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Adaptive topology optimization

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Abstract Topology optimization of continuum structures is often reduced to a material distribution problem. Up to now this optimization problem has been solved following a rigid scheme. A design space is parametrized by design patches, which are fixed during the optimization process and are identical to the finite element discretization. The structural layout is determined, whether or not there is material in the design patches. Since many design patches are necessary to describe approximately the structural layout, this procedure leads to a large number of optimization variables. Furthermore, due to a lack of clearness and smoothness, the results obtained can often only be used as a conceptual design idea.

To overcome these shortcomings adaptive techniques, which decrease the number of optimization variables and generate smooth results, are introduced. First, the use of pure mesh refinement in topology optimization is discussed. Since this technique still leads to unsatisfactory results, a new method is proposed that adapts the effective design space of each design cycle to the present material distribution. Since the effective design space is approximated by cubic or Bézier splines, this procedure does not only decrease the number of design variables and lead to smooth results, but can be directly joined to conventional shape optimization. With examples for maximum stiffness problems of elastic structures the quality of the proposed techniques is demonstrated.

1 Introduction

The determination of the structural layout is the first and basic problem in the course of a design process. This problem can be solved by the most universal kind of structural optimization, i.e. optimizing the topology of a structure. During the last years numerous methods for topology optimization of discrete and continuum structures have been introduced and applied to a broad range of design problems. An overview can be found in the proceedings edited by Bendsøe and Mota Soares (1993). However, since these methods are only able to outline roughly the form of the body, their results must be processed again and improved by further design tools, such as shape optimization techniques. Therefore, the objective of the present paper is to introduce adaptive techniques into topology optimization of continuum structures. These techniques improve the efficiency of the conventional topology optimization procedure and the quality of its results. Moreover, the proposed method provides an automatic link between topology and shape optimization. First, the conventional optimization procedure is shown and its shortcomings are discussed. An alternative design concept is presented which separates design and analysis models. An artificial orthotropic approach is introduced solving the material distribution problem. The application of mesh refinement to topology optimization is explained and discussed. Finally, an adaptive technique is presented, which orientates the active optimization variables in the analysis model on the material distribution in the design space. The features of the proposed techniques are demonstrated with maximum stiffness problems of two-dimensional elastic structures solved by optimality criteria and mathematical programming methods.

2 Topology optimization

The topology of a structure is defined as a spatial arrangement of structural members and joints or internal boundaries. Consequently, topology optimization means varying the connectivity between structural members of discrete structures or between domains of continuum structures, as can be seen in Fig. 1. For discrete structures, such as trusses, the variation of connectivity means to generate or to eliminate structural members between existing joints, but also to define new joints or to remove existing joints. Analogously, for continuum structures the variation of connectivity means to separate or to join together structural domains and to generate or to reduce structural domains. However, in the case of continuum structures it is not sufficient to only indicate where cuts must be made to change the structural topology. In addition, the shapes of the cuts must be determined to define the new structural layout. Therefore, Rozvany *et al.* (1995) call optimizing the topology of continuum structures generalized shape optimization. Since discrete structures can be understood as a special case of continuum structures, topology optimization only in the sense of general shape optimization is investigated in this study.

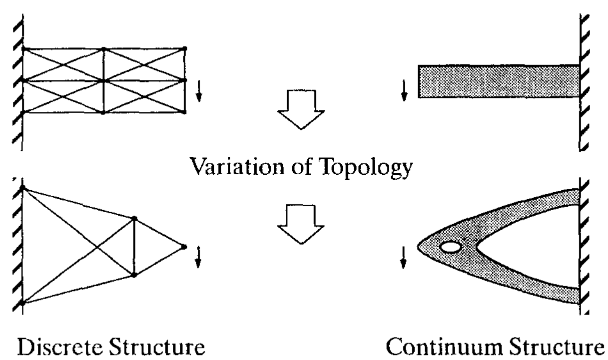


Fig. 1. Variation of topology

The entire form of a structure changes by a variation of the topology. Therefore, the geometric model of the optimization procedure must describe the body of the structure in a general way. In principle, there are two different possibilities, which are illustrated in Fig. 2. Geometrically a structure is defined by internal and external boundaries ("geometrical description"). From a material point of view the topology is defined by a simple 1/0 decision for each point in the design space Ω, whether there is material (1) or not (0) ("material description"). It is obvious that different optimization problems result, depending on which possibility is chosen.

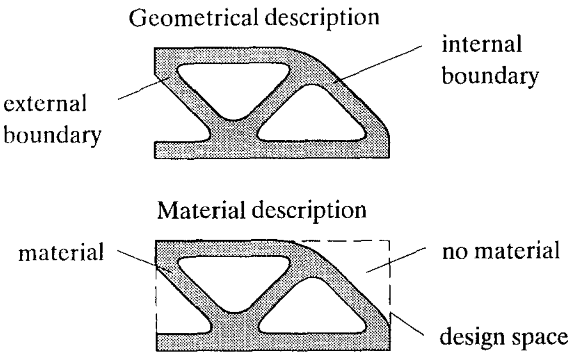


Fig. 2. Geometrical and material description

Up to now, only a few approaches based on a geometrical description are known in topology optimization, e.g. the "bubble method" by Eschenauer *et al.* (1993) or a heuristic approach by Rosen and Grosse (1992) in topology optimization are known. The main difficulty of these methods comes up when holes must be generated in a continuum structure, which means a violation of the continuum assumption and, therefore, a nondifferentiable step in the optimization procedure. Therefore, the predominant number of topology optimization methods use the material description, which will be also applied in the following investigations.

3 Material-based topology optimization

If only one isotropic material is used, the material description leads to a discrete value parameter function χ (Bendsøe 1989)

$$\chi_{(\mathbf{x})} = \begin{cases} 0 & \rightarrow \text{ no material} \\ 1 & \rightarrow \text{ material} \end{cases} \quad \mathbf{x} \in \Omega\,. \tag{1}$$

Up to now, using this material-based design model, the optimum layout of a structure is determined by a rather rigid procedure as shown schematically in Fig. 3. Since it is not possible to use each point of the design space as an optimization variable, first the design space must be parametrized by design patches. This parametrization is fixed during the optimization process and is identical to the finite element discretization, which is used for structural and sensitivity analyses. In the following procedure it is determined whether or not there is material in a design patch by optimizing the material distribution in the design space. As a result of the optimization process, the design space is more or less distinctly divided into voids (white patches) and structural elements (dark patches). Additionally, this optimized material distribution is lumped and smoothed in a postprocessing step to obtain the final layout.

Using the parametrized indicator function χ, the design problem yields an integer optimization problem, where the

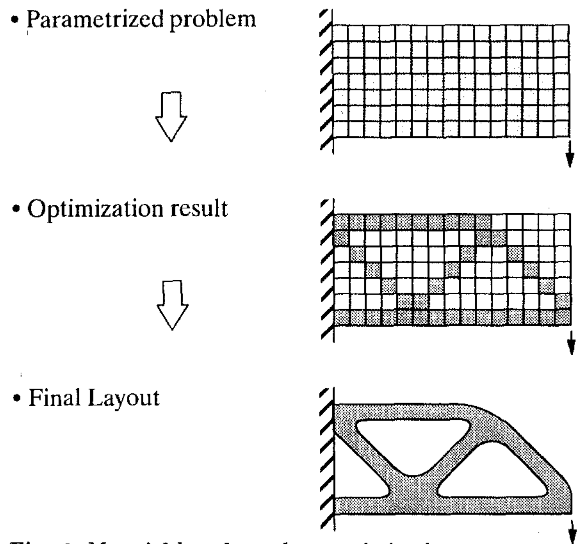


Fig. 3. Material-based topology optimization

indicator parameters of the parametrized problem are the optimization variables. The search for the optimum material distribution in the design space corresponds to a combinatorial problem. However, Kohn and Strang (1986) stated that this formulation of a general shape optimization problem is not convex and, therefore, not well-posed. This means, on the one hand, that the optimization results are strongly dependent on the chosen parametrization and, on the other hand, that the integer formulation comprises many artificial local minima. Moreover, since the number of design variables is preferably large to describe at least approximately the basic layout of a structure, the related integer problem is very costly to solve.

