 f_y$, $c_3 = 0.6678 f^2_y/f_c$, $\ell_4 = c_4 z_{1b}$, $c_4 = 0.2 f_c$,
where $M^{em}_{\max}$ and $V^{em}_{\max}$ are, respectively, the maximum moment and maximum shear force in the e-th element.

The uniaxial bending strength constraint for a column in accordance with the LSD of CEB/FIP is

$$q^c_3 = \frac{|P^c_{\max}|}{P^c_{dr}} - 1.0 \le 0\,, \tag{6}$$

where

$$P^c_{dr} = \frac{P_{cr}}{1 + \dfrac{e_d}{e_b}\left(\dfrac{P_{cr}}{P_{nb}} - 1\right)}\,, \tag{7}$$

if failure at ULS of collapse is initiated by the crushing of concrete (compression failure), and

$$P^c_{dr} = g_1(z^c_2 + d'_c)\left\{-\left(\frac{2e_d}{z^c_2 + d'_c} - 1 + z^c_3\right) + \left[\left(\frac{2e_d}{z^c_2 + d'_c} - 1 + z^c_3\right)^2 + 2z^c_3\left[\overline{g}_2\left(1 - \frac{2d'_c}{z^c_2 + d'_c}\right) + 1 - 0.5z^c_3\right]\right]^{\frac{1}{2}}\right\}\,, \tag{8}$$

if primary failure at ULS is initiated by the yielding of tensile steel followed by the crushing of concrete.

The section capacity for uniaxial load P_{cr}, the section capacity for balanced failure P_{nb} and its eccentricity e_b in (7) are given as

$$P_{cr} = z_{1c}(z_2^c + d_c')[0.85 f_{cd} - 0.85 f_{cd} z_3^c + f_{yd} z_3^c]\,, \tag{9}$$

$$P_{nb} = 0.42 f_{cd} z_{1c} z_2^c + 0.5 z_3^c z_{1c}(z_2^c + d_c')(f_s' - 0.85 f_{cd} - f_{yd}), \tag{10}$$

$$e_b = \Big\{0.5 z_{1c} z_3^c[(z_2^c)^2 - (d_c')^2](f_{yd} + f_s' - 0.85 f_{cd}) + 0.42 f_{cd} z_{1c} z_2^c (0.37 z_2^c + d_c')\Big\}/(2P_{nb})\,. \tag{11}$$

The stress f_s' and strain ε_s' of the compressive steel in a column are

$$f_s' = E_s \varepsilon_s' \leq f_{yd}\,, \tag{12}$$

$$\varepsilon_s' = \frac{0.63 z_2^c - d_c'}{0.63 z_2^c}\varepsilon_{cu}\,, \tag{13}$$

where $\varepsilon_{cu} = 0.0035$ is the strain in concrete at ULS and,

$$g_1 = 0.425 z_{1c} f_{cd}\,, \tag{14}$$

$$\overline{g}_2 = 2\left(\frac{f_{yd}}{0.85 f_{cd}} - 0.5\right)\,, \tag{15}$$

$f_{cd} = 0.667 f_c$, $f_{yd} = 0.87 f_y$ are the design compressive strength of concrete and design yield strength of reinforcing steel. The eccentricities e_b and e_d (design eccentricity) are measured from the plastic centroid of the cross-section to the location of the axial force in question, and for symmetric steel reinforcement these coincide with the principal axis of bending (Fig. 1); $P_{\max}^c$ is the maximum compressive force in the column due to the design action.

If the design action in the column results in axial tension, then

$$P_{dr}^c = \frac{z_{1c} z_3^c f_{yd}[(z_2^c)^2 - (d_c')^2]}{2e_d + z_2^c - 2d_c'}\,. \tag{16}$$

Recalling the notations used by Adamu and Karihaloo (1994a) to denote the location of $M_{\max}^{em}$ and $V_{\max}^{em}$ for a beam element from the left end as x_m^{em} and x_v^{em}, respectively,

$$M_{\max}^{em} = M_A^{em} - V_A^{em} x_m^{em} + \frac{p^{em}(x_m^{em})^2}{2}, \tag{17}$$

and

$$V_{\max}^{em} = V_A^{em} - p^{em} x_v^{em}\,. \tag{18}$$

For a vertical or inclined column the maximum compressive force is always at its lower end and is given by (including selfweight)

$$P_{\max}^c = P_A^c\,. \tag{19}$$

The strength constraints may now be rewritten as follows: for a beam section

$$q_1^{em} = \frac{\operatorname{sgn}(M_{\max}^{em})}{z_{1b}(z_2^m)^2 z_3^{em}(c_2 - c_3 z_3^{em})}\left[M_A^{em} - V_A^{em} x_m^{em} + \frac{p^{em}(x_m^{em})^2}{2}\right] - 1.0 \leq 0\,, \tag{20}$$

$$q_2^{em} = \frac{\operatorname{sgn}(V_{\max}^{em})}{\ell_4(z_2^m)}\,[V_A^{em} - p^{em} x_v^{em}] - 1.0 \leq 0\,, \tag{21}$$

and for a column

$$q_3^c = \frac{\operatorname{sgn}(P_{\max}^c)}{P_{dr}^c}\,[P_A^c] - 1.0 \leq 0\,. \tag{22}$$

These constraints may be expressed in a matrix form

$$q_1^{em} = \{\mathbf{R}_1^{em}\}^T \begin{Bmatrix} P_A^{em} \\ V_A^{em} \\ M_A^{em} \end{Bmatrix} + \{\hat{\mathbf{R}}_1^{em}\}^T \begin{Bmatrix} 0 \\ -p^{em} x_m^{em} \\ \dfrac{p^{em}(x_m^{em})^2}{2} \end{Bmatrix} - 1.0 \leq 0\,, \tag{23}$$

$$\mathbf{R}_1^{em} = \begin{Bmatrix} 0 \\ -\dfrac{x_m^{em}\operatorname{sgn}(M_{\max}^{em})}{z_{1b}(z_2^m)^2 z_3^{em}(c_2 - c_3 z_3^{em})} \\ \dfrac{\operatorname{sgn}(M_{\max}^{em})}{z_{1b}(z_2^m)^2 z_3^{em}(c_2 - c_3 z_3^{em})} \end{Bmatrix},$$

$$\{\hat{\mathbf{R}}_1^{em}\} = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{\operatorname{sgn}(M_{\max}^{em})}{z_{1b}(z_2^m)^2 z_3^{em}(c_2 - c_3 z_3^{em})} \end{Bmatrix},$$

$$q_2^{em} = \{\mathbf{R}_2^{em}\}^T \begin{Bmatrix} P_A^{em} \\ V_A^{em} \\ M_A^{em} \end{Bmatrix} + \{\hat{\mathbf{R}}_2^{em}\}^T \begin{Bmatrix} 0 \\ -p^{em} x_v^{em} \\ \dfrac{p^{em}(x_v^{em})^2}{2} \end{Bmatrix}, \tag{24}$$

$$\{\mathbf{R}_2^{em}\}^T = \begin{Bmatrix} 0 \\ \dfrac{\operatorname{sgn}(V_{\max}^{em})}{\ell_4(z_2^m)} \\ 0 \end{Bmatrix},\quad \{\hat{\mathbf{R}}_2^{em}\} = \begin{Bmatrix} 0 \\ \dfrac{\operatorname{sgn}(V_{\max}^{em})}{\ell_4(z_2^m)} \\ 0 \end{Bmatrix}$$

and

$$q_3^c = \{\mathbf{R}_3^c\}^T \begin{Bmatrix} P_A^c \\ V_A^c \\ M_A^c \end{Bmatrix}, \tag{25}$$

$$\{\mathbf{R}_3^c\} = \begin{Bmatrix} \dfrac{\operatorname{sgn}(P_{\max}^c)}{P_{dr}^c} \\ 0 \\ 0 \end{Bmatrix}.$$

The minimum cost design problem of an RC frame structure including the constraints and bounds may be mathematically stated using the augmented Lagrangian as follows:

minimize $\overline{\overline{\mathbf{\Phi}}} =$

$$\phi(\mathbf{z}) + \mu\left[\{\overline{\mathbf{F}}_f\}^T[\mathbf{f}]\{\mathbf{F}_f\} + \{\overline{\mathbf{F}}_f\}^T\{\hat{\mathbf{u}}_f\} - \Delta_{\text{al}} + \eta\right] + \sum_{m=1}^{N_b}\sum_{e=1}^{N_e}\sum_{j=1}^{J_e}\lambda_j^{em}\left[\{\overline{\overline{\mathbf{R}}}_j^{em}\}^T\{\mathbf{F}_f\} + \{\hat{\mathbf{R}}_j^{em}\}^T\{\hat{\mathbf{F}}_j^{em}\} - 1.0 + \omega_j^{em}\right] + \sum_{c=1}^{N_c}\lambda_3^c\left[\{\overline{\overline{\mathbf{R}}}_3^c\}^T\{\mathbf{F}_f\} - 1.0 + \omega_3^c\right] +$$

$$\{\alpha^r\}^T\left[\{\mathbf{P}\}-[\mathbf{B}]\{\mathbf{F}_f\}\right]+\{\alpha^v\}^T\left[\{\overline{\mathbf{P}}^v\}-[\mathbf{B}]\{\overline{\mathbf{F}}_f\}\right]+$$
$$\sum_{m=1}^{N_b}\sum_{e=1}^{N_e}\sum_{i=2}^{3}[\beta_i^{em}(-z_i^{em}+z_{i\ell b}+\underline{s}_{ib}^{em})+$$
$$\gamma_i^{em}(z_i^{em}-z_{iub}+\overline{s}_i^{em})]+$$
$$\sum_{c=1}^{N_c}\sum_{i=2}^{3}[\beta_i^c(-z_i^c+z_{i\ell c}+\underline{s}_i^c)+\gamma_i^c(z_i^c-z_{iuc}+\overline{s}_i^c)]\,. \quad (26)$$

3 Optimality criteria

3.1 Mathematical derivation

Based on (1), (3), (23), (24) and (25) the optimality criteria flowing from the variation of $\overline{\Phi}$ with respect to the depth $[z_2^m$ or $z_2^c]$ and the steel ratio $[z_3^{em}$ or $z_3^c]$ are given by

$$\sum_{e=1}^{N_e}\left\{\left[z_3^{em}m_{2b}+m_{1b}\right]L^{em}-\right.$$
$$\left.\mu\left[\frac{Q_1^{em}}{\ell_{2b}(z_2^m)^2}+\frac{3Q_2^{em}}{\ell_{2b}(z_2^m)^4}\right]\right\}+$$
$$\sum_{e=1}^{N_e}\lambda_1^{em}\left[-\frac{2}{z_2^m}\{\mathbf{R}_1^{em}\}^T\{\mathbf{F}_A^{em}\}-\frac{2}{z_2^m}\{\hat{\mathbf{R}}_1^{em}\}^T\{\hat{\mathbf{F}}_1^{em}\}\right]+$$
$$\sum_{e=1}^{N_e}\left\{\lambda_2^{em}\left[-\frac{1}{z_2^m}\{\mathbf{R}_2^{em}\}^T\{\mathbf{F}_A^{em}\}-\frac{1}{z_2^m}\{\hat{\mathbf{R}}_2^{em}\}^T\{\hat{\mathbf{F}}_2^{em}\}\right]\right\}-$$
$$(\beta_2^{em}-\gamma_2^{em})+\{\alpha^r\}^T\left\{\frac{\partial\mathbf{P}}{\partial z_2^m}\right\}=0\,,\quad m=1,\ldots,N_b\,, \quad (27)$$

or

$$(z_3^cm_{2c}+m_{1c})\,L^c-\mu\left[\frac{Q_1^c}{\ell_{2c}(z_2^c)^2}+\frac{3Q_2^c}{\ell_{2c}(z_2^c)^4}\right]+$$
$$\lambda_3^c\left[-\frac{1}{P_{dr}^c}\frac{\partial P_{dr}^c}{\partial z_2^c}\{\mathbf{R}_3^c\}^T\{\mathbf{F}_A^c\}\right]+\{\alpha^r\}^T\left\{\frac{\partial\mathbf{P}}{\partial z_2^c}\right\}-$$
$$(\beta_2^c-\gamma_2^c)=0\,,\quad c=1,\ldots,N_c\,, \quad (28)$$

and

$$\left\{(z_2^m m_{2b}L^{em})+\right.$$
$$\lambda_1^{em}\left[-\frac{(c_2-2c_3z_3^{em})}{z_3^{em}(c_2-c_3z_3^{em})}\{\mathbf{R}_1^{em}\}^T\{\mathbf{F}_A^{em}\}-\right.$$
$$\left.\left.\frac{(c_2-2c_3z_3^{em})}{z_3^{em}(c_2-c_3z_3^{em})}\{\hat{\mathbf{R}}_1^{em}\}^T\{\hat{\mathbf{F}}_1^e\}\right]\right\}-(\beta_3^{em}-\gamma_3^{em})=0\,,$$
$$e=1,\ldots,N_e;\ m=1,\ldots,N_b\,, \quad (29)$$

or

$$m_{2c}L^c(z_2^c+d_c')+\lambda_3^c\left[-\frac{1}{P_{dr}^c}\frac{\partial P_{dr}^c}{\partial z_3^c}\left\{\mathbf{R}_3^c\right\}^T\left\{\mathbf{F}_A^c\right\}\right]-$$
$$(\beta_3^c-\gamma_3^c)=0\,. \quad (30)$$

