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Generalized topology design of structures with a buckling load criterion

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Abstract Material based models for topology optimization of linear elastic solids with a low volume constraint generate very slender structures composed mainly of bars and beam elements. For this type of structure the value of the buckling critical load becomes one of the most important design criteria and so its control is very important for meaningful practical designs. This paper tries to address this problem, presenting an approach to introduce the possibility of critical load control into the topology optimization model.

Using the material based formulation for topology design of structures, the problem of optimal structural reinforcement for a critical load criterion is formulated. The stability problem is conveniently reduced to a linearized eigenvalue problem assuming only material effective properties and macroscopic instability modes. The respective optimality criteria are presented by introducing the Lagrangian associated with the optimization problem. Based on this Lagrangian a first-order method is used as a basis for the numerical update scheme. Two numerical examples to validate the developments are presented and analysed.

1 Introduction

This work is concerned with the development of an analytical model and a procedure for the computational solution of the topology design problem of a two-dimensional structure with a buckling load criterion.

Recently, the material model for topology optimization of structures using homogenization techniques (Bendsøe and Kikuchi 1988) has been extended to different criteria required by structural design applications (e.g. Bendsøe and Mota Soares 1993).

From these works it has been observed that as the admissible material volume decreases, the method tends to generate very slender structures composed mainly of thin bars and beam type elements. For these structural components the value of the buckling load becomes one of the most important design criteria and so its control is very important for meaningful practical designs. We try to address this problem by presenting an approach to introducing the possibility of buckling load control into the material based topology optimization model.

This article is organized as follows. Section 2 describes the mechanical model, which is a linearized buckling model where the displacement before buckling is assumed to be small and linear elastic. This simple model, although with limited practical applicability in structural analysis, originates a relatively simple structural optimization problem, which can be used as a basis for the development of new topology optimization methods assuming more elaborate mechanical models. This section ends with a description of the finite element model used to solve the problem computationally. In its discrete version the stability problem reduces to a generalized eigenvalue problem whose eigenvalues are the finite element approximations of the buckling loads of the structure.

In Sections 3 and 4 the optimal design problem is stated and the respective optimality conditions presented. Generalized gradients are used to deal with the nonsmooth character of the optimization problem if repeated eigenvalues (buckling loads) occur.

Finally, in Sections 5 and 6 the computational model is described and its validity is assessed in two numerical examples. From these examples it is observed that the final topology obtained can be used to identify practical frame type structures with much higher buckling loads.

2 The stability problem

To introduce the material based formulation consider an elastic structure made of a porous material obtained from the introduction of very small voids (square holes in our case) in an isotropic base material. This will permit a continuous variation of material "density", with high density values representing solid material and low density values representing the existence of voids. The interested reader is referred to the paper by Bendsøe *et al.* (1993) for a complete description of this model applied to compliance optimization of linear elastic structures. Also the work of Rozvany *et al.* (1993) and Mlejnek (1993) present alternative models, not based on porous materials, for compliance optimization problems.

Assuming this type of material, the structure is fixed in the boundary Γ_u and subject to a proportional surface loading in boundary Γ_t (see Fig. 1). By proportional loading it is meant that the applied load is a function of one parameter, i.e. $\bar{\mathbf{t}} = P\mathbf{t}$ where P is the load factor parameter.

Let us increase gradually the load factor P. The displacement remains unique as long as P is below a certain value. When it reaches this value (denoted by $P_{\rm cr}$, critical load factor) the displacement solution is no longer unique. For $P = P_{\rm cr}$ we will look for the possibility of a bifurcation in the solution where two neighbouring equilibrium positions exist corresponding to an infinitesimal increment in the load factor value.

Let $P < P_{\rm cr}$, in this situation we will assume that the structure exhibits linear elastic behaviour with a unique small displacement $\mathbf{u}^{\varepsilon 0}$. In the previous definition the superscript

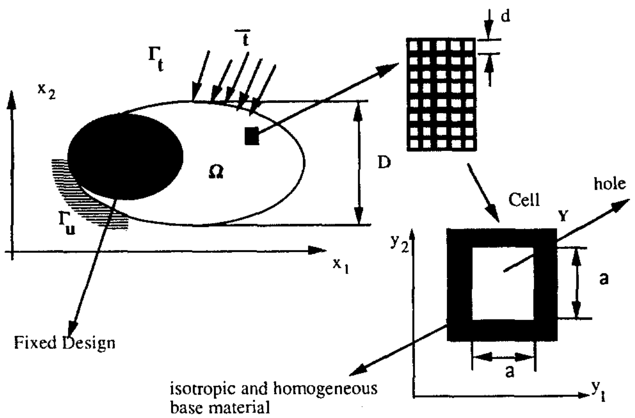


Fig. 1. Nomenclature

ε ($\varepsilon = d/D$ is the microstructure size parameter) identifies the displacement dependence on the material microstructure.