These problems can be overcome by transferring the integer problem into a continuous problem using the method of relaxation. The discrete value parameter function χ becomes a continuous distribution of a new parameter $\overline{\chi}$. Two different types of relaxation methods can be distinguished: the physical and the artificial approach. In the case of physical relaxation, intermediate values of the parameter function $\overline{\chi}$ are admissible at the final result, because these values can be interpreted physically as special material described by certain microstructure models, e.g. the "microhole approach" by Bendsøe and Kikuchi (1988). The macroscopic material behaviour is derived by the method of homogenization (Sanchez-Palencia 1980). In contrast to this, in the case of artificial relaxation the intermediate values are not admissible at the optimum. Following Rozvany *et al.* (1992), values between 0 and 1 can simply be seen as an intermediate stage of the optimization process. Consequently, the optimization problem must be posed such that the final optimized design contains only or at least approximately 1/0 values. In this case artificial approaches are introduced (Rozvany *et al.* 1992; Ramm and Maute 1994).

Based on a physical or an artificial approach numerous design problems have been solved. For example, Bendsøe and Kikuchi (1988) determined the optimum layout for plane stress structures, maximizing the structural stiffness. Suzuki and Kikuchi (1991) extended this method to three-dimensional shell structures. Moreover, the optimum layout with respect to natural frequencies was determined by Díaz and Kikuchi (1992) and the topology of two-dimensional elastic structures with critical load constraints was optimized by Neves *et al.* (1993). To solve diverse design problems material-based topology optimization was embedded in a mathematical programming scheme by Tenek and Hagiwara (1993) and Maute and Ramm (1994).

The main advantages of material-based topology optimization methods are their simplicity and stability. Topology optimization can be handled as a simple sizing problem varying material parameters of special materials. However, they all have the same basic shortcomings. Since the body of a structure is described by a 1/0 decision in the entire design space, a large number of optimization variables (> 100) is necessary to fix at least approximately the layout of the structure. Void areas are analysed and "optimized", since the number of optimization variables is constant. In domains of the design space, where the parametrization is too coarse, the structural layout cannot be clearly identified, since there intermediate values of $\overline{\chi}$ still remain at the optimum. Moreover, since a fixed scanning is used to describe the form of the body, the optimization process leads to jagged boundaries, which must be smoothed in an additional postprocessing step. Consequently, due to the lack of clearness and smoothness, the results obtained by conventional topology optimization can often only be used as a conceptional design idea instead of a clearly defined structural layout. Furthermore, local quantities, such as stresses, cannot be controlled during the optimization process. Olhoff *et al.* (1991) proposed the use of material-based topology optimization as a preprocessor and to define exactly the structural shape by conventional shape optimization. A similar approach was presented by Hinton and Sienz (1994). However, since topology and shape depend on each other, this procedure generally does not find the optimum layout. Moreover, as the examples by Olhoff *et al.* (1991) demonstrate, due to the lack of clearness of the results obtained by conventional topology optimization, it can be very difficult to determine even interactively an initial design for the shape optimization step. Moreover, a simultaneous application of conventional topology optimization methods and boundary variation techniques presented by Maute and Ramm (1994) still leads to unsatisfactory results, since only boundaries present in the initial design can be smoothed by this method.

4 Alternative design concepts

In structural optimization two kinds of models can be distinguished. In the design model the geometry is described and the design parameters, i.e. the optimization variables, are defined. The analysis model is used to determine the structural behaviour and its sensitivity with respect to the optimization variables based on a finite element analysis. The inflexibility of the conventional procedure can be reduced to the rigid coupling of design model and analysis model. To obtain a flexible and efficient method of topology optimization, it is necessary to separate these models for geometry and analysis, as is already done in pure shape optimization (Bletzinger *et al.* 1991). Since the pixel-like scanning of a structure is a very simple and flexible technique, the form of the structure is still parametrized by design patches in a design model as in the conventional procedure (Fig. 4). The material parameters of the design patches are the optimization variables. Depending on the material distribution in the design space an analysis model can be generated by different techniques, which will be discussed in the following sections. In this analysis model the finite element discretization is carried out. The optimization variables of the patches in the design model are linked to the material parameters of the finite elements in the analysis model. The material parameters of each finite element in the analysis model are the so-called active optimization variables, which are processed explicitly by the optimization algorithm. Since the parametrization of the design model is fixed during the optimization process, continuity is guaranteed. Since the active set of optimization variables is defined in the adapted analysis model, flexibility is provided.

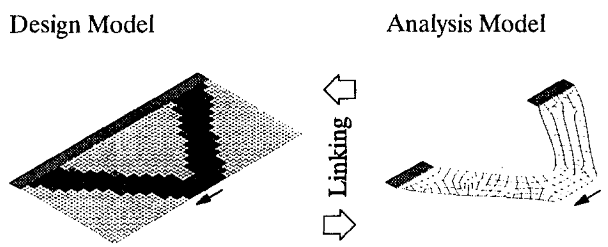


Fig. 4. Alternative design modelling

At the beginning of each design cycle an analysis model is generated with respect to the material distribution in the design model. Material data are mapped from the design model onto the analysis model. An improved or an optimum material distribution in the analysis model is determined by conventional topology optimization. At the end of each cycle the new material distribution of the analysis model is remapped onto the design model. Consequently, this mapping and remapping correspond to an implicit linking between the dependent optimization variables $\mathbf{s}_0$, i.e. the material parameters of the patches in the design model, and the independent optimization variables $\mathbf{s}$, i.e. the active material parameters of the finite elements in the analysis model,

$$\mathbf{s} = \mathbf{L}\mathbf{s}_0 , \tag{2}$$

where $\mathbf{L}$ is the linking matrix. It is obvious that the number of design patches, which finally decides on the resolution, i.e. on the clearness and smoothness, of the optimization result, can be increased without influencing the size of the active optimization problem. Consequently, the discretization in the design model is usually much finer than the discretization in the analysis model. Concerning the mapping process this means that the material properties s^j of a finite element in the analysis model at the beginning of each cycle are an average with respect to the volume of the properties s_0 of the related patches n in the design model,

$$\mathrm{s}^j = \frac{1}{V} \sum_n \mathrm{s}_0^n V_0^n . \tag{3}$$

The remapping corresponds to a direct adoption of the improved material data of the finite elements by the design patches. The convergence of the optimization process is examined by the change of the material distribution in the design model.

Another consequence of the proposed modelling should also be pointed out. Using the conventional method all data are fixed with respect to a first and only discretization of the design space. For example, boundary conditions or loads can be allocated directly to the nodes of the finite element discretization. However, this usual way of defining input data for the finite element code must be changed, if the proposed alternative design modelling is used. Since the finite element mesh changes in each design cycle, it is no longer possible to define structural data, such as loads or supports, and local optimization data, such as stress limits or deflection constraints, in the analysis model at the beginning of the optimization process. All data must be defined in the design model only in relation to geometrical quantities, e.g. boundary conditions must be defined assigning supported domains.

5 Orthotropic approach

Before discussing different techniques to generate an adaptive analysis model, the approach used in the present study to solve the material distribution problem is briefly explained, since some special features are required by these flexible adaptive methods. Usually in topology optimization only one fixed parametrization of the design space is used to optimize the material distribution. The dependencies of the result on mesh refinement, distortion and orientation are not noticed. However, using the adaptive method described above in conjunction with a full automatic mesh generator, the finite element mesh changes completely in each cycle. Since it is not desired that changes of the parametrization in the analysis model influence the result of the optimization process, a mesh-independent approach is required.