Note that β_2^{em} and γ_2^{em} in (27) could be simply rewritten as β_2^m and γ_2^m, respectively, in view of the fact that z_2^m is uniform in the span m. The terms $\{\alpha^r\}^T\{\frac{\partial\mathbf{P}}{\partial z_2^m}\}$ and $\{\alpha^r\}^T\{\frac{\partial\mathbf{P}}{\partial z_2^c}\}$ appear in (27) and (28) due to the inclusion of selfweight in the equilibrium equation of the real system. As discussed in the works of Zhou and Rozvany (1992), and Adamu and Karihaloo (1994a), $\{\alpha^r\}$ is a scaled adjoint displacement vector given by

$$\{\alpha^r\}^T=\mu\{\overline{\mathbf{u}}\}^T\,. \quad (31)$$

As the partial derivatives of the augmented Lagrangian are considered until now at the beam element/or column level, these terms may be written explicitly as follows:

(a) for beams

$$\{\alpha^r\}^T\left\{\frac{\partial\mathbf{P}}{\partial z_2^m}\right\}=\sum_{e=1}^{N_e}\mu\{\overline{\mathbf{u}}^{em}\}^T\left\{\frac{\partial\mathbf{P}^{em}}{\partial z_2^m}\right\}\,, \quad (32)$$

(b) for columns

$$\{\alpha^r\}^T\left\{\frac{\partial\mathbf{P}}{\partial z_2^c}\right\}=\mu\{\overline{\mathbf{u}}^c\}^T\left\{\frac{\partial\mathbf{P}^c}{\partial z_2^c}\right\}\,. \quad (33)$$

Alternatively, if one uses (4), (5) and (6) instead of (23), (24) and (25), then (27) or (28), and (29) or (30) may be rewritten as

$$\sum_{e=1}^{N_e}\left\{(z_3^{em}m_{2b}+m_{1b})L^{em}-\mu\left[\frac{Q_1^{em}}{\ell_{2b}(z_2^{em})^2}+\frac{3Q_2^{em}}{\ell_{2b}(z_2^m)^4}\right]-\right.$$
$$\lambda_1^{em}\frac{2|M_{\max}^{em}|}{z_{1b}(z_2^m)^3z_3^{em}(c_2-c_3z_3^{em})}-$$
$$\left.\lambda_2^{em}\frac{|V_{\max}^{em}|}{\ell_4(z_2^m)^2}+\frac{3\mu}{\ell_{2b}}\overline{u}^{em}\right\}-(\beta_2^m-\gamma_2^m)=0\,,$$
$$m=1,\ldots,N_b\,, \quad (34)$$

or

$$(z_3^cm_{2c}+m_{1c})L^c-\mu\left[\frac{Q_1^c}{\ell_{2c}(z_2^c)^2}+\frac{3Q_2^c}{\ell_{2c}(z_2^c)^4}\right]-$$
$$\lambda_3^c\frac{|P_{\max}^c|}{(P_{dr}^c)^2}\frac{\partial P_{dr}^c}{\partial z_2^c}+\frac{3\mu}{\ell_{2c}}\overline{u}^c-(\beta_2^c-\gamma_2^c)=0\,,$$
$$c=1,\ldots,N_c\,, \quad (35)$$

and

$$\left\{z_2^m m_{2b}L^{em}-\lambda_1^{em}\frac{(c_2-2c_3z_3^{em})|M_{\max}|}{z_{1b}(z_3^{em}z_2^m)^2(c_2-c_3z_3^{em})^2}-\right.$$
$$\left.(\beta_3^{em}-\gamma_3^{em})\right\}=0\,,\quad e=1,\ldots,N_e;\ m=1,\ldots,N_b\,, \quad (36)$$

or

$$m_{2c}L^c(z_2^c+d_c')-\lambda_3^c\frac{|P_{\max}^c|}{(P_{dr}^c)^2}\frac{\partial P_{dr}^c}{\partial z_3^c}-(\beta_3^c-\gamma_3^c)=0\,, \quad (37)$$

where

$$\overline{u}^{em}=-\frac{\ell_{2b}z_{1b}w_cL^{em}}{36}\Big(6\overline{w}_A^{em}\sin\theta+6\overline{v}_A^{em}\cos\theta+$$
$$\overline{\theta}_A^{em}L^{em}\cos\theta+6\overline{w}_B^{em}\sin\theta+6\overline{v}_B^{em}\cos\theta-\overline{\theta}_B^{em}L^{em}\cos\theta\Big)\,, \quad (38)$$

$$\overline{u}^c=-\frac{\ell_{2c}z_{1c}w_cL^c}{36}\Big(6\overline{w}_A^c\sin\theta+6\overline{v}_A^c\cos\theta+\overline{\theta}_AL^c\cos\theta+$$
$$6\overline{w}_B^c\sin\theta+6\overline{v}_B^c\cos\theta-\overline{\theta}_B^cL^c\cos\theta\Big)\,, \quad (39)$$

in which $\overline{u}^{em}$ and $\overline{u}^{c}$ are related to the effect of selfweight resulting from the equilibrium equation. Details of their calculation are given in Appendix B. Other details are available in the thesis by Adamu (1995).

For the satisfaction of the compatibility condition of the adjoint system, the adjoint initial displacements may be obtained from (23), (24) or (25). In the notation of Adamu and Karihaloo (1994a) these are:

for beams

$$\left\{\hat{\mathbf{u}}_{fA}^{em}\right\} =$$

$$\left\{\begin{array}{c}\hat{w}_{fA}^{em}\\ \hat{v}_{fA}^{em}\\ \hat{\theta}_{fA}^{em}\end{array}\right\} = \frac{\lambda_1^{em}}{\mu}\frac{\operatorname{sgn}(M_{\max}^{em})}{z_{1b}(z_2^m)^2 z_3^{em}(c_2 - c_3 z_3^{em})}\left\{\begin{array}{c}0\\ -x_m^{em}\\ 1\end{array}\right\} +$$

$$\frac{\lambda_2^{em}}{\mu}\frac{\operatorname{sgn}(V_{\max}^{em})}{\ell_4 z_2^m}\left\{\begin{array}{c}0\\ 1\\ 0\end{array}\right\}, \tag{40}$$

and for columns

$$\left\{\hat{\mathbf{u}}_{fA}^{c}\right\} = \left\{\begin{array}{c}\hat{w}_{fA}^{c}\\ \hat{v}_{fA}^{c}\\ \hat{\theta}_{fA}^{c}\end{array}\right\} = \frac{\lambda_3^c}{\mu}\frac{\operatorname{sgn}(P_{\max}^c)}{P_{dr}^c}\left\{\begin{array}{c}1\\ 0\\ 0\end{array}\right\}. \tag{41}$$

The fixed-end forces corresponding to these adjoint displacements are computed using standard formulae.

3.2 Iterative procedure

The procedure for solving the minimum cost design problem of an RC frame structure using the DCOC-based method involves the analysis of the real and adjoint structures followed by the updating of the design variables and Lagrange multipliers, as appropriate, and finally a check on the convergence based on changes in cost of construction and cross-sectional parameters (design variables) in successive iterations. These three major steps are repeated until the convergence criteria are satisfied. The analysis of both structural systems is carried out using the standard stiffness method. The updating part of the computations is described below.

3.2.1 Updating the design variables

Details of the possible combinations of the behavioural constraints controlling the distribution of the design variables - the depth and the steel ratio for the beams and columns of an RC frame structure are given by Adamu (1995).

For instance, in beams consider elements controlled by deflection and flexural strength constraints. If the depth z_2^m is governed by the deflection constraint, whereas the steel ratio of each of the elements in span m is controlled by the flexural strength constraint, (34) becomes

$$\sum_{e=1}^{N_e}\left\{(z_3^{em}m_{2b} + m_{1b})L^{em} - \mu\left[\frac{Q_1^{em}}{\ell_{2b}(z_2^m)^2} + \frac{3Q_2^{em}}{\ell_{2b}(z_2^m)^4}\right] - \right.$$

$$\left.\lambda_1^{em}\frac{2|M_{\max}^{em}|}{z_{1b}(z_2^m)^3 z_3^{em}(c_2 - c_3 z_3^{em})} + \frac{3\mu}{\ell_{2b}}\overline{u}^{em}\right\} = 0. \tag{42}$$

Following the same notation as Adamu and Karihaloo (1994b), let $\bar{e}$ denote the element in span m with absolute maximum bending moment $|M_{\max}^{\bar{e}m}|$, and assume that the depth of this element satisfies the flexural strength (4), so that

$$z_2^m = \sqrt{\frac{|M_{\max}^{\bar{e}m}|}{z_{1b}z_3^{\bar{e}m}(c_2 - c_3 z_3^{\bar{e}m})}}. \tag{43}$$

Substituting (43) in the third and fifth terms of (42) and extracting the highest power of z_2^m from the fourth term gives

$$(z_2^m)^4 = \frac{\sum_{e=1}^{N_e}\hat{\mu}Q_2^{em}}{\sum_{e=1}^{N_e}\left\{z_3^{em}\overline{m}_{2b}L^{em} + L^{em} - \hat{\mu}(LAML) - \frac{\lambda_1^{em}}{m_{1b}}(LAM)\right\}}, \tag{44}$$

where

$$LAML = \frac{\ell_{2b}z_{1b}z_3^{\bar{e}m}(c_2 - c_3 z_3^{\bar{e}m})Q_1^{em}}{3\ell_{2b}|M_{\max}^{\bar{e}m}|} - \overline{u}^{em}, \tag{45}$$

$$LAM = \frac{2\sqrt{z_{1b}}|M_{\max}^{em}|[z_3^{\bar{e}m}(c_2 - c_3 z_3^{\bar{e}m})]^{\frac{3}{2}}}{z_3^{em}(c_2 - c_3 z_3^{em})(|M_{\max}^{\bar{e}m}|)^{\frac{3}{2}}}, \tag{46}$$

and

$$\hat{\mu} = \frac{3\mu}{(\ell_{2b}m_{1b})}, \quad \overline{m}_{2b} = \frac{m_{2b}}{m_{1b}}.$$

The steel ratio for each element e of span m may be obtained from (4)

$$z_3^{em} = \frac{1}{2}\left\{\left(\frac{c_2}{c_3}\right) - \sqrt{\left(\frac{c_2}{c_3}\right)^2 - \frac{4|M_{\max}^{em}|}{z_{1b}(z_2^m)^2 c_3}}\right\}. \tag{47}$$

The Lagrange multipliers λ_1^{em} for elements satisfying the flexural strength constraint are obtained from (36) as

$$\lambda_1^{em} = \frac{m_{2b}L^{em}z_{1b}(z_2^m)^3(z_3^{em})^2(c_2 - c_3 z_3^{em})^2}{|M_{\max}^{em}|(c_2 - 2c_3 z_3^{em})}, \tag{48}$$

or

$$\frac{\lambda_1^{em}}{m_{1b}} = \frac{\overline{m}_{2b}L^{em}z_{1b}(z_2^m)^3(z_3^{em})^2(c_2 - c_3 z_3^{em})^2}{|M_{\max}^{em}|(c_2 - 2c_3 z_3^{em})}. \tag{49}$$

Note that the Lagrange multiplier λ_1^{em} for each element with active flexural strength constraint is affected by all other elements of this span with uniform z_2^m through the summation in (44).