Now let $P = P_{\mathrm{cr}}$. At this point there are two possible infinitely close equilibrium positions: the initial, with displacement $\mathbf{u}^{\varepsilon 0}$ and a secondary, with displacements $\mathbf{u}^{\varepsilon}$. For the latter let us denote the displacement by

$$\mathbf{u}^{\varepsilon} = \mathbf{u}^{\varepsilon 0} + \alpha \mathbf{u}^{\varepsilon 1}, \tag{1}$$

where α is an infinitesimal real parameter and $\mathbf{u}^{\varepsilon 1}$ identifies the displacement in the structure when it shifts from the initial to the secondary equilibrium positions.

These equilibrium positions are characterized by the stationary conditions of the total potential energy functional,

$$\Pi(\mathbf{u}^{\varepsilon}) = A(\mathbf{u}^{\varepsilon}) - R(\mathbf{u}^{\varepsilon}), \tag{2}$$

where $A(\mathbf{u}^{\varepsilon})$ is the elastic energy,

$$A(\mathbf{u}^{\varepsilon}) = \frac{1}{2}\int_{\Omega^{\varepsilon}} \sigma_{ij}(\mathbf{u}^{\varepsilon}) e_{ij}(\mathbf{u}^{\varepsilon}) \mathrm{d}\Omega, \tag{3}$$

and $R(\mathbf{u}^{\varepsilon})$ is the force potential,

$$R(\mathbf{u}^{\varepsilon}) = P\int_{\Gamma_t} t_i u_i^{\varepsilon} \mathrm{d}\Gamma. \tag{4}$$

Assuming a linear elastic material, we have

$$\sigma_{ij}(\mathbf{u}^{\varepsilon}) = E_{ijk\ell} e_{k\ell}(\mathbf{u}^{\varepsilon}), \tag{5}$$

and substituting the constitutive equation (5) in expression (3) we obtain the elastic energy defined as

$$A(\mathbf{u}^{\varepsilon}) = \frac{1}{2}\int_{\Omega^{\varepsilon}} E_{ijk\ell} e_{k\ell}(\mathbf{u}^{\varepsilon}) e_{ij}(\mathbf{u}^{\varepsilon}) \mathrm{d}\Omega, \tag{6}$$

with

$$e_{ij}(\mathbf{u}^{\varepsilon}) = \frac{1}{2}\left(\frac{\partial u_i^{\varepsilon}}{\partial x_j} + \frac{\partial u_j^{\varepsilon}}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_k^{\varepsilon}}{\partial x_j}\frac{\partial u_k^{\varepsilon}}{\partial x_i}\right). \tag{7}$$

Until now no assumptions were made about the dependence of displacement fields on the material microstructure. Obviously to tackle the problem directly, without simplifying assumptions, would be impossible due to the complex shape of the microstructure.

Following the homogenization method hypotheses, based on the local periodicity of the material microstructure, we assume that the displacement fields can be represented as asymptotic expansions in terms of the cell size parameter ε,

$$\mathbf{u}^{\varepsilon 0}(\mathbf{x}) = \mathbf{u}^{00}(\mathbf{x}) + \varepsilon \mathbf{u}^{01}(\mathbf{x},\mathbf{y}) + \varepsilon^2 \mathbf{u}^{02}(\mathbf{x},\mathbf{y}) + \dots, \mathbf{y} = \frac{\mathbf{x}}{\varepsilon}, \tag{8}$$

and

$$\mathbf{u}^{\varepsilon 1}(\mathbf{x}) = \mathbf{u}^{10}(\mathbf{x}) + \varepsilon \mathbf{u}^{11}(\mathbf{x},\mathbf{y}) + \varepsilon^2 \mathbf{u}^{12}(\mathbf{x},\mathbf{y}) + \dots, \mathbf{y} = \frac{\mathbf{x}}{\varepsilon}. \tag{9}$$

In the previous expressions the various expansion terms are functions of the macroscopic variable $\mathbf{x}$ and periodic functions of the microscopic variable $\mathbf{y}$ (see e.g. Guedes and Kikuchi 1990).