As mentioned already by Kikuchi (1992), isotropic approaches used by Mlejnek *et al.* (1991), Rozvany *et al.* (1992) or Maute and Ramm (1994) sometimes show considerable dependencies on meshes. Orthotropic approaches, such as the rectangular microhole model introduced by Bendsøe and Kikuchi (1988) or the rank two layered model used by Bendsøe (1989), possess better convergence properties. The advantageous features of the orthotropic approaches can be traced back to their greater adaptability concerning mesh distortion and orientation, since these approaches have three independent parameters defining the material properties. However, since most orthotropic models arising from a physical relaxation process are based on periodically structured micromodels, they need the method of homogenization to obtain the relationship between material stiffness and optimization variables. This yields complex, in general implicit relations between macroscopic and microscopic parameters, which must be approximated explicitly. A simple and clear method can be derived from an isotropic approach, which belongs to the artificial type of relaxation. Bendsøe (1989) introduced the isotropic direct approach

$$\rho_{(\overline{\chi})} = \rho^{\circ}\overline{\chi}\,, \quad E_{(\overline{\chi})} = E^{\circ}\overline{\chi}^{\mu}\,, \quad \mu = 1,\ldots,9\,, \tag{4}$$

where ρ and E are the density and Young's modulus, respectively. The corresponding values of the homogeneous basic material are denoted by ρ° and E°. The exponent μ is artificially introduced in order to obtain clearly structured results. This approach can be easily extended to an orthotropic material model to obtain results almost independent of meshes. The material matrix $\mathbf{C}$ is defined

$$\mathbf{C} = \frac{1}{1-\nu^2}\begin{bmatrix} E_1 & \nu\sqrt{E_1E_2} & 0 \\ \nu\sqrt{E_1E_2} & E_2 & 0 \\ 0 & 0 & G \end{bmatrix}, \tag{5}$$

with

$$G = \frac{(1-\nu)}{2}\sqrt{E_1E_2}\,, \quad \overline{\chi} = (\overline{\chi}_1+\overline{\chi}_2-\overline{\chi}_1\overline{\chi}_2)\,, \quad \rho_{(\overline{\chi})} = \rho^{\circ}\overline{\chi}\,,$$

$$E_{i(\overline{\chi})} = E^{\circ}\overline{\chi}_i^{\mu}\,, \quad \mu = 1,\ldots,9\,,$$

where $\overline{\chi}_1, \overline{\chi}_2$ define independent material parameters and the angle Θ determines the orientation of the material with respect to a frame of reference. Poisson's ratio of the homogeneous material is denoted by ν. The material described by (5) is weak in shear, since the shear modulus G vanishes, if either E_1 or E_2 is equal to zero. Since the relationship between these parameters and the material stiffness is very similar to the one obtained by the rectangular microhole model, this approach is also successful in topology optimization. Moreover, due to its simplicity this model can be easily applied to other mechanical problems, such as slab and shell structures. The quality of the material approach chosen is verified by the following example.

The design problem is to find the structural layout of maximum stiffness in a plane design space. The mass in the design space is restricted. The structural situation is shown in Fig. 5. At the beginning of the optimization process the design space consists of equally distributed material. The initial material data are given in Table 1. The orientation Θ of the orthotropic material refers to the x-axis of the given coordinate system in Fig. 5. Firstly, a mesh I parallel to the edges of the design space is generated and the material distribution for the isotropic direct approach (4) and for the proposed orthotropic approach (5) is optimized. Although there are some differences in detail, both approaches lead to nearly the same result. However, the main shortcoming of the isotropic approach appears if another mesh is used to solve the design problem. In the second mesh II only the orientation of the elements is changed by 45°. In the case of the isotropic approach a totally different result is obtained. In contrast to this, the orthotropic approach essentially leads to the same result obtained before.

Table 1. Initial material data for comparison between isotropic and orthotropic approaches

Initial data	Mesh I	Mesh II
Number of elements	216	400
Isotropic approach		
$\overline{\chi}^{\circ}$	0.4375	0.229
μ	3.0	3.0
Orthotropic approach		
$\overline{\chi}_1^{\circ} = \overline{\chi}_2^{\circ}$	0.25	0.1225
Θ	0	0
μ	3.0	3.0

6 Mesh refinement in topology optimization

If the analysis model needs to be adapted to the present stage of optimization in each cycle, it is obvious to use mesh refining techniques analogous to the methods of h-type adaptivity

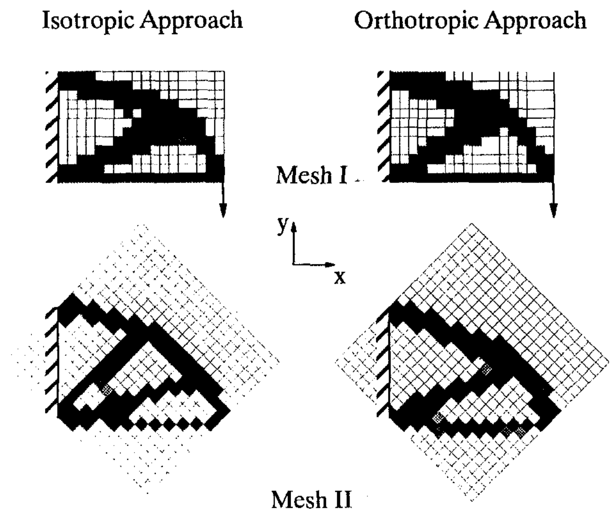


Fig. 5. Isotropic and orthotropic material approaches

used in the finite element method to reduce the error in the energy norm. In topology optimization the objective of mesh refinement in the analysis model is to decrease altogether the number of finite elements in the analysis model and, therefore, the number of active optimization variables, on the one hand, and to improve in detail the clearness of the structural layout, on the other hand. In the present study the procedure applied is evident. In domains of the design space, where there is no material and, therefore, no stiffness, i.e. $\overline{\chi}_i \approx 0$, a coarse parametrization is used. However, in domains of the design space, where the structural layout is not clear, i.e. $0 < \overline{\chi}_i < 1$, a fine finite element mesh is generated. Moreover, areas with $\overline{\chi}_i \approx 1$ are refined, since there new holes of small size can emerge.

The local refinement of the finite element mesh is controlled by the material distribution in the design model. Assuming that the areas of the patches in the design model are of equal size, the local refining indicator η is defined with respect to each design patch by

$$\eta = \begin{cases} \alpha(\overline{\chi} - \overline{\chi}_{\lim})^2 + \eta_{\max} & \forall\, \overline{\chi} < \overline{\chi}_{\lim} \\ \eta_{\max} & \forall\, \overline{\chi} \geq \overline{\chi}_{\lim} \end{cases}, \qquad (6)$$

with

$$\alpha = -\frac{\eta_{\max} - \eta_{\min}}{\overline{\chi}^2_{\lim}},$$

where $\eta_{\max}$ and $\eta_{\min}$ are upper and lower bounds of the refining indicator. Above the limit value $\overline{\chi}_{\lim}$ the refining indicator is set to the upper bound $\eta_{\max}$. With respect to this distribution of refining indicators a new quadrilateral mesh is generated by a full automatic mesh generator based on the advancing front method (Peraire *et al.* 1987). The size of the finite elements in the analysis model is controlled by the refining indicator in conjunction with the size of the patches in the design model. This means that for a refining indicator $\eta = 1.0$ an element is generated, whose size is equal to the size of the design patch. For $\eta > 1.0$ the finite element generated is smaller, for $\eta < 1.0$ larger than the corresponding design patch.

The advantages and disadvantages of the proposed method are discussed by a simple example. The structural situation is shown in Fig. 6. In a plane rectangular design