On the other hand, as the deflection constraint is always active ($\mu > 0$), consider a column in which the design is controlled by deflection and uniaxial bending strength constraints. For this column, the OC (35) and (37) become

$$(z_3^c m_{2c} + m_{1c})L^c - \mu\left[\frac{Q_1^c}{\ell_{2c}(z_2^c)^2} + \frac{3Q_2^c}{\ell_{2c}(z_2^c)^4}\right] -$$

$$\lambda_3^c\frac{|P_{\max}^c|}{(P_{dr}^c)^2}\frac{\partial P_{dr}^c}{\partial z_2^c} + \frac{3\mu}{\ell_{2c}}\overline{u}^c = 0, \tag{50}$$

$$m_{2c}L^c(z_2^c + d_c') - \lambda_3^c\frac{|P_{\max}^c|}{(P_{dr}^c)^2}\frac{\partial P_{dr}^c}{\partial z_3^c} = 0. \tag{51}$$

From (51)

$$\lambda_3^c = \frac{(P_{dr}^c)^2 m_{2c}L^c(z_2^c + d_c')}{|P_{\max}^c|(\partial P_{dr}^c/\partial z_3^c)}. \tag{52}$$

Substituting (52) into (50) and simplfying further, one obtains

$$z_3^c \overline{m}_{2c} L^c + L^c - \hat{\overline{\mu}} \ell_{2m} \left[\frac{\ell_{2c} Q_1^c}{3 \overline{\ell}_{2c} (z_2^c)^2} + \frac{Q_2^c}{(z_2^c)^4} \right] -$$

$$\frac{\overline{m}_{2c} L^c (z_2^c + d_c')(\partial P_{dr}^c / \partial z_2^c)}{(\partial P_{dr}^c / \partial z_3^c)} + \hat{\overline{\mu}} \overline{\ell}_{2m} \overline{u}^c = 0 \,, \tag{53}$$

where

$$\overline{m}_{2c} = \frac{m_{2c}}{m_{1c}} \,,$$

and

$$\overline{\ell}_{2m} = \frac{\ell_{2b} m_{1b}}{\ell_{2c} m_{1c}} \,.$$

Further, the uniaxial bending strength constraint (22) requires that

$$P_{dr}^c = |P_{\max}^c| = \begin{cases} \dfrac{P_{cr}}{1 + \frac{e_d}{e_b}\left(\frac{P_{cr}}{P_{nb}} - 1\right)}, & \text{if compression failure governs} \\ g_1 (z_2^c + d_c') \left[-B + \sqrt{BAC} \right], & \text{if tension failure governs} \\ \dfrac{z_{1c} z_3^c f_{yd} [(z_2^c)^2 - (d_c')^2]}{2e_d + z_2^c - 2d_c'}, & \text{if axial tensile force prevails} \end{cases} \,, \tag{54}$$

where

$$B = \frac{2e_d}{z_2^c + d_c'} - 1 + z_3^c \,,$$

$$BAC = B^2 + 2 z_3^c \left[\overline{g}_2 \left(1 - \frac{2d_c'}{z_2^c + d_c'} \right) + 1 - 0.5 z_3^c \right] \,,$$

where P_{cr}, P_{nb}, e_b, g_1, and $\overline{g}_2$ are defined in (9), (10), (11), (14) and (15), respectively, with e_d the design eccentricity. The two design variables (depth z_2^c and steel ratio z_3^c) for this case are computed from (53) and (54), using an iterative procedure.

In a similar manner, the other possible combinations governing the design are considered. Again the Lagrange multipliers corresponding to the strength-controlled elements in a span will be influenced by all the elements of this span because of uniform depth z_2^m.

Finally, designating the depths z_2^m computed from the deflection and flexural strength constraints, the shear and flexural strength constraints, the flexural strength and upper bound on the steel ratio, and a lower bound on depth combinations as z_{2d}^m, z_{2v}^m, z_{2m}^m and $z_{2\ell b}$, respectively, the resizing rule for beam depth is

$$z_2^m = \max\{ \; z_{2d}^m, \;\; z_{2v}^m, \;\; z_{2m}^m, \;\; z_{2\ell b} \; \} \,, \tag{55}$$

whereas that of the steel ratio z_3^{em} is

$$z_3^{em} = \begin{cases} z_{3m}^{em}, \text{ if } z_{3\ell b} < z_{3m}^{em} < z_{3ub} \\ z_{3\ell b}, \text{ if } z_{3m}^{em} \le z_{3\ell b} \\ z_{3ub}, \text{ if } z_{3m}^{em} \ge z_{3ub} \end{cases} \,, \tag{56}$$

where z_{3m}^{em} is computed using (47).

Likewise, designating z_2^c computed from the deflection and flexural strength constraints, the deflection constraint and lower bound on the steel ratio, the combined axial and bending strength with an upper bound on the steel ratio and a lower bound on both combinations of variables as z_{2a}^c, z_{2d}^c, z_{2au}^c and $z_{2\ell c}$, respectively, the resizing rule for column depth is

$$z_2^c = \max\{ \; z_{2a}^c, \;\; z_{2d}^c, \;\; z_{2au}^c, \;\; z_{2\ell c} \; \} \tag{57}$$

and the gross steel ratio z_3^c is

$$z_3^c = \begin{cases} z_{3ad}^c, \text{ if } z_{3\ell c} < z_{3ad}^c < z_{3uc} \text{ and } z_2^c = z_{2ad}^c > z_{2\ell c} \\ z_{3\ell a}^c, \text{ if } z_{3\ell c} \le z_{3\ell a}^c \le z_{3uc} \text{ and } z_2^c = z_{2\ell c} \\ z_{3uc}, \text{ if } z_2^c = z_{2\ell u}^c \\ z_{3\ell c}, \text{ if } z_2^c = z_{2d\ell}^c \text{ or } z_2^c = z_{2\ell c} \end{cases} \,, \tag{58}$$

where z_{3ad}^c and $z_{3\ell a}^c$ are computed using (53) and (54), respectively, with the latter computed for $z_2^c = z_{2\ell c}$. If the former governs the design, the latter is omitted. Note that for each of the cases delineated above the two basic design variables must be computed iteratively.

3.2.2 Computation of Lagrange multipliers and fixed-end forces

As discussed above in the updating part, suitable expressions of the Lagrange multipliers are needed for strength controlled elements in order to calculate the fixed-end forces resulting from the prestrains.

For instance, in beams consider elements whose depth is controlled by the deflection constraint and steel ratio by the flexural strength constraint. Equation (48) yields

$$\frac{\lambda_1^{em}}{\mu} = \frac{3 \overline{m}_{2b} L^{em} z_{1b} (z_2^m)^3 (z_3^{em})^2 (c_2 - c_3 z_3^{em})^2}{\ell_{2b} \hat{\overline{\mu}} |M_{\max}^{em}| (c_2 - 2c_3 z_3^{em})} \,, \tag{59}$$

so that the fixed-end forces due to the prestrains are

$$\left\{ \overline{\mathbf{F}}_F^{em} \right\} = - \frac{\lambda_1^{em}}{\mu} \frac{\ell_{2b} z_2^m \operatorname{sgn}(M_{\max}^{em})}{z_{1b} z_3^{em} (c_2 - c_3 z_3^{em})(L^{em})^3} \times \begin{Bmatrix} 0 \\ -12 x_m^{em} + 6L^{em} \\ -6L^{em} x_m^{em} + 4(L^{em})^2 \\ 0 \\ 12 x_m^{em} - 6L^{em} \\ -6L^{em} x_m^{em} + 2(L^{em})^2 \end{Bmatrix} . \tag{60}$$

Note that λ_1^{em} is influenced by all elements of the span m through the summation involved in the determination of z_2^m [see (44)].

Likewise for columns whose design is controlled by deflection and a uniaxial bending strength constraints combination, (52) yields

$$\frac{\lambda_3^c}{\mu} = \frac{3 (P_{dr}^c)^2 \overline{m}_{2c} L^c (z_2^c + d_c')}{\ell_{2c} \hat{\overline{\mu}} \overline{\ell}_{2m} |P_{\max}^c| (\partial P_{dr}^c / \partial z_3^c)} \,, \tag{61}$$

so that the fixed-end forces due to the prestrains are

$$\left\{ \overline{\mathbf{F}}_F^c \right\} = -[\mathbf{s}^c]\{\hat{\overline{\mathbf{u}}}^c\} = - \frac{\lambda_3^c}{\mu} \frac{\overline{\ell}_{2c} z_2^c \operatorname{sgn}(P_{\max}^c)}{L^c P_{dr}^c} \begin{Bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{Bmatrix} . \tag{62}$$

The Lagrange multiplier $\hat{\overline{\mu}}$ is computed from the deflection constraint. For beams, let us denote the spans whose design is controlled by the deflection constraint, i.e. by (44) as

N_m^D and the remaining spans by N_m^L so that $N_m^L + N_m^D = N_b$. The contribution from beams to the deflection is

$$u_m^b = \sum_{N_m^D} \sum_{e=1}^{N_e} \left[\frac{Q_1^{em}}{\bar{\ell}_{2b}(\hat{\bar{\mu}} \bar{z}_{2m})^{\frac{1}{2}}} + \frac{Q_2^{em}}{\ell_{2b}(\hat{\bar{\mu}} \bar{z}_{2m})^{\frac{3}{4}}} \right] +$$

$$\sum_{N_m^L} \sum_{e=1}^{N_e} \left[\frac{Q_1^{em}}{\bar{\ell}_{2b} z_2^m} + \frac{Q_2^{c}}{\ell_{2b}(z_2^m)^3} \right] , \tag{63}$$

where

$$\bar{z}_{2m} = \frac{\sum_{e=1}^{N_e} Q_2^{em}}{\sum_{e=1}^{N_e} \left\{ z_3^{em} \overline{m}_{2b} L^{em} + L^{em} - \hat{\bar{\mu}}(LAML) - \frac{\lambda_1^{em}}{m_{1b}}(LAM) \right\}} , \tag{64}$$

with $LAML$ and LAM given by (45) and (46), respectively.

As regards the columns, the design is said to be controlled by the deflection constraint, if the pair of design variables is obtained from (53) and (54), or from the deflection constraint and lower bound on the steel ratio combination. To simplify the computation of the Lagrange multiplier $\hat{\bar{\mu}}$, the depths given by these two sets of combinations may be written as

$$(z_2^c)^4 = \frac{\hat{\bar{\mu}} \bar{\ell}_{2m} \left[\frac{\ell_{2c} Q_1^c (z_2^c)^2}{3 \bar{\ell}_{2c}} + Q_2^c \right]}{\left[(z_3^c \overline{m}_{2c} + 1) L^c + \hat{\bar{\mu}} \bar{\ell}_{2m} \overline{u}^c - \frac{\overline{m}_{2c} L^c (z_2^c + d_c') (\partial P_{dr}^c / \partial z_2^c)}{\partial P_{dr}^c / \partial z_3^c)} \right]} , \tag{65}$$

and

$$(z_2^c)^4 = \frac{\hat{\bar{\mu}} \bar{\ell}_{2m} \left[\frac{\ell_{2c} Q_1^c (z_2^c)^2}{3 \bar{\ell}_{2c}} + Q_2^c \right]}{\left[(z_3^c \overline{m}_{2c} + 1) L^c + \hat{\bar{\mu}} \bar{\ell}_{2m} \overline{u}^c \right]} , \tag{66}$$

respectively. Denoting the columns whose design is controlled by the deflection constraint in combination with uniaxial bending strength by N_{e1}^D and those in combination with the lower bound on steel ratio by N_{e2}^D, and the remaining by N_e^L, ($N_c = N_{e1}^D + N_{e2}^D + N_e^L$), the contribution at the specified degree of freedom from the columns is

$$u_m^c = \sum_{N_{e1}^D} \left[\frac{Q_1^c}{\bar{\ell}_{2c}(\hat{\bar{\mu}} \bar{z}_{2m1})^{\frac{1}{2}}} + \frac{Q_2^c}{\ell_{2c}(\hat{\bar{\mu}} \bar{z}_{2m1})^{\frac{3}{4}}} \right] +$$

$$\sum_{N_{e2}^D} \left[\frac{Q_1^c}{\bar{\ell}_{2c}(\hat{\bar{\mu}} \bar{z}_{2m2})^{\frac{1}{2}}} + \frac{Q_2^c}{\ell_{2c}(\hat{\bar{\mu}} \bar{z}_{2m2})^{\frac{3}{4}}} \right] +$$

$$\sum_{N_e^L} \left[\frac{Q_1^c}{\bar{\ell}_{2c} z_2^c} + \frac{Q_2^c}{\ell_{2c}(z_2^c)^3} \right] , \tag{67}$$

where

$$\bar{z}_{2m1} = \frac{\bar{\ell}_{2m} \left[\frac{\ell_{2c} Q_1^c (z_2^c)^2}{3 \bar{\ell}_{2c}} + Q_2^c \right]}{\left[(z_3^c \overline{m}_{2c} + 1) L^c + \hat{\bar{\mu}} \bar{\ell}_{2m} \overline{u}^c - \frac{\overline{m}_{2c} L^c (z_2^c + d_c') (\partial P_{dr}^c / \partial z_2^c)}{(\partial P_{dr}^c / \partial z_3^c)} \right]} , \tag{68}$$

$$\bar{z}_{2m2} = \frac{\bar{\ell}_{2m} \left[\frac{\ell_{2c} Q_1^c (z_2^c)^2}{3 \bar{\ell}_{2c}} + Q_2^c \right]}{\left[(z_3^c \overline{m}_{2c} + 1) L^c + \hat{\bar{\mu}} \bar{\ell}_{2m} \overline{u}^c \right]} . \tag{69}$$

Finally, the deflection constraint (3) may be rewritten as

$$\sum_{N_m^D} \sum_{e=1}^{N_e} \left[\frac{Q_1^{em}}{\bar{\ell}_{2b}(\hat{\bar{\mu}} \bar{z}_{2m})^{\frac{1}{2}}} + \frac{Q_2^{em}}{\ell_{2b}(\hat{\bar{\mu}} \bar{z}_{2m})^{\frac{3}{4}}} \right] +$$

$$\sum_{N_{e1}^D} \left[\frac{Q_1^c}{\bar{\ell}_{2c}(\hat{\bar{\mu}} \bar{z}_{2m1})^{\frac{1}{2}}} + \frac{Q_2^c}{\ell_{2c}(\hat{\bar{\mu}} \bar{z}_{2m1})^{\frac{3}{4}}} \right] +$$

$$\sum_{N_{e2}^D} \left[\frac{Q_1^c}{\bar{\ell}_{2c}(\hat{\bar{\mu}} \bar{z}_{2m2})^{\frac{1}{2}}} + \frac{Q_2^c}{\ell_{2c}(\hat{\bar{\mu}} \bar{z}_{2m2})^{\frac{3}{4}}} \right] ,$$

$$\sum_{N_m^L} \sum_{e=1}^{N_e} \left[\frac{Q_1^{em}}{\bar{\ell}_{2b} z_2^m} + \frac{Q_2^{em}}{\ell_{2b}(z_2^m)^3} \right] + \sum_{N_e^L} \left[\frac{Q_1^c}{\bar{\ell}_{2c} z_2^c} + \frac{Q_2^c}{\ell_{2c}(z_2^c)^3} \right] -$$

$$\Delta_{\rm al} = 0 , \tag{70}$$

where $\hat{\bar{\mu}}$ is computed from (70) using an iterative procedure. Details of the computational procedure including the evaluation of the approximate $\hat{\bar{\mu}}_{\rm int}$, are described by Adamu (1995).