One important assumption made in (9) is that the displacement $\mathbf{u}^{10}$, the first term in the bifurcated solution expansion, is only a function of the macroscopic variable $\mathbf{x}$. This implies that the model will detect only macroscopic buckling modes, i.e. no cell buckling. This is justified if one is interested only in the problem of optimal design for critical loads associated with macroscopic modes. If this is not the case, then expansion (9) is not valid since no periodicity assumption could be made for $\mathbf{u}^{\varepsilon 1}$.

Using the above expansions in the definition of $\mathbf{u}^{\varepsilon}(\mathbf{x})$, substituting it into the strain expression (7) and assuming only a nonlinear contribution from the term $\mathbf{u}^{10}(\mathbf{x})$, we obtain

$$e_{ij}(\mathbf{u}^{\varepsilon}) = \frac{1}{2}\left(\frac{\partial u_i^{00}}{\partial x_j} + \frac{\partial u_j^{00}}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i^{01}}{\partial y_j} + \frac{\partial u_j^{01}}{\partial y_i}\right) +$$
$$\alpha\left\{\frac{1}{2}\left(\frac{\partial u_i^{10}}{\partial x_j} + \frac{\partial u_j^{10}}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i^{11}}{\partial y_j} + \frac{\partial u_j^{11}}{\partial y_i}\right)\right\} +$$
$$\alpha^2\left(\frac{1}{2}\frac{\partial u_k^{10}}{\partial x_j}\frac{\partial u_k^{10}}{\partial x_i}\right) + \varepsilon(\dots). \tag{10}$$

Substituting (10) into the potential energy functional, we have

$$\Pi(\mathbf{u}^{\varepsilon}) = \frac{1}{2}\int_{\Omega^{\varepsilon}} E_{ijkm}\left(\frac{\partial u_k^{00}}{\partial x_m} + \frac{\partial u_k^{01}}{\partial y_m}\right)\left(\frac{\partial u_i^{00}}{\partial x_j} + \frac{\partial u_i^{01}}{\partial y_j}\right)\mathrm{d}\Omega -$$
$$P\int_{\Gamma_t} t_i u_i^{00}\mathrm{d}\Gamma + \alpha\left\{\int_{\Omega^{\varepsilon}} E_{ijkm}\left(\frac{\partial u_k^{00}}{\partial x_m} + \frac{\partial u_k^{01}}{\partial y_m}\right)\left(\frac{\partial u_i^{10}}{\partial x_j} +\right.\right.$$
$$\left.\left.\frac{\partial u_i^{11}}{\partial y_j}\right)\mathrm{d}\Omega - P\int_{\Gamma_t} t_i u_i^{10}\mathrm{d}\Gamma\right\} +$$
$$\alpha^2\left\{\frac{1}{2}\int_{\Omega^{\varepsilon}} E_{ijkm}\left[\left(\frac{\partial u_k^{10}}{\partial x_m} + \frac{\partial u_k^{11}}{\partial y_m}\right)\left(\frac{\partial u_i^{10}}{\partial x_j} + \frac{\partial u_i^{11}}{\partial y_j}\right) +\right.\right.$$
$$\left.\left.\left(\frac{\partial u_s^{10}}{\partial x_k}\frac{\partial u_s^{10}}{\partial x_m}\right)\left(\frac{\partial u_i^{00}}{\partial x_j} + \frac{\partial u_i^{01}}{\partial y_j}\right)\right]\mathrm{d}\Omega\right\} + \alpha^3(\dots) + \varepsilon(\dots). \tag{11}$$

Stationarity of the total potential energy equilibrium requires

$$\delta\Pi(\mathbf{u}^{\varepsilon}) = \delta A(\mathbf{u}^{\varepsilon}) - \delta R(\mathbf{u}^{\varepsilon}) = 0. \tag{12}$$

Since $\mathbf{u}^{\varepsilon 0}$ is known, the variation $\delta\mathbf{u}^{\varepsilon}$ is defined as $\delta\mathbf{u}^{\varepsilon} = [\mathbf{v}^{10}(\mathbf{x}) + \varepsilon\mathbf{v}^{11}(\mathbf{x},\mathbf{y}) + \varepsilon^2(\dots)]$ with