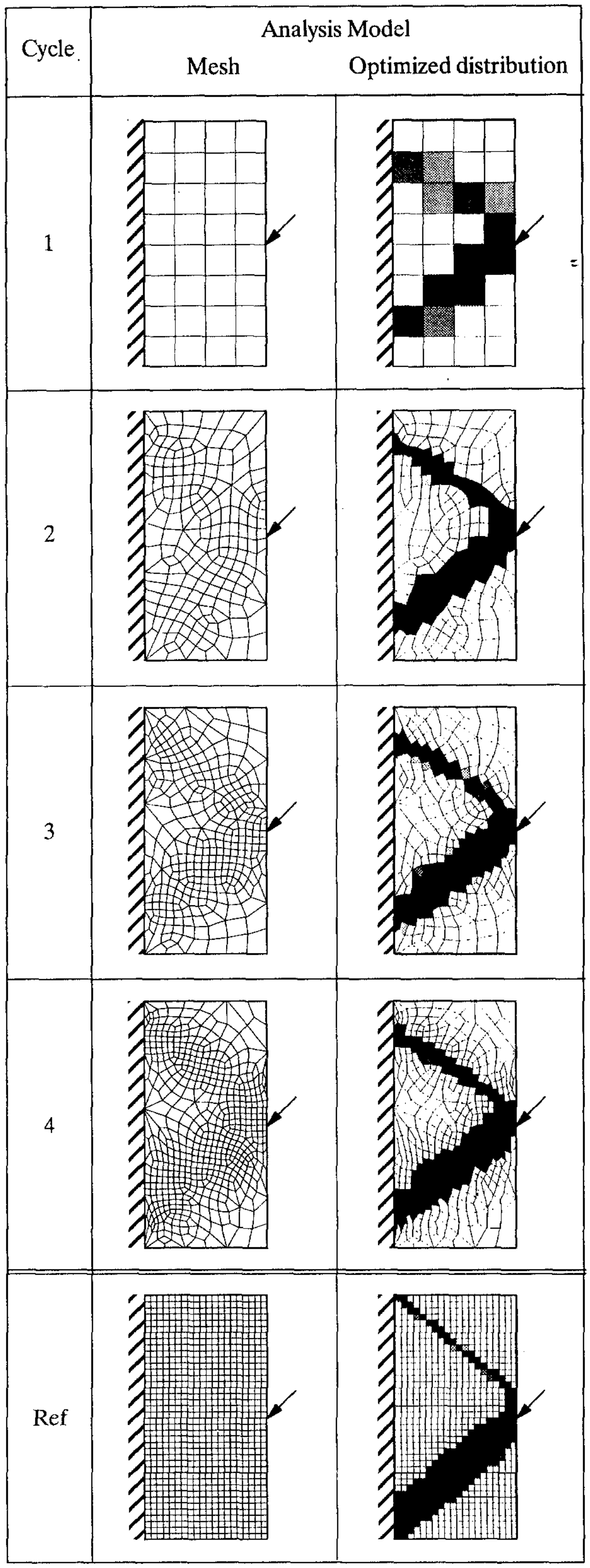


Fig. 6. Mesh refinement in topology optimization

space a concentrated load applied under 45° must be transferred to the opposite clamped edge. The structural layout of

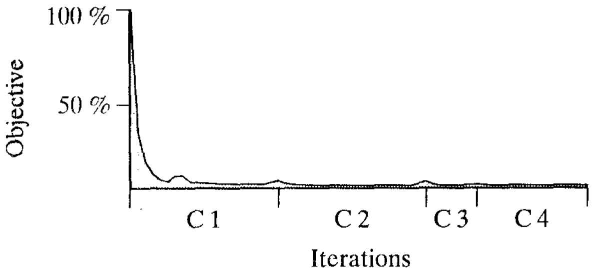


Fig. 7. Mesh refinement in topology: history of objective w.r.t. initial objective value

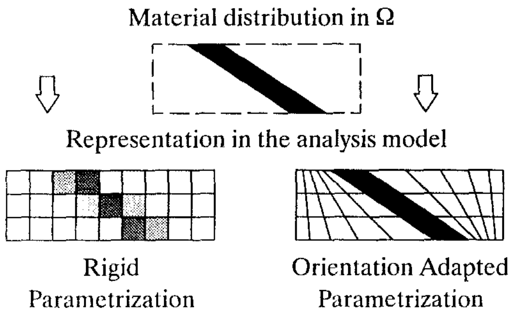


Fig. 8. Parametrized material distribution

maximum stiffness must be found restricting the mass to 25 percent of the maximum possible mass in the design space. The design model consists of 800 square elements. Firstly, a coarse uniform mesh of 32 eight-node plane stress elements is generated. At the beginning of the first design cycle the analysis model consists of equally distributed material. Based on the orthotropic approach ($\mu = 2.5$) the optimum material distribution is approximately determined using an optimality criteria method, which corresponds to the algorithm by Bendsøe and Kikuchi (1988). Optimality criteria methods can be also used for several types of constraints (Rozvany *et al.* 1993). The optimization step is finished, if the difference between the objective values of two successive iterations is less than 0.1 percent. The improved material distribution is remapped onto the design model. Based on the present material values $\overline{\chi}$ of the patches in the design model local refining indicators η for each patch are determined by (6) with $\overline{\chi}_{\text{lim}} = 0.1$ and $\eta_{\text{max}} = 0.6$. Controlled by these local refining indicators η a new finite element mesh is generated for the second cycle. As shown in Fig. 6 the void areas in the right lower and upper corners as well as the void area in the left centre of the design space are meshed coarsely. Structural domains with $\overline{\chi}_i \approx 1$ are refined. Consequently, in the following optimization step the number of active design variables can be reduced in comparison to a uniform fine mesh and the structural layout can be determined in detail, where it is necessary.

After a new mesh is generated, the optimized material distribution of cycle 1, which is saved in the design model, is mapped onto the new analysis model. The mapped material distribution serves as basis for the next optimization step. To obtain an increasingly detailed structural layout, the adaptive procedure is repeated, increasing gradually the maximum refining indicator until the size of the smallest finite elements in the analysis model is equal to the size of the design patches. In the final cycle the resolution of the structural layout in the analysis model corresponds to the resolution in the design model. The finite element mesh and the optimized material distribution of each analysis model are shown in Fig. 6. The number of finite elements in each mesh, the number of iterations used to optimize the material distribution and the maximum refining indicator as well as the refining limit are listed in Table 2. The minimization of the strain energy in the design space, i.e. the maximization of the structural stiffness, is shown in Fig. 7. At the end of the final cycle the objective, i.e. the strain energy, is reduced to $z_{\text{opt}} = 1.12$ percent referred to the initial design.

Table 2. Mesh refinement in topology optimization: iteration data

Cycle	1	2	3	4
Number of elements	32	194	317	524
Number of iterations	19	19	6	16
Max. refinement η_{max}	-	0.6	0.8	1.0
Refinement limit $\overline{\chi}_{\text{lim}}$	-	0.1	0.25	0.25

Refining the mesh only where it is necessary, the structural layout can be determined efficiently and in detail. For comparison, if the design problem is solved in only one cycle using directly the uniform fine discretization of the design model and the same optimization algorithm as for the adaptive procedure, on the one hand, almost the same result ($z_{\text{ref,opt}} = 1.09$ percent) with the same resolution is obtained in 36 iterations (Fig. 6). On the other hand, the numerical effort is nearly twice as high in comparison to the adaptive method using a mesh refining technique. The numerical effort e_{num} is estimated by the sum over all cycles n_c of the product of the number of design variables n_{dva} and the number of iterations n_{iter},

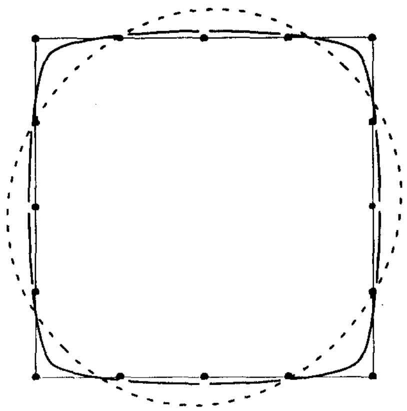
$$e_{\text{num}} = \sum_{n_c} n_{\text{dva}} n_{\text{iter}} \, . \qquad (7)$$

However, this example also shows important shortcomings of the applied procedure. One problem is that the convergence of the optimization process is disrupted, generating new meshes for the analysis model and mapping material data from the old mesh onto the new one as shown in Fig. 7. In general, the number of iterations needed in each cycle does not considerably decrease in the course of the optimization process, although the optimized material distribution of the previous cycle is used as a basis for the next cycle. Furthermore, avoiding serious mesh distortion, the size of the finite elements from coarse to fine meshes should only change smoothly. Consequently, a local refinement yields large areas with small finite elements, i.e. also void areas have a fine mesh and the number of optimization variables increases unnecessarily. In particular, since only the size of the finite elements is adapted to the material distribution, the orientation of the elements is arbitrary. Although the optimization results of the orthotropic approach are almost independent of the element orientation, a detailed layout can only be determined if the finite elements of the analysis model are aligned with respect to the material distribution as shown in Fig. 8. Using a material description an arbitrary material distribution can only be described exactly, if *size* and *orientation* of the finite elements together are adapted. To overcome these shortcomings the following method is introduced.

7 ATO - Adaptive topology optimization

Since optimum topology influences optimum shape and vice versa, it is in general important to determine the structural layout in detail. Therefore, size and orientation of the finite elements of the analysis model are adapted to the material distribution of the design model by the following technique, which will be briefly outlined.