The computational aspects of the iterative procedure for obtaining the optimum design of an RC frame structure using DCOC are summarized below.

1. Analysis. The procedure commences with an analysis of the real and adjoint systems for a known material distribution. The selfweight effect in the real system and the prestrain effect in the adjoint system are included in this analysis. During the first iteration, the design values are arbitrarily chosen. In the subsequent iterations they are supplied from the design part of the procedure. Any prestrain effect in the analysis of the adjoint system is disregarded during the first iteration because the controlling constraints are not yet identified. However, in the subsequent iterations, the adjoint load vector includes, in addition to the virtual load vector, the equivalent nodal loads caused by adjoint initial displacements of a beam element or column whose design is controlled by the strength constraints.

2. Updating of the design variables and Lagrange multipliers. The design variables are updated in accordance with (55) and (56) for beam elements, and (57) and (58) for columns. The Lagrange multipliers λ_j^i ($j = 1, 2, 3$ and i stands for em or c) are updated according to the type of the strength constraint controlling the design, as discussed above. The Lagrange multiplier $\hat{\bar{\mu}}$ is evaluated using (70) in each iteration.

3. Check change in the design domain. Step (2) alone is repeated if there is a change in the beam element or column domain controlled by displacement and strength constraints.

4. Convergence criteria. Convergence on the change in the cost of construction and design variables is checked using

$$\frac{|\phi_{\rm new} - \phi_{\rm old}|}{\phi_{\rm new}} \le \varepsilon_{\rm cost} , \tag{71}$$

and

$$\frac{|z^i_{k,\text{new}} - z_{k,\text{old}}|}{z^i_{k,\text{new}}} \leq \varepsilon_{\text{cros}}\,, \tag{72}$$

where $\varepsilon_{\text{cost}}$ and $\varepsilon_{\text{cros}}$ are set equal to 10^{-4} and 10^{-3}, respectively.

5. Steps 1 through 4 are repeated until the convergence criteria (step 4) are simultaneously satisfied.

3.3 Numerical example

To illustrate the above procedure, several examples are considered. The material data and design constants used are as follows.

1. *Material*
   concrete
   $f_c = 25$ MPa,
   $E_c = 26,000$ MPa.
   $w_c = 24.5$ kN/m^3
   reinforcing steel
   $f_y = 400$ MPa,
   $E_s = 200,000$ MPa
2. *Relative cost* (with $c_c = 130\$/\text{m}^3$, $c_s = 9590\$/\text{m}^3$ and $c_f = 55\$/\text{m}^3$)
   $c_{sc} = 66$
   $c_{fc} = 0.42$.

3.3.1 Example 1. A portal frame

It is required to obtain the minimum cost design of the portal frame shown in Fig. 2a. The factored design load due to selfweight is included during the design process. The flexural and shear strength constraints (with the shear only against web crushing) for beams, and the uniaxial bending strength constraint for columns have been considered in accordance with the CEB/FIP model code requirements. The maximum deflection at the midpoint of the beam due to the design loads must be less than or equal to 20 mm (i.e. the allowable span to deflection ratio is 300). The additional data regarding beam and column dimensions are: $z_{2\ell b} = z_{2\ell c} = 200$ mm, $z_{3\ell b} = 0.0015$, $z_{3ub} = 0.2605 f_c/f_y$, $z_{3\ell c} = 0.01$, $z_{3uc} = 0.020$ [a reduced value has been adopted, as recommended in the code, and by Ferguson (1979) to provide adequate spacing between bars], $d'_b = 50$ mm and $d'_c = 40$ mm.

The width of the beam was prescribed to be 250 mm, and that of the columns was 200 mm. The initial steel ratios for the beam and columns were chosen to be 0.01 and 0.012, respectively. The optimum design is summarized in Table 1, and the *AF*, *SF* and *BM* diagrams are shown in Figs. 2b, c and d, respectively. The elevation and transverse sections of the frame indicating the optimum distribution of the design variables are shown in Fig. 3.

The optimum design of the beam was governed by a combination of deflection and flexural strength constraints, whereas that of the left column was controlled by a lower bound on the depth and a uniaxial bending strength constraint on the steel ratio, and that of the right column was controlled by a uniaxial bending strength constraint on the depth and an upper bound on the steel ratio. The mode of failure marked by T in Table 1 is to show that the design is governed by tension failure, i.e. failure is initiated by the yielding of tensile steel followed by the crushing of concrete.

The initial design cost of 11.2252 converged to an optimum cost of 9.2474. These values must be multiplied by c_c to obtain the actual cost of construction. The CPU time used on a micro VAX/VMS was 3.86 seconds. The Lagrange multiplier $\bar{\bar{\mu}}$ converged to 0.5161. The *AF*, *SF* and *BM* diagrams in Fig. 2 are for the optimum frame and include the effect of the factored selfweight loads shown in Table 1.

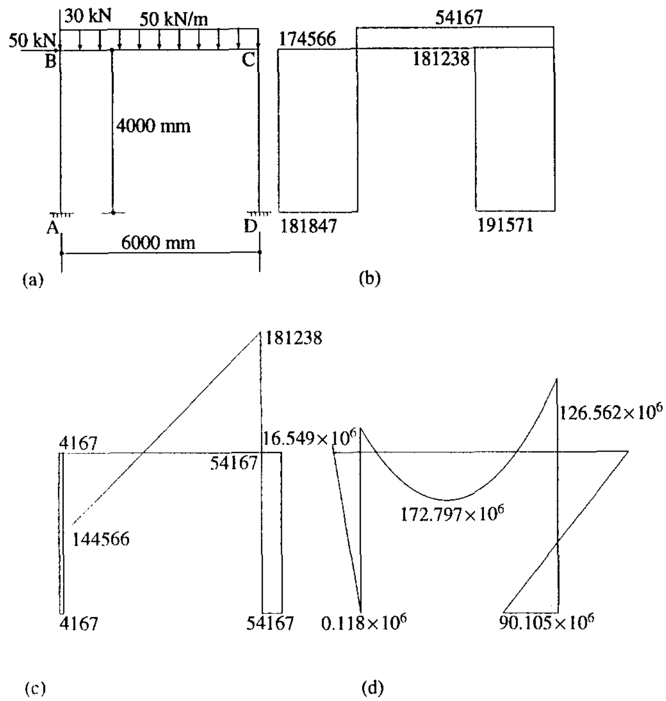

Fig. 2. A portal frame: (a) geometry and loading; (b) *AFD* (N); (c) *SFD* (N); (d) *BMD* (N-mm) for optimum design

Table 1. Minimum cost design of a portal frame

Column	Initial z_2^c mm	Optimum z_2^c mm	Gross steel area mm^2	Mode of failure	Selfweight load kN/m
AB	500	235	1101	T	1.82
DC	500	351	1562	T	2.58

Beam	Initial z_2^c mm	Optimum z_2^c mm	A_{s1} mm^2	A_{s2} mm^2	A_{s3} mm^2	A_{s4} mm^2	Selfweight load kN/m
BC	500	470	944	1209	1209	849	4.30

3.3.2 Example 2. One-bay, two-storey frame

The one-bay, two-storey frame shown in Fig. 4a is subjected to the factored design loads in addition to selfweight. The strength constraints are the same as for Example 1. The maximum deflection at the midpoint of the top beam must be less than or equal to 10 mm. The bounds on design variables and other pertinent data remain the same as for Example 1, except that $z_{2c\ell} = 210$ mm, $z_{1b} = z_{1c} = 250$ mm.

The initial steel ratio for the beams was chosen to be 0.01 and that for the columns was 0.015. The results are summarized in Table 2. The end forces of each member and the midspan bending moment of the beams, as obtained from the analysis part of the solution are given in Table 3 in which $AM1$, $AM2$, and $AM3$ represent the axial force, shear force and bending moment at the left end for a beam or the lower end for a column, and $AM4$, $AM5$ and $AM6$ at the right or upper end, respectively. The midspan bending moment for each beam is given in the last column of Table 3. The (+) and

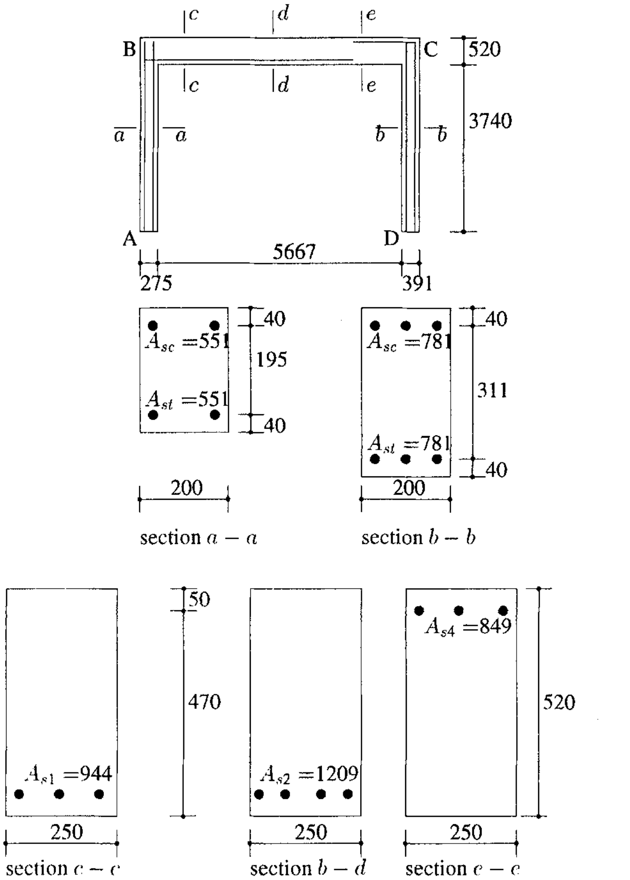

Fig. 3. Elevation and cross-sections of minimum cost RC portal frame

Table 2. Minimum cost design of a one-bay, two-storey frame

Column	Initial z_2^c mm	Optimum z_2^c mm	Gross steel area mm^2	Mode of failure	Selfweight load kN/m
AC	300	210	625	C	2.07
BD	300	210	1082	T	2.07
CE	300	306	1131	T	2.07
DF	300	325	1115	T	2.07

Beam	Initial z_2^c mm	Optimum z_2^c mm	A_{s1} mm^2	A_{s2} mm^2	A_{s3} mm^2	A_{s4} mm^2	Selfweight load kN/m
CD	400	325	964	936	936	1322	3.10
EF	400	366	544	558	558	652	3.44

(−) signs of the end forces correspond to the sign convention adopted in Fig. 1. The AF, SF and BM diagrams are shown in Figs. 4b, c and d, respectively with the cross-sections of the members of the optimum frame shown in Fig. 5.

The design of the top beam is governed by the combination of deflection and flexural strength constraints, whereas that of the lower beam is governed by the flexural strength constraint and upper bound on the steel ratio, or by the flexural strength constraint alone. The design of the two top columns is controlled by the combination of deflection and uniaxial bending strength constraints, whereas that of the two lower columns is governed by the the lower bound on the depth and the uniaxial bending strength constraint on the steel ratio.

The initial design cost of 15.2302 converged to the optimum cost of 13.8066. These values must be multiplied by c_C to obtain the actual cost of construction. The CPU time used on a micro VAX/VMS was 5.05 seconds. The Lagrange multiplier $\bar{\bar{\mu}}$ converged to 0.9029.