- Points of isoline



Approximation

- - - - by 2 Bézier splines : err = 33.8%

—— by 4 Bézier splines : err = 16.9%

Fig. 9. Adaptive approximation of a polygon

The fundamental idea of the proposed technique is to align the finite elements to isolines of the material distribution. First, based on the material distribution of the design model, the isolines for one or more levels are determined and approximated by cubic or Bézier splines. Domains with density values below a certain threshold value are considered as voids and are therefore neglected in the following optimization step. Domains with density values above this threshold value are considered as potential structural areas and are carried on being analysed and optimized. In addition, the local mesh refinement discussed before is possible in the remaining domains. Before the method is illustrated and verified by a few examples, some aspects are discussed in more detail.

7.1 Adaptive approximation of isolines

Assuming an arbitrary material distribution in the design model, in a first step the elemental data of the design patches are transferred to nodal data by an averaging algorithm. The points of one isoline are determined on the edges of the design patches by linear interpolation of the nodal density values. To obtain the shape of the isolines these points can simply be connected by polygons or used as points of a cubic or Bézier approximation (Farin 1988) leading to smooth outlines. A cubic or a Bézier parametrization of the isolines is not only advantageous for the quality of the finite element meshes and, therefore, for the quality of the results of the following optimization step; it also provides the possibility to use the design model of this topology optimization procedure in conventional shape optimization.

The parametrization of the isolines by shape functions leads to a quadratic optimization problem. If $\mathbf{p}_i$ denotes the points of the isolines and $\mathbf{q}_i$ denotes the corresponding points of the spline, the coordinates of the control points of the approximation splines can be obtained as a result of the following minimization problem:

$$\sum_i \| \mathbf{p}_i - \mathbf{q}_i \| \rightarrow \min , \tag{8}$$

where $\| \quad \|$ defines the Euclidian norm. Since $\mathbf{q}_i$ can be expressed in terms of the control points $\mathbf{r}$ of the approximation splines,

$$\mathbf{q}_i = \sum_{n=1}^{4} \Phi_n(t_i)\mathbf{r}_n , \quad j = 1,2,3 , \tag{9}$$

which leads to

$$\sum_{k=1}^{3i} (b_k - \mathbf{A}_k \mathbf{r})^2 \rightarrow \min , \tag{10}$$

where the matrix $\mathbf{A}$ contains the values of the shape functions $\Phi(t_i)$. Different shape functions Φ must be applied depending on which type of approximation is used. The local coordinates t_i of each point i are calculated by

$$t_i = \frac{\ell_i}{\ell_n} , \quad \ell_i = \sum_{j=1}^{i} \| \mathbf{p}_{j-1} - \mathbf{p}_j \| . \tag{11}$$

The solution of the quadratic minimization problem yields a system of linear equations

$$\mathbf{A}^T \mathbf{A} \mathbf{r} = \mathbf{A}^T \mathbf{b} , \tag{12}$$

which can be solved very efficiently by Householder transformation. The approximation algorithm begins with only one spline and determines the optimum location of the control points. If the approximation error, which can be calculated by

$$\text{err} = \frac{1}{\ell_n} \sum_i \| \mathbf{p}_i - \mathbf{q}_i \| , \tag{13}$$

is larger than a given limit, the number of approximation splines is increased and the procedure is repeated. To obtain smooth isolines C1-continuity between adjoining splines must be required. This results in a dependency of certain control points $\mathbf{r}_D$ from independent control points $\mathbf{r}_i$, which can be expressed by a linking rule

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_D \\ \mathbf{r}_\ell \end{bmatrix} = \begin{bmatrix} \mathbf{r}_D^\circ \\ \mathbf{r}_\ell^\circ \end{bmatrix} + \begin{bmatrix} \mathbf{L} \\ \boldsymbol{\ell} \end{bmatrix} \mathbf{r}_\ell , \tag{14}$$

where $\mathbf{r}^\circ$ denotes the constant parts of $\mathbf{r}$, $\mathbf{L}$ is the linking matrix and $\boldsymbol{\ell}$ the unit matrix. Consequently, the approximation algorithm must not be changed due to C1 requirements. Only the composition of the matrix $\mathbf{A}$ in (12) must be modified. In Fig. 9 the result of this adaptive approximation procedure is shown for a closed isoline. The polygon describing a square with 16 points is approximated by 2 and by 4 Bézier splines. The more splines that are used, the more the approximation error decreases.

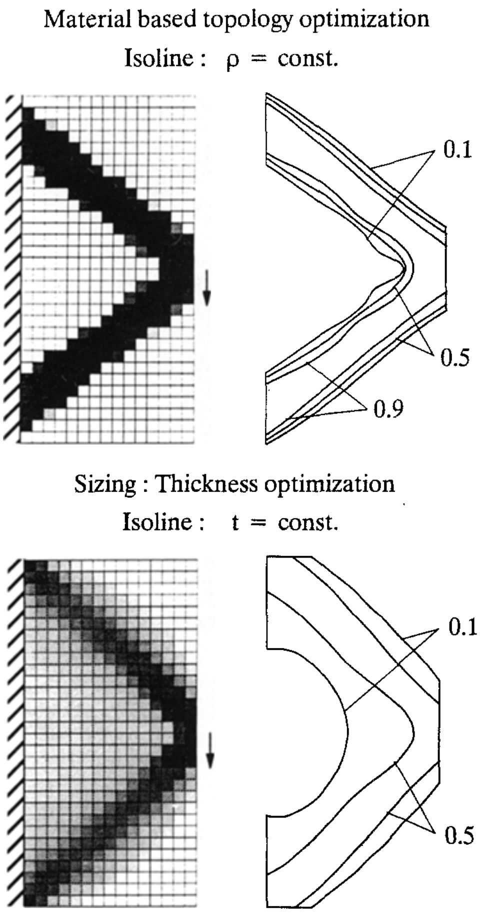


Fig. 10. Material distribution for topology and thickness optimization

7.2 Neglecting void domains

An important aspect of the proposed method is that void areas can be neglected in the analysis model. In this way the number of optimization variables is considerably reduced, and it is possible to confine the optimization process to nonvoid areas. Areas with density values below a certain threshold level ρ_{lim} are neglected. Since in contrast to usual sizing problems, such as thickness optimization, the material gradients between void and nonvoid areas are large in topology optimization, the determination of this threshold level ρ_{lim} is not problematic. As shown in Fig. 10, the isolines of a material distribution generated by topology optimization are close together. Therefore, it does not matter which level is chosen for the threshold value ρ_{lim}, as long as it is small enough,

$$\frac{\rho_{\text{lim}}}{\rho^{\circ}} < 0.1 \dots 0.5 . \tag{15}$$

Furthermore, neglecting void areas does not lead to an irreversible optimization process within the proposed method, since the original optimization variables are kept in the design model. This means that areas that are assumed to be void in one cycle are not automatically neglected in all following cycles. Instead of this, an area once neglected can be analysed and optimized in the following cycles, if the optimum structure seems to be in the corresponding part of the design space. Consequently, it is possible that material can accumulate again in already void domains. This feature of ATO is explained in detail and verified by the example in Section 7.5.

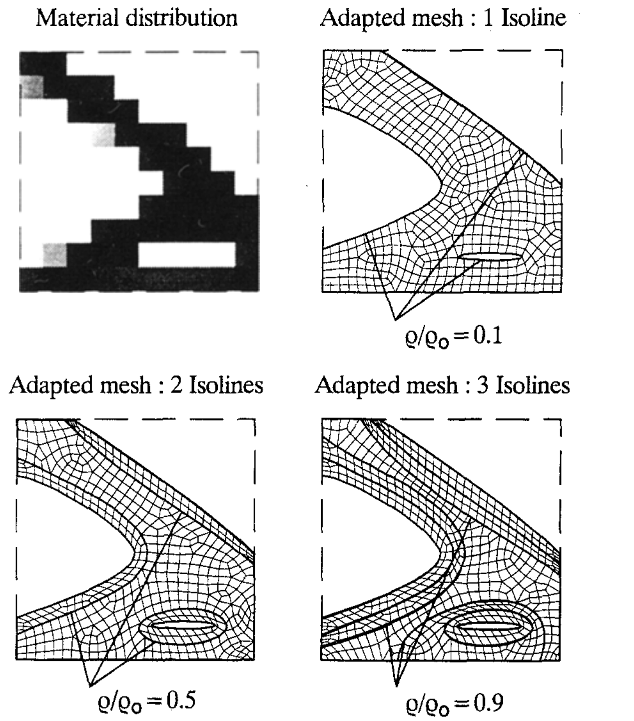


Fig. 11. Mesh refining by multiple isolines

In the present study the following procedure is used to select nonvoid areas. After the isolines of one or more levels are approximated by cubic or Bézier splines, the design patches are assigned to the generated isosurfaces. Two criteria are available to decide whether an isosurface will be neglected or not. On the one hand, the decision depends on the maximum density value of each isosurface. If this limit is smaller than the threshold value, the related domain of the design space is assumed to be void. On the other hand, the criteria are directed to an average density of the isosurface. Since it is often not intended to generate structures with holes below a certain size, the minimum area of neglected domains can additionally be set.

7.3 Mesh refinement and orientation

Local mesh refinement by refining indicators is possible in this method as well. In addition, the present method provides a further possibility to generate local mesh refinements and to improve the orientation of the finite elements inside the mesh. As shown in Fig. 11, due to the approximation of the structure by isolines, the finite element mesh is well-oriented in the periphery of the analysis model. If not only one but more isolines are used to adapt the finite element mesh to the material distribution, it is possible to fit the size and the orientation of elements also inside the analysis model. Consequently, the finite element meshes are built up in layers, which can be processed locally by the optimization algorithm. However, the number and the levels of the isolines must be chosen carefully to avoid an excessive number of finite elements, i.e. active optimization variables.

Mesh refinement discussed above belongs only to geometrical criteria. However, controlling the refinement of the finite element discretization, local refining indicators can additionally consider mechanical aspects, e.g. based on the error in the energy norm. Consequently, the number of finite elements of a certain area can be increased, if the structural layout must be determined in more detail or (and) the local error of the finite element analysis exceeds a given limit.

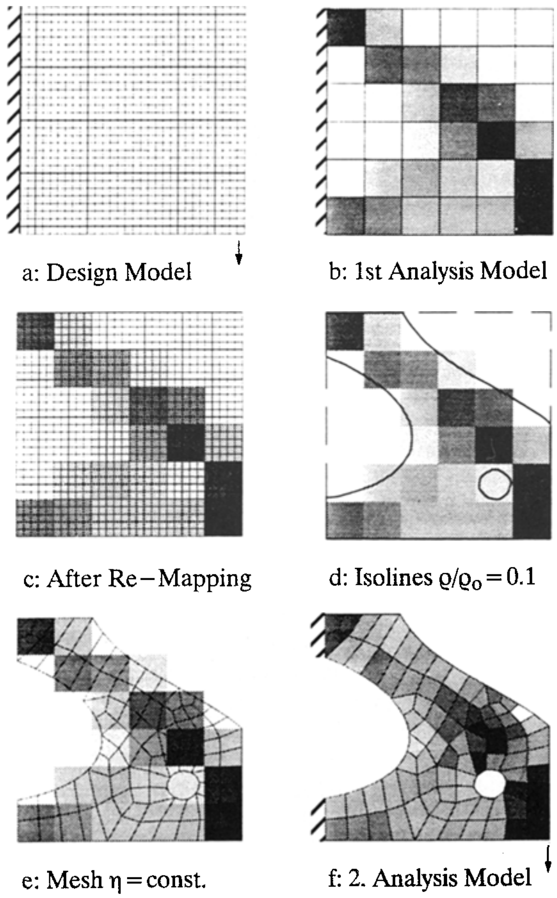


Fig. 12 First example: steps during the 1st cycle

7.4 A first example

The following example is chosen to illustrate the procedure of the proposed method. A square wall structure, which is identical to the design space, is clamped on its left edge as shown in Fig. 12a. The structure is loaded by a vertical load in the lower right corner. The objective of the optimization problem is to find the structural layout of maximum stiffness. The available mass is restricted to 25 percent of the maximum possible mass in the design space. The corresponding design model is discretized by 900 square design patches.

At the beginning of the optimization process the design space consists of equally distributed material using the orthotropic approach discussed before ($\mu = 2.0$). Consequently, a uniform finite element mesh is generated for the first analysis model. The optimized material distribution based on this discretization is shown in Fig. 12b. The material distribution problem was solved by the same optimality criteria method used in the example of Section 6. The optimization step is finished if the change of the objective is less than 0.1 percent. After the material distribution is remapped onto the design model (Fig. 12c), the isolines for a density ratio ρ/ρ_o of 10 percent are determined allowing an interactively chosen approximation error e_{app} of 150 percent (Fig. 12d). Since the discretization of the analysis model in the first cycle is very rough, this high approximation error is necessary to generate smooth surfaces. Neglecting void domains, i.e. areas with a maximum density ratio $(\rho/\rho_o)_{\max}$ below 20 percent are assumed to be void, the remaining effective design space is remeshed. A uniform refining indicator $\eta = 0.33$ is used (Fig. 12e). Finally, a new analysis model is built up mapping the material distribution from the design model onto the new mesh, updating load and support conditions and defining the new optimization problem (Fig. 12f).

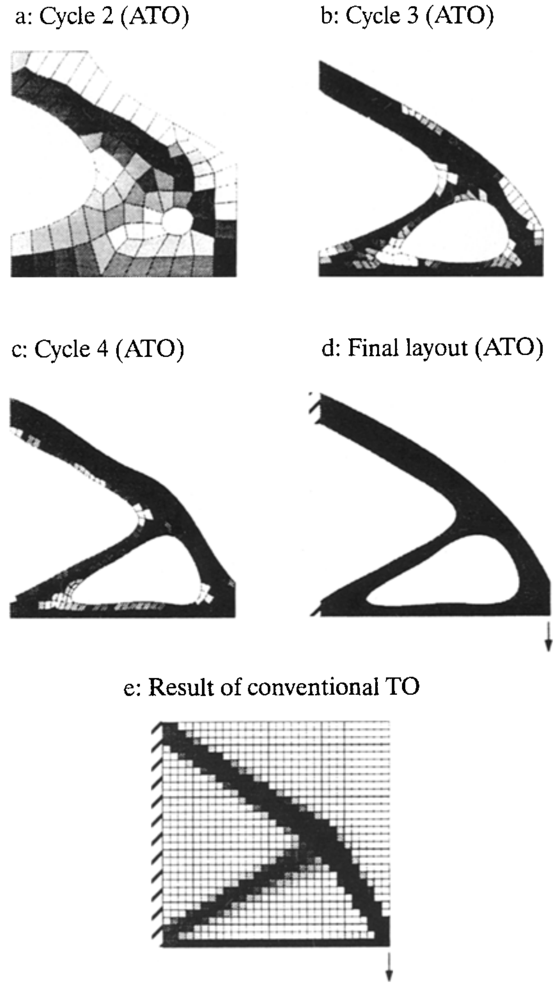


Fig. 13. First example: optimized material distribution

The optimized material distributions of the following cycles and the final layout, i.e. the analysis model generated on the basis of the result of cycle 4, are shown in Figs. 13a-d. The corresponding iteration data are given in Table 3. The minimization of the objective, i.e. the strain energy in the effective design space, in the course of the optimization process can be seen in Fig. 14. In the final layout the objective is reduced to 4.31 percent referred to the design space filled up with equally distributed material. Using ATO the structural layout can be found directly and efficiently. The number of finite elements can be reduced significantly neglecting void domains without distinctly disturbing or influencing the optimization process. The approximation of the effective design space leads automatically to a smooth and exactly defined structural layout. However, even in ATO, as mentioned before, the generation of new meshes slightly disturbs the convergence of the optimization process.