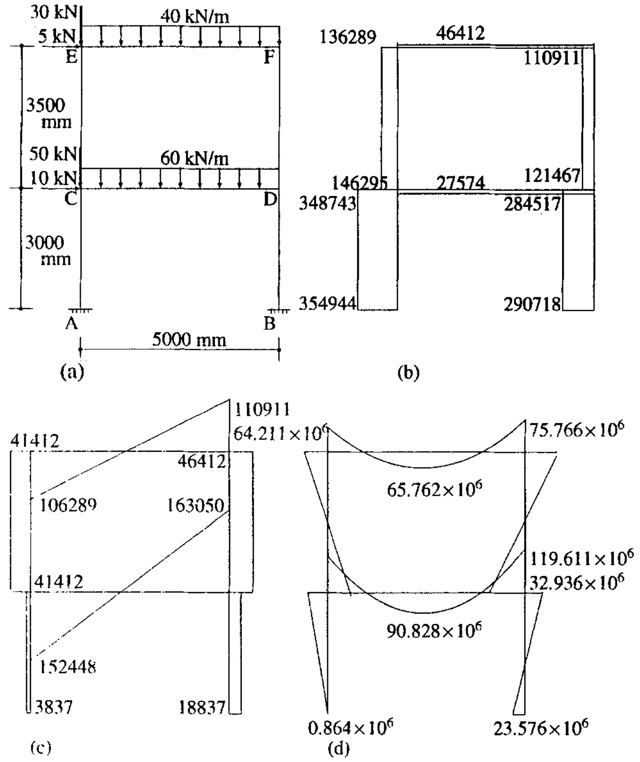

Fig. 4. A one-bay, two-storey frame: (a) geometry and loading; (b) AFD (N); (c) SFD (N); (d) BMD (N-mm) for optimum design

3.4 Regular multibay and multistorey frames

As indicated in the previous sections, the formwork cost constitutes the major cost of RC construction. In RC multibay and multistorey frames, the beam formwork is re-used from floor to floor, if column sizes are kept constant. In other words, it is common practice to keep the column size constant over each floor or several floors and to make adjustment for the differing loads in each storey with reinforcing steel. Moreover, this facilitates supervision of construction. For the same reasons, it is normal to keep the beam depth and width uniform in a given storey or several storeys. These observations will be exploited in the formulation of the optimization problem, in which it will be assumed that all beams at a given floor level have the same cross-section and all columns in a given storey have the same cross-section.

Let us designate the number of storeys by N_s, so that the total number of beams is

$$N_b = \sum_{k=1}^{N_s} N_b^k \,, \tag{73}$$

where N_b^k is the number of bays in the floor, and the total number of columns is

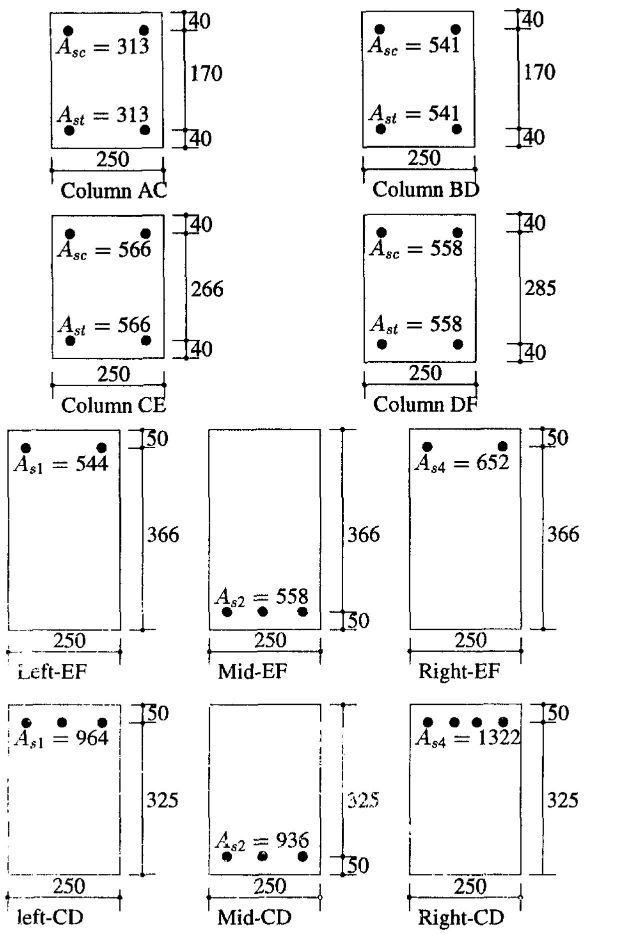

Fig. 5. Optimum distribution of the design variables at various sections of the one-bay, two-storey frame of Fig. 4

$$N_c = \sum_{k=1}^{N_s} \left(N_b^k + 1\right) . \tag{74}$$

In this section, the effective depth z_{2b}^k, $(k = 1, \ldots, N_s)$, of the beam group in each storey is the unknown design variable. However, the steel ratio is allowed to vary along the span of each beam in the storey. If a beam is discretized into N_e beam elements, the steel ratio may be designated by z_{3b}^{emk} $(e = 1, \ldots, N_e;\ m = 1, \ldots, N_b^k,\ k = 1, \ldots, N_s)$.

Table 3. End forces (N) and midspan moment (N-mm) of the members of the one-bay, two-storey frame

Member	$AM1$	$AM2$	$AM3$	$AM4$	$AM5$	$AM6$	MSM
AC	354944	-3837	864083	-348743	3837	-12376040	
BD	290718	18837	23576153	-284517	-18837	32935803	
CD	-27574	152448	93106316	27574	163050	-119610927	90827691
CE	146295	-41412	-80730276	-136289	41412	-64210540	
DF	121467	46412	86675124	-110911	-46412	75765692	
EF	46412	106289	64210541	-46412	110911	-75765692	65762138

The effective depth z_{2c}^k $(k = 1, \ldots, N_s)$, of the column group in each floor is also an unknown design variable, whereas the gross steel ratio z_{3c}^{nk}, $(n = 1, \ldots, N_b^k + 1;\ k = 1, \ldots, N_s)$ in each of the columns of the given floor is permitted to vary freely.

Hence, an RC frame structure of N_s storeys with N_b^k bays in each storey will involve a total of $2N_s + \sum_{k=1}^{N_s} \left[N_b^k(N_e + 1) + 1\right]$ unknown design variables.

The objective function for multibay and multistorey RC frame may then be written as

$$\phi = \sum_{k=1}^{N_s} \sum_{m=1}^{N_b^k} \sum_{e=1}^{N_e} \left[z_{2b}^k \left(z_{3b}^{emk} m_{2b} + m_{1b}\right) + \bar{c}_{1b}\right] L^{emk} + \sum_{k=1}^{N_s} \sum_{n=1}^{N_b^k+1} \left[z_{2c}^k \left(z_{3c}^{nk} m_{2c} + m_{1c}\right) + z_{3c}^{nk} m_{2c} d_c' + \bar{c}_{2c}\right] L^{nk} , \tag{75}$$

where m_{1b}, m_{2b}, $\bar{c}_{1b}$, m_{1c}, m_{2c}, $\bar{c}_{2c}$ remain the same as given in (1).

The listing of the design constraints and problem formulation also remain unchanged from those in the previous section, with superscripts or subscripts appropriately modified to reflect the new groupings of design variables.

The optimality criteria (34)-(37) for the variation of the depth z_{2b}^k or z_{2c}^k in each floor and the steel ratio z_{3b}^{emk} or z_{3c}^{nk} in each beam element or column become

$$\sum_{m=1}^{N_b^k} \sum_{e=1}^{N_e} \left\{ \left[z_{3b}^{emk} m_{2b} + m_{1b}\right] L^{emk} - \mu \left[\frac{Q_1^{emk}}{\bar{\ell}_{2b}(z_{2b}^k)^2} + \frac{3Q_2^{emk}}{\ell_{2b}(z_{2b}^k)^4}\right] - \lambda_1^{emk} \frac{2|M_{\max}^{emk}|}{z_{1b}(z_{2b}^k)^3 z_{3b}^{emk}(c_2 - c_3 z_{3b}^{emk})} - \lambda_2^{emk} \frac{|V_{\max}^{emk}|}{\ell_4(z_{2b})^2} + \frac{3\mu}{\ell_{2b}} \bar{u}^{emk} \right\} - \left(\beta_2^{emk} - \gamma_2^{emk}\right) = 0 ,$$

$$k = 1, \ldots, N_s , \tag{76}$$

or

$$\sum_{n=1}^{N_b^k+1} \left\{ \left(z_{3c}^{nk} m_{2c} + m_{1c}\right) L^{nk} - \mu \left[\frac{Q_1^{nk}}{\bar{\ell}_{2c}(z_{2c}^k)^2} + \frac{3Q_2^{nk}}{\ell_{2c}(z_{2c}^k)^4}\right] - \lambda_3^{nk} \frac{|P_{\max}^{nk}|}{(P_{dr}^{nk})^2} \frac{\partial P_{dr}^{nk}}{\partial z_{2c}^k} + \frac{3\mu}{\ell_{2c}} \bar{u}^{nk} \right\} - \left(\beta_2^{nk} - \gamma_2^{nk}\right) = 0 ,$$

$$k = 1, \ldots, N_s , \tag{77}$$

and

$$\left\{ z_{2b}^k m_{2b} L^{emk} - \lambda_1^{emk} \frac{(c_2 - 2c_3 z_{3b}^{emk})|M_{\max}^{emk}|}{z_{1b}(z_{3b}^{emk} z_{2b}^k)^2 (c_2 - c_3 z_{3b}^{emk})^2} - \left(\beta_3^{emk} - \gamma_3^{emk}\right) \right\} = 0 , \tag{78}$$

or

$$\left\{ m_{2c} L^{nk} (z_{2c}^k + d_c') - \lambda_3^{nk} \frac{|P_{\max}^{nk}|}{(P_{dr}^{nk})^2} \frac{\partial P_{dr}^{nk}}{\partial z_{3c}^{nk}} - \left(\beta_3^{nk} - \gamma_3^{nk}\right) \right\} = 0 , \tag{79}$$

in which $\overline{u}^{emk}$ and $\overline{u}^{nk}$ are identical to $\overline{u}^{em}$ and $\overline{u}^{c}$ of (38) and (39), respectively, with em replaced by emk and c replaced by nk. Note that β_2^{emk} and γ_2^{emk} in (76) are independent of em because the depth z_{2b}^k does not vary with e (element) or m (span) at a given floor level (k). Likewise, β_2^{nk} and γ_2^{nk} in (77) are independent of n.

As in the previous section, the procedure for updating commences with the development of explicit expressions for the evaluation of design variables as controlled by various design constraint combinations. Some of the basic changes are listed below.

If the design of the beam is governed by deflection and flexural strength constraints, the expression of the depth for the beam group in a storey equivalent to (44) is given by

$$(z_{2b}^k)^4 = \frac{\sum_{m=1}^{N_b^k}\sum_{e=1}^{N_e} \hat{\bar{\mu}} Q_2^{emk}}{\sum_{m=1}^{N_b^k}\sum_{e=1}^{N_e}\left\{(z_{3b}^{emk}\overline{m}_{2b}+1)L^{emk} - \hat{\bar{\mu}}(LAML) - \frac{\lambda_1^{emk}}{m_{1b}}(LAM)\right\}}, \tag{80}$$

$$LAML = \frac{\ell_{2b} z_{1b} \bar{z}_{3b}^k (c_2 - c_3 \bar{z}_{3b}^k) Q_1^{emk}}{3\bar{\ell}_{2b}|M_{\max}^{\bar{k}}|} - \overline{u}^{emk}, \tag{81}$$

$$LAM = \frac{2\sqrt{z_{1b}}|M_{\max}^{emk}|\left[\bar{z}_{3b}^k(c_2 - c_3\bar{z}_{3b}^k)\right]^{\frac{3}{2}}}{z_3^{emk}(c_2 - c_3 z_{3b}^{emk})(|M_{\max}^{\bar{k}}|)^{\frac{3}{2}}}, \tag{82}$$

where $M_{\max}^{\bar{k}}$ is the maximum bending moment among the beams in the considered storey and $\bar{z}_{3b}^k$ is the corresponding desired steel ratio. The steel ratio z_{3b}^{emk}, the Lagrange multipliers λ_1^{emk} and $\frac{\lambda_1^{emk}}{m_{1b}}$ are computed using (47), (48) and (49), respectively, with an appropriate change in notation. If the steel ratio in any of the elements of a beam group is smaller than the lower bound $z_{3b\ell}$, the $\frac{\lambda_1^{emk}}{m_{1b}}$ in (80) is set equal to zero during the computation of z_{2b}^k. Otherwise, the evaluation of the design variable when the other constraint combinations govern remains the same, as given in Section 3.2.

Regarding a column group in a floor, if the deflection and uniaxial bending strength constraints govern the design, then the depth is given

$$z_{2c}^k = \sqrt{\frac{-\overline{zb} \pm \sqrt{(\overline{zb})^2 - 4(\overline{za})(\overline{zc})}}{2(\overline{za})}}, \tag{83}$$

in which

$$\overline{za} = \sum_{n=1}^{N_b^k+1}\left\{\left(z_{3c}^{nk}\overline{m}_{2c}+1\right)L^{nk} + \hat{\bar{\mu}}\ell_{2m}\overline{u}^{nk} + \overline{za1}\right\}, \tag{84}$$

$$\overline{za1} = -\frac{\overline{m}_{2c}L^{nk}(z_{2c}^k + d_c')\partial P_{dr}^{nk}/\partial z_{2c}^k}{\partial P_{dr}^{nk}/\partial z_{3b}^{nk}}, \tag{85}$$

$$\overline{zb} = \sum_{n=1}^{N_b^k+1} -\hat{\bar{\mu}}\ell_{2m}\frac{\ell_{2c}Q_1^{nk}}{3\bar{\ell}_{2c}}, \tag{86}$$

$$\overline{zc} = \sum_{n=1}^{N_b^k+1} -\hat{\bar{\mu}}\bar{\ell}_{2m}Q_2^{nk}. \tag{87}$$

The steel ratio for each column in this floor is evaluated using (54). If the steel ratio for any column is smaller than $z_{3c\ell}$, it assumes its lower bound, and during the computation of z_{2c}^k from (83), $\overline{za1}$ becomes zero, because the uniaxial bending strength constraint is not active for that column.

If the depth z_{2c}^k using (83) is smaller than its lower bound, the steel ratio is assigned its upper bound and the depth is computed for one or more columns with critical load to satisfy the uniaxial bending strength constraint. The depth of the column group in this case is the largest among those computed for the respective critical forces. The steel ratio for the remaining columns in this group is determined as per the uniaxial bending strength requirement of each column. If the steel ratio for any of the columns in the same group is smaller than the lower bound, it is set equal to its lower bound.

The computation of the Lagrange multipliers and the corresponding fixed end actions for analysis of the adjoint system is identical to the one given in Section 3.2, but with one significant difference. The Lagrange multipliers of the strength-controlled elements at a given floor level are now influenced by all the elements at this level because the depth of all beams (all columns) is the same.

The Lagrange multiplier $\hat{\bar{\mu}}$ is computed from the deflection constraint in the same manner as the previous section

$$\hat{\bar{\mu}}^{\frac{3}{4}} = \frac{\sum_{N_m^D} Q_{ba}^k + \sum_{N_{e1}^D} Q_{ca}^k}{\Delta_{\mathrm{al}} - \left[\sum_{N_m^L} Q_{bp}^k + \sum_{N_e^L} Q_{cp}^k\right]}, \tag{88}$$

in which

$$Q_{ba}^k = \sum_{m=1}^{N_b^k}\sum_{e=1}^{N_e}\left(\frac{\hat{\bar{\mu}}^{\frac{1}{4}}Q_1^{emk}}{\bar{\ell}_{2b}\bar{z}_{2m}^{\frac{1}{2}}} + \frac{Q_2^{emk}}{\ell_{2b}\bar{z}_{2m}^{\frac{3}{4}}}\right),$$

$$Q_{ca}^k = \sum_{n=1}^{N_b^k+1}\left(\frac{\hat{\bar{\mu}}^{\frac{1}{4}}Q_1^{nk}}{\bar{\ell}_{2c}\bar{z}_{2m1}^{\frac{1}{2}}} + \frac{Q_2^{nk}}{\ell_{2c}\bar{z}_{2m1}^{\frac{3}{4}}}\right),$$

$$Q_{bp}^k = \sum_{m=1}^{N_b^k}\sum_{e=1}^{N_e}\left[\frac{Q_1^{emk}}{\bar{\ell}_{2b}(z_{2b}^k)} + \frac{Q_2^{emk}}{\ell_{2b}(z_{2b})^3}\right],$$

$$Q_{cp}^k = \sum_{n=1}^{N_b^k+1}\left[\frac{Q_1^{nk}}{\bar{\ell}_{2c}(z_{2c}^k)} + \frac{Q_2^{nk}}{\ell_{2c}(z_{2c}^k)^3}\right],$$

$$\bar{z}_{2m} = \frac{\sum_{m=1}^{N_b^k}\sum_{e=1}^{N_e} Q_2^{emk}}{\sum_{m=1}^{N_b^k}\sum_{e=1}^{N_e}\left\{(z_{3b}^{emk}\overline{m}_{2b}+1)L^{emk} - \hat{\bar{\mu}}(LAML) - \frac{\lambda_1^{emk}}{m_{1b}}(LAM)\right\}},$$

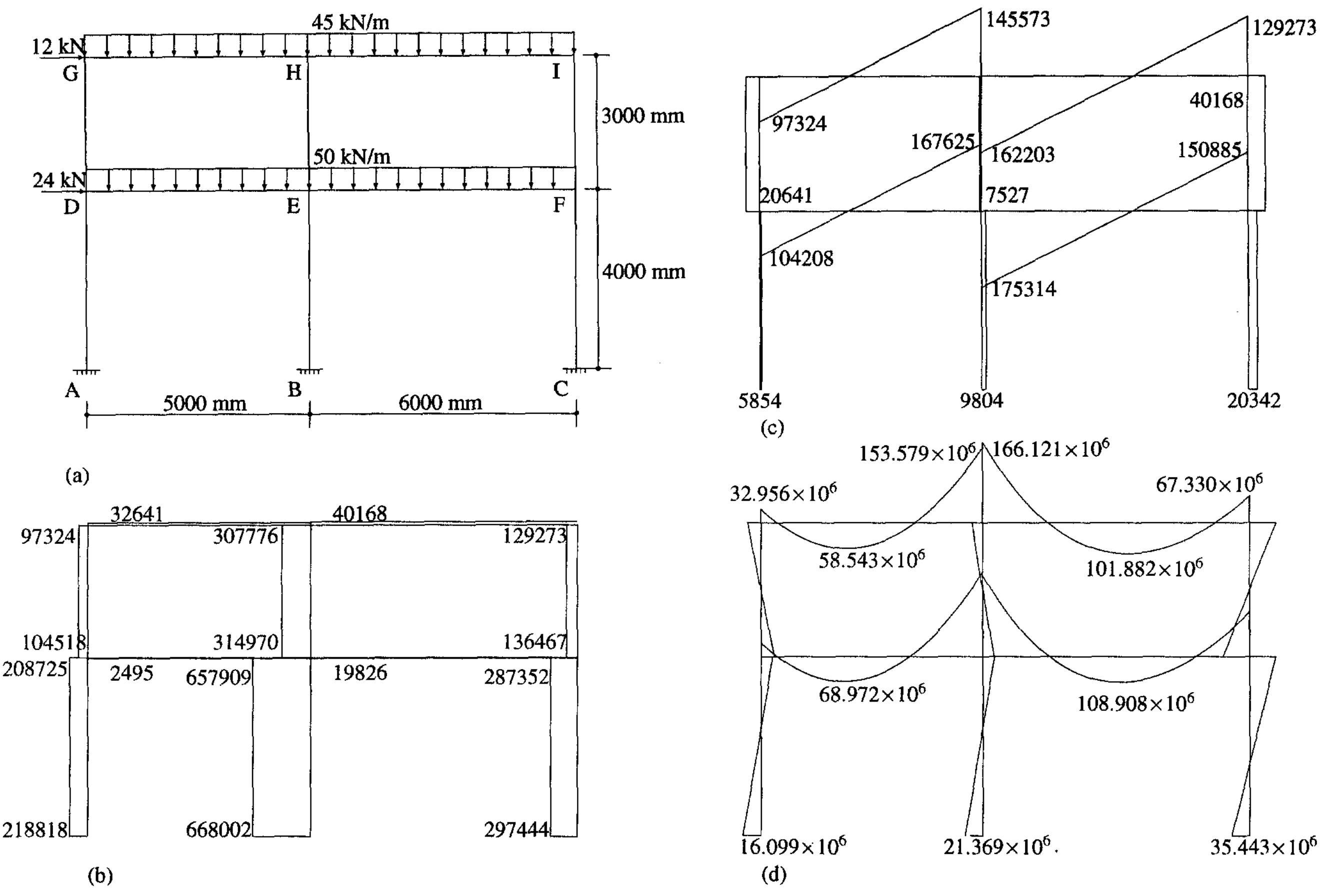

Fig. 6. Two-bay, two-storey RC frame: (a) geometry and loading; (b) AFD (N); (c) SFD (N); (d) BMD (N-mm)

$$\bar{z}_{2m1} = \frac{\bar{\ell}_{2m}\sum_{n=1}^{N_b^k+1}\left[\frac{\ell_{2c}Q_1^{nk}(z_{2c}^k)^2}{3\ell_{2c}} + Q_2^{nk}\right]}{\sum_{n=1}^{N_b^k+1}\left\{(z_{3c}^{nk}\overline{m}_{2c}+1)L^{nk} + \bar{\hat{\mu}}\ell_{2m}\overline{u}^{nk} + \overline{zal}\right\}},$$

where $\overline{zal}$ is given in (85). It beomes zero whenever the steel ratio in a column of a group assumes its lower bound.

Table 4. Minimum cost design of a two-bay, two-storey RC frame

Column	Initial z_{2c}^k mm	Optimum z_{2c}^k mm	Gross steel area mm^2	Mode of failure	Selfweight load kN/m
AD	400	265	763	T	2.52
BE	400	265	1537	C	2.52
CF	400	265	763	T	2.52
DG	400	250	725	T	2.40
EH	400	250	725	C	2.40
FI	400	250	725	T	2.40

Beam	Initial z_{2b}^k mm	Optimum z_{2b}^k mm	A_{s1} mm^2	A_{s2} mm^2	A_{s3} mm^2	A_{s4} mm^2	Selfweight load kN/m
DE	500	478	415	434	434	1242	4.37
EF	500	478	1179	706	706	637	4.37
GH	500	382	402	467	467	1406	3.58
HI	500	382	1558	859	859	584	3.58

3.4.1 Example 3. Multibay, multistorey frame

The two-bay, two-storey frame shown in Fig. 6a is subjected to the factored design loads in addition to selfweight. The strength constraints are the same as in the previous two examples. The maximum deflection at midpoint of beam EF must be less than or equal to 10 mm. The design data including the bounds and initial steel ratios remain the same as for Example 2, except that $z_{2c\ell} = 250$ mm.

The results are summarized in Table 4. The end forces of each member and the midspan bending moments of the beams, as obtained from the analysis part of the solution are given in Table 5 in the notation of Example 2. The AF, SF and BM diagrams are shown in Figs. 6b, c and d, respectively, with the cross-sections of the members of the optimum frame shown in Fig. 7.

The design of the top beams (GH and HI) in the upper floor is governed by the flexural strength constraint and upper bound on steel ratio, or by the flexural strength constraint alone, whereas those in the lower floor (DE and EF) are controlled by the combination of deflection and flexural strength constraints. The design of the columns in the top floor are governed by a lower bound on both design variables, whereas those in the ground floor are controlled by a uniaxial bending strength constraint on the depth and an upper bound on the steel ratio, or by a uniaxial bending strength constraint and a lower bound on the steel ratio, or by a uniaxial bending strength constraint alone. Further, the results show that the interior columns of both storeys are governed by the compression mode of failure, whereas all exterior columns are governed by the tension mode of failure. This is the most likely phenomenon to occur for such a geometry and loading combination, due to the fact that interior columns are sub-

Table 5. End forces (N) and midspan moment (N-mm) of the members of the two-bay, two-storey frame

Member	$AM1$	$AM2$	$AM3$	$AM4$	$AM5$	$AM6$	MSM
AD	218818	5854	16098535	-208725	-5854	7316566	
BE	668002	9804	21368740	-657909	-9804	17849160	
CF	297444	20342	35443284	-287352	-20342	45923712	
DE	-2495	104208	21651384	2495	167625	-180195100	68972192
EF	-19826	175314	172384380	19826	150885	-99097808	108908340
DG	104518	-20641	-28967950	-97324	20641	-32956076	
EH	314970	-7527	-10038437	-307776	7526	-12541389	
FI	136467	40168	53174092	-129273	-40168	67329760	
GH	32641	97324	32956076	-32641	145573	-153579440	58542636
HI	40168	162203	166120820	-40168	129273	-67329760	101881670

jected to a smaller bending moment and a larger axial force than the exterior columns.