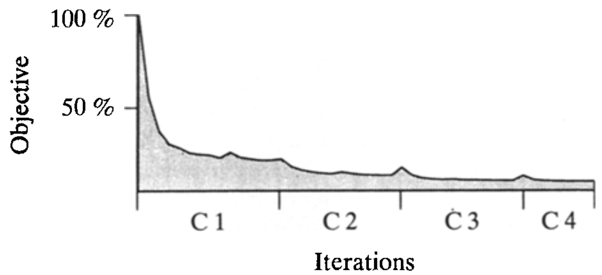


Fig. 14. First example: history of objective w.r.t. initial objective value

To compare the result obtained by ATO the design problem of the present example is solved by conventional topology optimization. The same optimality criteria method is used as applied in ATO before. For the conventional procedure the discretization of the design model is directly used to solve the material distribution problem (Fig. 13e). Based on this uniform fine discretization, the objective can be reduced to 6.24 percent in 13 iterations. If the numerical effort is estimated by (7), it is obvious that by using ATO, not only the quality of the optimization result can be improved, but also the numerical efficiency is increased considerably. In the present example, the numerical effort is decreased by more than 50 percent using ATO in comparison to the conventional procedure with the same resolution.

Table 3. First example: iteration data

Cycle	1	2	3	4	FL
Number of elements	36	85	197	225	-
Number of iterations	14	12	12	8	-
Isoline: ρ/ρ_o	-	0.1	0.2	0.3	0.4
Approx. error e_{app} %	-	150	100	75	50
Void domain $(\rho/\rho_o)_{max}$	-	0.2	0.4	0.6	0.8
Refining indicator η	-	0.33	0.67	0.83	1.00

7.5 Punched plate

The development of the structural layout in the previous example is straightforward. The accumulation of mass in the design space converges directly to the final layout. Once a domain is neglected, it never again becomes part of the effective design space in the following cycles. Neglected domains only expand in the course of the optimization procedure. Therefore, the present example is chosen to discuss qualitatively, how once neglected areas are again rematerialized in following cycles.

The design space is a rectangular plate, which is partly clamped on its left edge and partly loaded on its right edge (Fig. 15a). A design model consisting of 2 × 960 design patches is chosen. Due to symmetry of the problem only one half of the structure is analysed and optimized. The objective of the design problem is to find the structural layout of maximum stiffness, while the available mass in the design space is restricted. In contrast to the examples presented earlier, at the beginning of the optimization process the design space is not filled up with equally distributed material. Instead, a punched plate is chosen as the initial design. The hole inside the plate, as well as the areas along the upper and lower edges are firstly neglected. Since the punched plate consists of a material of maximum density, the first optimization step is skipped. Consequently, the material distribution of the initial analysis model is directly transferred into the design model. To introduce some kind of fuzziness, and, therefore, the possibility for variation in the following cycle, the material distribution in the design model is smoothed averaging elemental data of the design patches to nodal data and back to elemental data. The smoothing factor s_{mat} indicates how often this smoothing loop is passed. The degree of smoothing can be chosen according to how much the effective design space should be enlarged. The more the material distribution is smoothed, the larger the effective design space becomes, assuming that the density ratio (ρ/ρ_o) of the isolines is sufficiently small. Another technique to modify the effective design space is to allow larger errors for the approximation of the isolines.

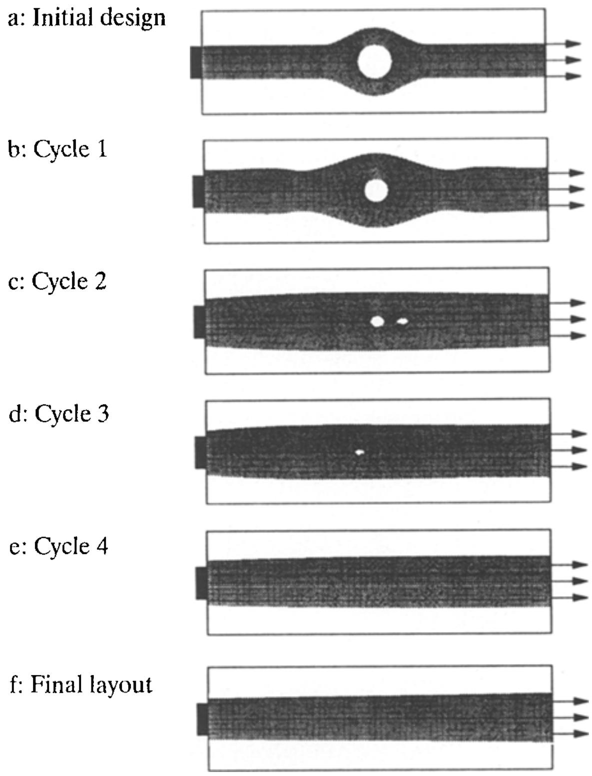


Fig. 15. Punched plate: effective design spaces in the course of the optimization process

Both techniques are used to generate a new design model. The effective design space of each cycle, the final layout and the iteration data are given in Figs. 15b-f and Table 4. In the course of the optimization process the structural layout converges more and more. Consequently, the fuzziness determining the effective design space can be gradually reduced and, therefore, the numerical efficiency can be increased. In the final cycle no smoothing is necessary and the approximation error for the isolines is small. A clearly defined structural layout with smooth boundaries is generated.

7.6 Beam-like structure

Up to now ATO seems "only" to provide the possibility to generate structures with smooth boundaries and to decrease the numerical effort. However, the following example shows that ATO is able to include additionally the interaction between optimum topology and the corresponding shape of a structure and vice versa. In an extended but still conventional version of topology optimization (e.g. Olhoff *et al.* 1991; Hinton and Sienz 1994), first the optimum material distribution in a design space for a certain design problem is found. This result is transferred interactively into a basic design for a following shape optimization step. The optimum shape is determined by traditional boundary variation techniques. Consequently, once the conceptual design is determined, variation of topology is no longer possible in the final shape optimization step, even if a modified shape necessitates a change of topology to obtain the optimum structural layout. As the following example shows, this shortcoming can be overcome using ATO.

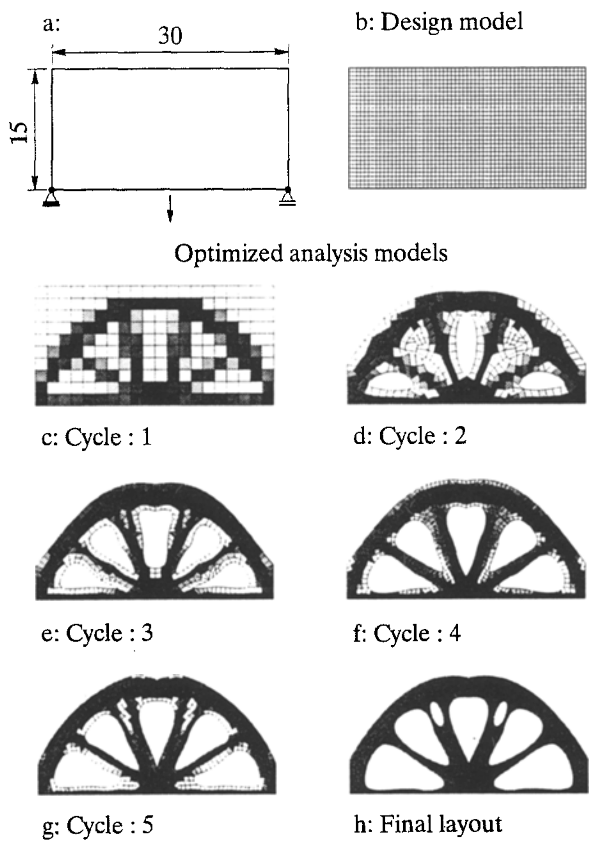


Fig. 16. Topology optimization of a beam-like structure

A rectangular wall structure, which is identical to the design space, is fixed on the lower left corner and vertically supported on the lower right corner, as shown in Fig. 16a. The structure is loaded by a vertical load in the centre of the lower edge. The objective of the optimization problem is to find the structural layout of maximum stiffness. The available mass is restricted to 40 percent of the maximum possible mass in the design space. Due to symmetry of the problem, only one half must be analysed and optimized. At the beginning of the optimization process the design space consists of equally distributed material using the orthotropic approach discussed before ($\mu = 2.0$). The design model of the design space is discretized by 2×900 square patches (Fig. 16b). The linear finite element analysis is carried out by 2×2 reduced integrated, eight-node, isoparametric plane stress elements. The material distribution problem is solved by the optimality criteria method used in the examples before. The optimized material distribution of each cycle is shown in Figs. 16c-g. The iteration data are listed in Table 5, where the number of finite elements of one half of the structure is denoted by n_{ele}, the values of the objective with respect to the initial design at the end of each optimization step by z_{opt}, the required accuracy by acc and the needed number of iterations by n_{iter}.