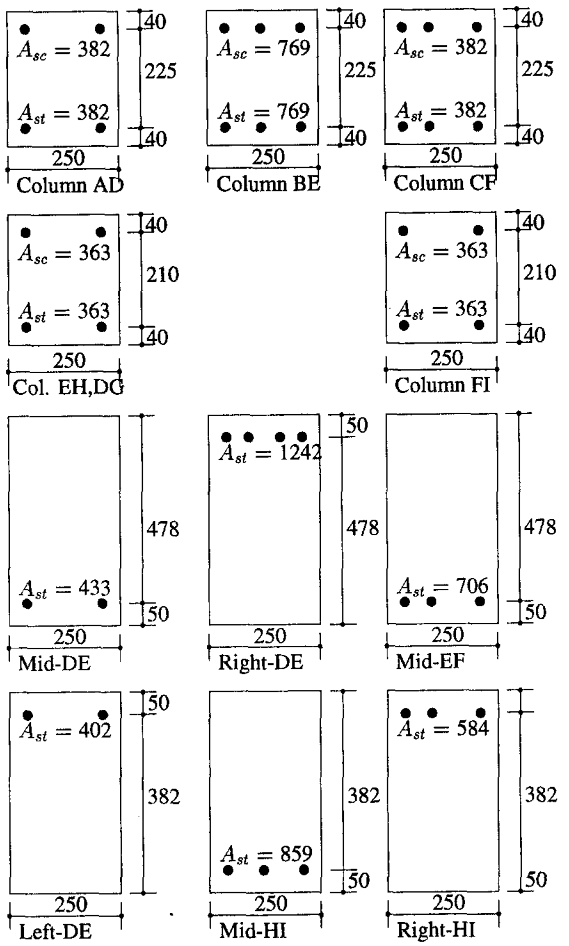

Fig. 7. Optimum distribution of the design variables at various sections of the two-bay, two-storey frame

The initial design cost of 34.0204 converged to the optimum cost of 27.4168. These values must be multiplied by c_C to obtain the actual cost of construction. The CPU time used on a micro VAX/VMS was 7.08 seconds. The Lagrange multiplier $\hat{\mu}$ converged to 1.9633.

3.4.2 Example 4. Seven-storey RC frame

A seven-storey RC frame is subjected to the factored design loads shown in Fig. 8 in addition to the selfweight. The maximum deflection at midpoint of beam FG in the bottom floor must be less than or equal to 14 mm. The strength constraints and other design information remain the same as for Example 3.

Table 6. Minimum cost design of the seven-storey RC frame

Column	Initial z_2^c mm	Optimum z_2^c mm	Gross steel area mm^2	Mode of failure	Selfweight load kN/m
AE	500	490	1324	C	4.38
BF	500	490	2610	C	4.38
CG	500	490	2651	C	4.38
DH	500	490	1324	C	4.38
EI	500	390	1075	C	3.55
FJ	500	390	1659	C	3.55
GK	500	390	2152	C	3.55
HL	500	390	1075	C	3.55
IM	500	344	961	C	3.18
JN	500	344	1265	C	3.18
KO	500	344	1924	C	3.18
LP	500	344	961	C	3.18
MQ	500	287	817	T	2.70
NR	500	287	1635	C	2.70
OS	500	287	1571	C	2.70
PT	500	287	817	T	2.70
RU	500	279	1397	C	2.64
SV	500	279	1583	C	2.64
UW	500	254	1209	C	2.43
VX	500	254	1469	C	2.43
WY	500	250	1056	T	2.40
XZ	500	250	1165	T	2.40

Beam	Initial z_2^c mm	Optimum z_2^c mm	A_{s1} mm^2	A_{s2} mm^2	A_{s3} mm^2	A_{s4} mm^2	Selfweight load kN/m
EF	500	521	667	699	699	1852	4.72
FG	500	521	1679	908	908	2020	4.72
GH	500	521	1103	540	540	986	4.72
IJ	500	433	684	569	569	1478	3.99
JK	500	433	1404	761	761	1763	3.99
KL	500	433	818	448	448	972	3.99
MN	500	423	675	580	580	1570	3.92
NO	500	423	1457	815	815	1724	3.92
OP	500	423	952	442	442	923	3.92
QR	500	446	743	1015	1015	1814	4.10
RS	500	446	1597	613	613	1762	4.10
ST	500	446	1176	850	850	602	4.10
UV	500	409	1110	1400	1400	1667	3.80
WX	500	409	1334	1293	1293	1663	3.79
YZ	500	500	1219	2034	2034	1182	4.55

The optimum distribution of the design variables is sum-

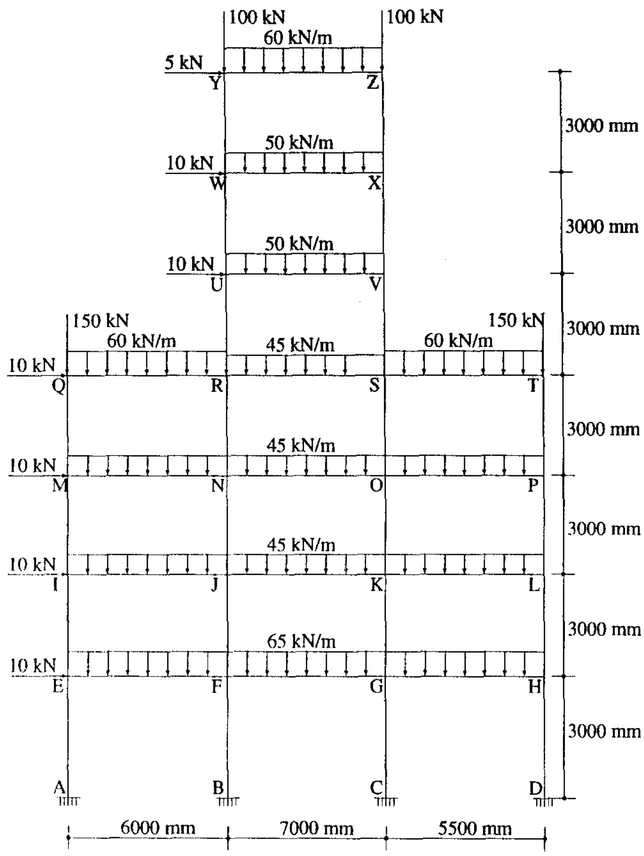

Fig. 8. The seven-storey RC frame

marized in Table 6. The end forces of each member and midspan bending moments of the beams as obtained from the analysis part of the solution are given in Table 7.

The designs of the bottom floor beams are controlled by the combination of deflection and flexural strength constraints, whereas all the other beams are governed by the flexural strength constraint and an upper bound on the steel ratio, or by the flexural strength constraint alone. The designs of the columns are governed by a uniaxial bending strength constraint and a lower bound on the steel ratio, a uniaxial bending strength constraint and an upper bound on the steel ratio, a lower bound on the depth and a uniaxial bending strength constraint on the steel ratio, or the uniaxial strength constraint alone.

The initial design cost of 139.7619 converged to the optimum cost of 120.2252. These values must be multiplied by c_C to obtain the actual cost of construction. The CPU time used on a micro VAX/VMS was only 3 minutes and 36.58 seconds. The Lagrange multiplier $\hat{\overline{\mu}}$ converged to 2.4628.

4 Conclusion

The minimum cost designs of RC frame structures are obtained using methods based on DCOC. As the design problem involves more complex behavioural constraints, the study in this paper commenced with the reformulation of the design problem, followed by a derivation of OC. The design problem was solved in a systematic manner by taking into account customs followed in the actual construction of these structures. Algorithms were developed and numerical procedures coded to obtain the solution to the minimum cost design problem of these structures whose cross-sectional parameters are uniform member-wise or uniform per storey. The variation of the force resultants in the beams was accounted for by allowing the steel ratio to vary freely along the spans, whereas the variation of the force resultants in the columns per storey was accounted for by using a different, but constant steel ratio in each column. In this part of the two-part paper, the columns of the frame are assumed to be subjected to uniaxial bending action. However, some of the columns, especially those of an edge frame, can be subjected to bending in both major axes of the cross-section. Hence, in Part II, columns under biaxial bending are considered. Several examples were chosen to demonstrate the usefulness of the algorithms developed. It was shown that the algorithms based on DCOC are extremely efficient as judged by the small amount of CPU time necessary even for reasonably large frame structures.

References

Adamu, A. 1995: *Minimum cost design of RC structures using methods based on optimality criteria.* Ph.D. Thesis, University of Sydney

Adamu, A.; Karihaloo, B.L. 1994a: Minimum cost design of RC beams using DCOC. Parts I: beams with freely-varying cross-sections. *Struct. Optim.* **7**, 237-251

Adamu, A.; Karihaloo, B.L. 1994b: Minimum cost design of RC beams using DCOC. Parts II: beams with uniform cross-sections. *Struct. Optim.* **7**, 252-259

Adamu, A.; Karihaloo, B.L. 1995: Minimum cost design of RC beams with segmentation using continuum-type optimality criteria. *Struct. Optim.* (in press)

Adamu, A.; Karihaloo, B.L.; Rozvany, G.I.N. 1994: Minimum cost design of reinforced concrete beams using continuum-type optimality criteria. *Struct. Optim.* **7**, 91-102

Comité Euro-International du Béton (CEB) 1990: *CEB-FIB Model Code.* Paris: Bulletin d'information Nos. 195 and 196

Ferguson, P.M. 1979: *Reinforced concrete fundamentals.* New York: John Wiley & Sons

Grierson, D.E.; Moharrami, H. 1993: Design optimization of reinforced concrete building frameworks. In: Rozvany, G.I.N. (ed.) *Optimization of large structural systems* (Proc. NATO/DFG ASI, held in Berchtesgaden, Germany, 1991), pp. 883-896. Dordrecht: Kluwer

Kanagasundaram, S.; Karihaloo, B.L. 1990: Minimum cost design of reinforced concrete structures. *Struct. Optim.* **2**, 173-184

Karihaloo, B.L. 1993: Minimum cost design of reinforced concrete members by nonlinear programming. In: Rozvany, G.I.N. (ed.) *Optimization of large structural systems* (Proc. NATO/DFG ASI, held in Berchtesgaden, Germany, 1991), pp. 927-950. Dordrecht: Kluwer

McGuire, W.; Gallagher, R.H. 1979: *Matrix structural analysis.* New York: John Wiley & Sons

Moharrami, H.; Grierson, D.E. 1993: Computer-automated design of reinforced concrete frameworks. *J. Struct. Eng. ASCE* **119**, 2036-2058

Table 7. End forces (N or N-mm) and midspan moment (N-mm) of the members of the seven-storey frame

Member	$AM1$	$AM2$	$AM3$	$AM4$	$AM5$	$AM6$	MSM
AE	806922	-14552	-1092805	-789402	14552	-57116956	
BF	2278733	11422	33633612	-2261213	-11422	12055904	
CG	2209631	29676	58136640	-2192111	-29676	60569200	
DH	819706	38454	69920896	-802186	-38454	83893504	
EF	-12885	181854	113382160	12885	236484	-277272420	118426400
FG	-19438	238264	256249660	19438	249797	-296615390	150621090
GH	-12261	194841	179233710	12261	188636	-162169070	92938856
EI	607548	-37438	-56265200	-596886	37438	-56047652	
FJ	1786464	4869	8966834	-1775802	-4869	5641041	
GK	1747472	36853	56812472	-1736810	-36853	53747844	
HL	613551	50715	78275568	-602888	-50715	73869088	
IJ	20116	131919	95060232	-20116	162049	-185451600	80220248
JK	15210	166528	177900880	-15210	176435	-212577680	104853280
KL	12046	131375	111866230	-12046	138096	-130346610	64154800
IM	464967	-27322	-39012584	-455435	27322	-42951984	
JN	1447225	-37	1909679	-1437694	37	-2020678	
KO	1428999	33690	46963604	-1419468	-33690	54106396	
LP	464793	38669	56477516	-455261	-38669	59528048	
MN	-3027	130477	91645712	3027	163014	-189257440	79666512
NO	10603	167653	178387260	-10603	174753	-203234640	108794220
OP	-11012	135139	124851620	11012	133894	-121428920	61820064
MQ	324959	-40349	-48693732	-316856	40349	-72351848	
NR	1107026	13593	12890845	-1098924	-13592	27886728	
OS	1109576	12076	24276618	-1101474	-12076	11950724	
PT	321367	49680	61900872	-313265	-49680	87139784	
QR	50349	166856	72351848	-50349	217730	-224971550	139777660
RS	14446	169553	204070110	-14446	174130	-220089360	88643288
ST	49680	189272	158660340	-49680	163265	-87139784	119469130
RU	711641	-22310	-6985283	-703733	22310	-59944380	
SV	738072	47310	49478296	-730163	-47310	92451360	
UV	-16524	180739	137102220	16524	195857	-190015790	165961940
UW	522994	-48834	-77157840	-515715	48834	-69344632	
VX	534306	63834	97564424	-527028	-63834	93938056	
WX	-6984	183999	159234530	6984	192542	-189137170	155287100
WY	331717	-65818	-89889888	-324523	65818	-107564600	
XZ	334486	70818	95199112	-327292	-70818	117255380	
YZ	70818	224523	107564610	-70818	227292	-117255380	282928260

Park, R.; Paulay, T. 1975: *Reinforced concrete structures.* New York: John Wiley & Sons

Rice, P.F.; Hoffman, E.S. 1972: *Structural design guide to the ACI building code.* New York: Litton Educational Publishing Inc.

Rozvany, G.I.N. 1989: *Structural design via optimality criteria.* Dordrecht: Kluwer

Rozvany, G.I.N.; Zhou, M.; Gollub, W. 1990: Continuum-type optimality criteria methods for large finite element systems with a displacement constraint. Part II. *Struct. Optim.* **2**, 77-104

Rozvany, G.I.N.; Zhou, M. 1993a: Continuum-based optimality criteria (COC) methods: an introduction. In: Rozvany, G.I.N. (ed.) *Optimization of large structural systems* (Proc. NATO/DFG ASI, held in Berchtesgaden, Germany, 1991), pp. 1-26. Dordrecht: Kluwer

Rozvany, G.I.N.; Zhou, M. 1993b: Optimality criteria methods for large structural systems. *A Research Project Report* (Ro 744/1) **58**, Essen, Germany

Standards Association of Australia. *AS 3600 1988: Concrete structures.* Sydney: SAA

Warner, R.F.; Rangan, B.V.; Hall, A.S. 1988: *Reinforced concrete* (third edition). Longman Australia

Weaver, W.; Gere, J.M. 1965: *Analysis of framed structures.* New York: Van Nostrand

Zhou, M.; Rozvany, G.I.N. 1992: DCOC: an optimality criteria method for large systems. Part I: theory. *Struct. Optim.* **5**, 12-25

Zhou, M.; Rozvany, G.I.N. 1993: DCOC: an optimality criteria method for large systems. Part II: algorithm. *Struct. Optim.* **6**, 250-262

Appendix A. The force-displacement relationships

In the notation of Zhou and Rozvany (1992, 1993), the force-displacement relationships for the frame element (McGuire and Gallagher 1979; Weaver and Gere 1965) shown in Fig. A.1a are

$$\{\mathbf{F}^e\} = [\mathbf{s}^e]\{\mathbf{u}^e\}\,, \tag{89}$$

where

$$[\mathbf{s}^e] = \frac{E}{(L^e)^3}\,.$$

$$\begin{bmatrix} A^e(L^e)^2 & 0 & 0 & -A^e(L^e)^2 & 0 & 0 \\ 0 & 12I^e & 6I^eL^e & 0 & -12I^e & 6I^eL^e \\ 0 & 6I^eL^e & 4I^e(L^e)^2 & 0 & -6I^eL^e & 2I^e(L^e)^2 \\ -A^e(L^e)^2 & 0 & 0 & A^e(L^e)^2 & 0 & 0 \\ 0 & -12I^e & -6I^eL^e & 0 & 12I^e & -6I^eL^e \\ 0 & 6I^eL^e & 2I^e(L^e)^2 & 0 & -6I^eL^e & 4I^e(L^e)^2 \end{bmatrix}, \tag{90}$$

$$\{\mathbf{F}^e\} = \{\ P^e_A \quad V^e_A \quad M^e_A \quad P^e_B \quad V^e_B \quad M^e_B\ \}^T, \tag{91}$$

$$\{\mathbf{u}^e\} = \{\ w^e_A \quad v^e_A \quad \theta^e_A \quad w^e_B \quad v^e_B \quad \theta^e_B\ \}^T. \tag{92}$$

Since the global coordinate system may be different from the element coordinate system, the orthogonal transformation matrix is introduced

$$[\mathbf{T}^e] = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{93}$$

where θ is the angle measured from the global x-axis X_g, to the longitudinal axis of the element, denoted X.

The element displacements and forces in the local coordinates are

$$\{\mathbf{u}^e\} = [\mathbf{T}^e]\{\mathbf{u}^e_g\}, \tag{94}$$

$$\{\mathbf{F}^e\} = [\mathbf{T}^e]\{\mathbf{F}^e_g\}, \tag{95}$$

where $\{\mathbf{u}^e_g\}$ and $\{\mathbf{F}^e_g\}$ are, respectively, element displacement and force vectors in the global coordinate system.

For the flexibility formulation, the stable statically determinate frame element chosen is the one supported as a cantilever beam with the built-in end at B. The force-displacement relationships for this element are given by

$$\{\mathbf{u}^e_{fA}\} \begin{Bmatrix} w^e_{fA} \\ v^e_{fA} \\ \theta^e_{fA} \end{Bmatrix} = [\mathbf{f}^e_A] \begin{Bmatrix} P^e_A \\ V^e_A \\ M^e_A \end{Bmatrix}, \tag{96}$$

with

$$[\mathbf{f}^e_A] = \frac{L^e}{E} \begin{bmatrix} \frac{1}{A^e} & 0 & 0 \\ 0 & \frac{(L^e)^2}{3I^e} & -\frac{L^e}{2I^e} \\ 0 & -\frac{L^e}{2I^e} & \frac{1}{I^e} \end{bmatrix}. \tag{97}$$

The transformation matrix for the degree of freedom can be written as

$$[\mathbf{\Upsilon}^e_A] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{98}$$

If there are loads applied on the frame element, as in Fig. A.1b, the fixed-end forces are

$$\{\mathbf{F}^e_F\} = \begin{Bmatrix} P^e_{FA} \\ V^e_{FA} \\ M^e_{FA} \\ P^e_{FB} \\ V^e_{FB} \\ M^e_{FB} \end{Bmatrix} = \begin{Bmatrix} \frac{q^eL^e}{2} \\ \frac{p^eL^e}{2} \\ \frac{p^e(L^e)^2}{12} \\ \frac{q^eL^e}{2} \\ \frac{p^eL^e}{2} \\ -\frac{p^e(L^e)^2}{12} \end{Bmatrix}, \tag{99}$$

where the equivalent nodal loads $\{\mathbf{P}^e_E\}$ are the fixed-end forces with the signs reversed. These can be transformed to the equivalent nodal loads $\{\mathbf{P}^e_{Eg}\}$ in the global coordinate system as follows:

$$\{\mathbf{P}^e_{Eg}\} = [\mathbf{T}^e]^T\{\mathbf{P}^e_E\} = -[\mathbf{T}^e]^T\{\mathbf{F}^e_F\}. \tag{100}$$

In this case (89) becomes

$$\{\mathbf{F}^e\} = [\mathbf{s}^e]\{\mathbf{u}^e\} + \{\mathbf{F}^e_F\}. \tag{101}$$

The initial relative displacements $\{\hat{\mathbf{u}}^e_{fA}\}$ caused by loads within an element may be computed using standard formulae, and (96) rewritten to read

$$\{\mathbf{u}^e_{fA}\} = [\mathbf{f}^e_A]\{\mathbf{F}^e_{fA}\} + \{\hat{\mathbf{u}}^e_{fA}\}, \tag{102}$$

where

$$\{\mathbf{F}^e_{fA}\}^T = \{\ P^e_A \quad V^e_A \quad M^e_A\ \}.$$

Appendix B. Inclusion of selfweight

The selfweight of a rectangular cross-section per unit length is

$$W_c = z_{1x}(z^i_2 + d'_x)w_c, \tag{103}$$

in which i stands for a beam element em or column c, and x for beam b or column c.

For a general frame element, such as the one shown in Fig. A.1b, the selfweight may be resolved into two components

$$W_v = p^i = W_c\cos\theta, \tag{104}$$

$$W_x = q^i = W_c\sin\theta. \tag{105}$$

The vector of nodal forces $\{\mathbf{P}\}$ due to the selfweight in the real system may be included in its equilibrium equation through equivalent nodal loads

$$\left\{\mathbf{P}^i_{EW}\right\} = -\left\{\mathbf{F}^i_{FW}\right\} = z_{1x}(z^i_2 + d'_x)w_c \begin{Bmatrix} \frac{L^i}{2}\sin\theta \\ \frac{L^i}{2}\cos\theta \\ \frac{(L^i)^2}{12}\cos\theta \\ \frac{L^i}{2}\sin\theta \\ \frac{L^i}{2}\cos\theta \\ \frac{(L^i)^2}{12}cos\theta \end{Bmatrix}. \tag{106}$$

If the unit considered is a horizontal beam element, (106) simplifies to

$$\{\mathbf{P}^{em}_{EW}\} = -\{\mathbf{F}^{em}_{FW}\} = -z_{1b}(z^m_2 + d'_b)w_c \begin{Bmatrix} 0 \\ L^{em}/2 \\ (L^{em})^2/12 \\ 0 \\ L^{em}/2 \\ -(L^{em})^2/12 \end{Bmatrix}. \tag{107}$$

On the other hand, if the unit is a vertical column, it simplifies to

$$\{\mathbf{P}^c_{EW}\} = -\{\mathbf{F}^c_{FW}\} = -z_{1c}(z^c_2 + d'_c)w_c \begin{Bmatrix} L^c/2 \\ 0 \\ 0 \\ L^c/2 \\ 0 \\ 0 \end{Bmatrix}. \tag{108}$$

The partial derivatives of (106) with respect to the design variables z^i_2 are

$$\left\{\frac{\partial\mathbf{P}^i_{EW}}{\partial z^i_2}\right\} = -z_{1x}w_c \begin{Bmatrix} \frac{L^i}{2}\sin\theta \\ \frac{L^i}{2}\cos\theta \\ \frac{(L^i)^2}{12}\cos\theta \\ \frac{L^i}{2}\sin\theta \\ \frac{L^i}{2}\cos\theta \\ -\frac{(L^i)^2}{12}\cos\theta \end{Bmatrix}. \tag{109}$$

The adjoint nodal displacement vector due to selfweight is

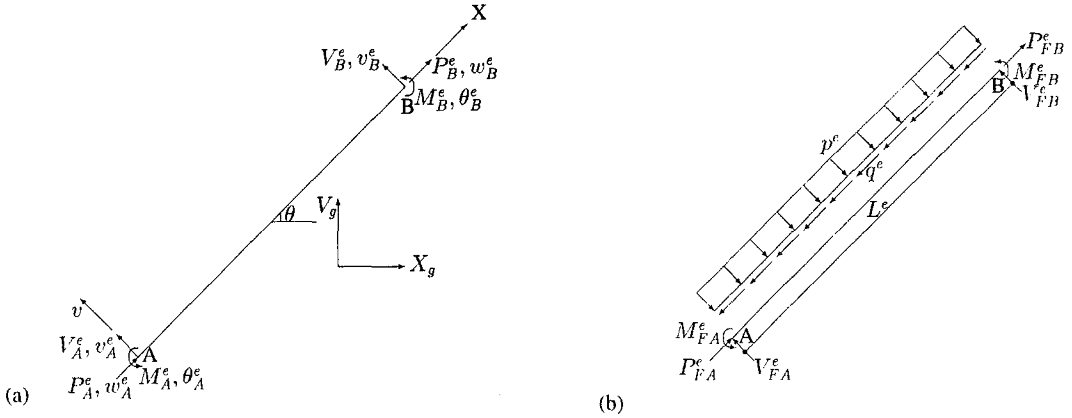

Fig. A.1 A frame element showing: (a) nodal force-displacement relationships; (b) fixed-end actions due to distributed loads between nodes

$$\{\overline{\mathbf{u}}^i\}^T = \left\{ \begin{array}{cccccc} \overline{w}^i_A & \overline{v}^i_A & \overline{\theta}^i_A & \overline{w}^i_B & \overline{v}^i_B & \overline{\theta}^i_B \end{array} \right\}, \tag{110}$$

so that for beam elements or columns

$$\mu\{\overline{\mathbf{u}}^i\}^T \left\{ \frac{\partial \mathbf{P}^i_{EW}}{\partial z^m_2} \right\} = -\mu \frac{z_{1x} w_c L^i}{12} \Big(6\overline{w}^i_A \sin\theta + 6\overline{v}^i_A \cos\theta +$$

$$\overline{\theta}^i_A L^i \cos\theta + 6\overline{w}^i_B \sin\theta + 6\overline{v}^i_B \cos\theta - \overline{\theta}^i_B L^i \cos\theta \Big) . \tag{111}$$

Denoting

$$\mu\{\overline{\mathbf{u}}^i\}^T \left\{ \frac{\partial \mathbf{P}^i_{EW}}{\partial z^i_2} \right\} = \frac{3\mu}{\ell_{2x}} \overline{u}^i , \tag{112}$$

gives

$$\overline{u}^i = -\frac{\ell_{2x} z_{1x} w_c L^i}{36} (6\overline{w}^i_A \sin\theta + 6\overline{v}^i_A \cos\theta + \overline{\theta}^i_A L^c \cos\theta +$$

$$6\overline{w}^i_B \sin\theta + 6\overline{v}^i_B \cos\theta - \overline{\theta}^i_B L^i \cos\theta) . \tag{113}$$

Equations (112) represent the additional terms in OC (32) and (33), from selfweight.

Received Nov. 3, 1994
Revised manuscript received March 1, 1995