Table 4. Punched plate: iteration data

Cycle	1	2	3	4	FL
Number of elements	133	288	220	257	-
Number of iterations	13	17	13	17	-
Isoline: ρ/ρ_0	0.1	0.2	0.3	0.4	0.5
Approx. error e_{app} %	100	100	100	75	50
Smoothing factor s_{mat}	3	2	1	0	0

Table 5. Beam-like structure: iteration data

Cycle	1	2	3	4	5
n_{ele}	100	202	291	402	365
n_{iter}	24	17	12	10	29
z_{opt} %	12.0	10.4	9.9	10.2	9.6
acc	10^{-3}	10^{-3}	10^{-4}	10^{-4}	10^{-5}

Based on a first indistinct result of cycle 1, the analysis model is adapted to the optimized material distribution. In the following optimization cycles the contours of the structure become increasingly clear. Until cycle 4 the topology of the structure does not change and only the shape of external and internal boundaries is determined in detail. However, since topology and shape depend on each other, the topology of the structure changes in cycle 5, improving the objective of the design problem. This would not be possible if the conventional procedure were used. In contrast, using ATO a variation of topology and shape can be carried out during the entire optimization process.

7.7 Slab structure

In a last example, it is shown that ATO can not only be applied to plane stress problems, but also to slab structures in its present stage of development. For a square design space clamped on two opposite edges and loaded in its centre, the structure of minimum weight must be found (Fig. 17a). The maximum displacement of the loaded node is restricted. Since only the displacement of the loaded node is constrained, i.e. the minimum stiffness of the structure is given, this design problem corresponds to a maximum stiffness problem where the mass for the structure is restricted. Due to symmetry of the problem, only one quarter of the design space must be analysed and optimized. The linear finite element analysis is carried out by 2×2 reduced integrated, eight-node, isoparametric plate elements.

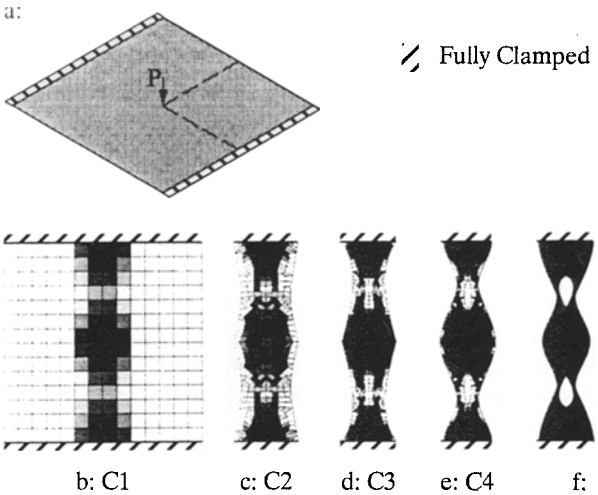


Fig. 17. Topology optimization of a slab structure

At the beginning of the optimization process, the design space consists of equally distributed material of maximum density. For the analysis model in the first cycle, a coarse uniform mesh is generated and the material distribution is optimized. Analogously to the examples discussed previously, the structural layout is determined in detail during the following cycles. The optimized material distribution of each cycle is shown in Figs. 17b-e, the final layout in Fig. 17f. The iteration data are given in Fig. 18 and Table 6. In this optimization procedure the maximum number of iterations in each optimization step is set to 20.

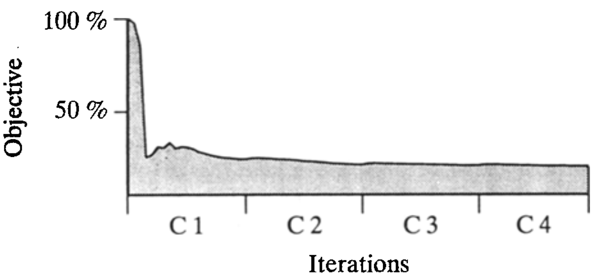


Fig. 18. Slab structure: history of objective w.r.t. initial objective value

Table 6. Slab structure: iteration data

Cycle	1	2	3	4	FL
Number of elements	49	94	127	139	-
Number of iterations	20	20	20	20	-
Isoline: ρ/ρ_0	-	0.1	0.2	0.3	0.4
Approx. error e_{app} %	-	150	100	50	50
Void domain$(\rho/\rho_0)_{max}$	-	0.2	0.4	0.5	0.5

Finally, this example is used to present an overview of different optimization methods to solve a design problem. Firstly, a pure boundary variation technique is applied based on the design element concept introduced in structural optimization by Braibant and Fleury (1986). The free edge of a quarter of the design space is parametrized by a four-node Bézier spline. Considering the continuity of the symmetric problem only 3 design nodes can be varied independently during the optimization process. The result of the shape optimization is shown in Fig. 19a. Only the shape of the structure has changed, but not its topology. To provide the possibility for a change of topology as well, the design problem is solved by conventional topology optimization. However, the result shown in Fig. 19b contains a not clearly defined structural layout with jagged boundaries. Combining both previous techniques it is possible to obtain results with a new topology and, at least, smooth boundaries, which are present in the initial structural layout. For this a variable shape and a variable density distribution are introduced. However, the result of the combined optimization method shown in Fig. 19c still contains not clearly defined structural domains around the additionally inserted holes. This problem can be overcome using ATO. The final result of the adaptive optimization process discussed before is shown in Fig. 19d. Consequently, concerning the quality of the optimization results and the numerical effort ATO is an efficient method for optimizing the topology and shape of structures. In contrast to the previous examples, these optimization problems were solved by an SQP method (Schittkowski 1981).

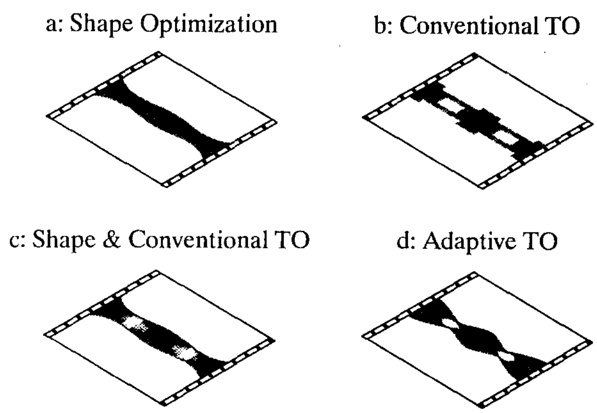


Fig. 19. Comparison of optimization methods

8 Conclusion

Material-based topology optimization provides a general tool in structural design. However, since its results are usually lacking in clearness and smoothness, the optimized layouts can often only be used as conceptual design idea instead of a clearly defined form of the structure. Therefore, the application of adaptive techniques in topology optimization is discussed and a method to generate smooth and well-defined two-dimensional structures is introduced.

The proposed methods are based on a division of design and analysis model. In the design model the structural layout is described by a pixel-like scanning, as usually used in material-based topology optimization. These design patches are linked to the finite elements of the analysis model. Their material parameters are the active optimization variables of the corresponding cycle. The parametrization of the analysis model is adapted to the material distribution of each design cycle. Applying a simple mesh refining technique as used in h-type adaptive finite element analysis, on the one hand, the number of optimization variables can be reduced, since void domains of the design space are meshed coarser than non-void domains. On the other hand, the optimization results still contain jagged boundaries. Therefore, a method is proposed that adapts size and orientation of the finite elements in the analysis model to the material distribution in the design model. Consequently, a smooth and clearly defined structural layout can be generated by topology optimization. In addition, since in this method void areas are neglected during the optimization step, the number of active optimization variables is considerably reduced.

The features of the proposed method are demonstrated by optimizing the layout of two-dimensional elastic structures with respect to maximum stiffness by an optimality criteria method. However, since adaptive topology optimization is also embedded in a mathematical programming scheme, a broad range of design problems can be solved efficiently. Moreover, the applied approximation technique of isolines allows one to include conventional boundary variation techniques to improve once more the quality of the optimization results and the numerical efficiency of the optimization procedure. Material-based topology optimization and shape optimization can be used sequentially or simultaneously. The implementation of shape optimization into this concept is currently underway.

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