$\mathbf{v}^{10} \in V_{\Omega} = \{\mathbf{v} : \mathbf{v}|_{\Gamma_{\mathbf{u}}} = \mathbf{0}\}$, $\mathbf{v}^{11} \in V_{\Omega\times Y} =$

$\{\mathbf{v}(\mathbf{x},\mathbf{y}) : \mathbf{v}|_{\Gamma_{\mathbf{u}}} = \mathbf{0},\ \mathbf{v}$ is Y−periodic in $\mathbf{y}\}$,

and the first variation of the potential energy is given by

$\delta\Pi(\mathbf{u}^{\varepsilon}) =$

$$\alpha\left\{\int_{\Omega^\varepsilon} E_{ijkm}\left(\frac{\partial u_k^{00}}{\partial x_m}+\frac{\partial u_k^{01}}{\partial y_m}\right)\left(\frac{\partial v_i^{10}}{\partial x_j}+\frac{\partial v_i^{11}}{\partial y_j}\right)\mathrm{d}\Omega-\right.$$

$$\left.P\int_{\Gamma_t} t_i v_i^{10}\mathrm{d}\Gamma\right\}+$$

$$\alpha^2\left\{\int_{\Omega^\varepsilon} E_{ijkm}\left[\left(\frac{\partial u_k^{10}}{\partial x_m}+\frac{\partial u_k^{11}}{\partial y_m}\right)\left(\frac{\partial v_i^{10}}{\partial x_j}\frac{\partial v_i^{11}}{\partial y_j}\right)+\right.\right.$$

$$\left.\left.\left(\frac{\partial u_s^{10}}{\partial x_k}\frac{\partial v_s^{10}}{\partial x_m}\right)\left(\frac{\partial u_i^{00}}{\partial x_j}+\frac{\partial u_i^{01}}{\partial y_j}\right)\right]\mathrm{d}\Omega\right\}+$$

$$\alpha^3(\ldots)+\varepsilon(\ldots)=0\,,\quad \forall \mathbf{v}^{10}\in V_\Omega\,,\mathbf{v}^{11}\in V_{\Omega\times Y}\,. \tag{13}$$

Equating to zero each α power term one obtains the equations of equilibrium

$$\int_{\Omega^\varepsilon} E_{ijkm}\left(\frac{\partial u_k^{00}}{\partial x_m}+\frac{\partial u_k^{01}}{\partial y_m}\right)\left(\frac{\partial v_i^{10}}{\partial x_j}+\frac{\partial v_i^{11}}{\partial y_j}\right)\mathrm{d}\Omega-$$

$$P\int_{\Gamma_t} t_i v_i^{10}\mathrm{d}\Gamma=0\,,\quad \forall\, \mathbf{v}^{10}\in V_\Omega\,,\quad \mathbf{v}^{11}\in V_{\Omega\times Y}\,, \tag{14}$$

$$\int_{\Omega^\varepsilon} E_{ijkm}\left[\left(\frac{\partial u_k^{10}}{\partial x_m}+\frac{\partial u_k^{11}}{\partial y_m}\right)\left(\frac{\partial v_i^{10}}{\partial x_j}+\frac{\partial v_i^{11}}{\partial y_j}\right)+\right.$$

$$\left.\left(\frac{\partial u_s^{10}}{\partial x_k}\frac{\partial v_s^{10}}{\partial x_m}\right)\left(\frac{\partial u_i^{00}}{\partial x_j}+\frac{\partial u_i^{01}}{\partial y_j}\right)\right]\mathrm{d}\Omega=0\,,$$

$$\forall\, \mathbf{v}^{10}\in V_\Omega\,,\quad \mathbf{v}^{11}\in V_{\Omega\times Y}\,. \tag{15}$$

Now let us assume that the microstructure size parameter ε is infinitesimal, i.e. the material heterogenities have a characteristic dimension (d) much smaller than the global dimension of the structure (D) (see Fig. 1). Thus taking the limit as $\varepsilon\to 0$ we obtain the homogenized equilibrium equations defining the displacements $\mathbf{u}^{00}$ and $\mathbf{u}^{01}$,
$\mathbf{u}^{00}$ solution of

$$\int_{\Omega} E_{ijk\ell}^{H}\varepsilon_{ij}(\mathbf{u}^{00})\varepsilon_{k\ell}^{0}(\mathbf{v})\mathrm{d}\Omega-\int_{\Gamma_t} t_i v_i \mathrm{d}\Gamma=0\,,\quad \forall \mathbf{v}=0 \text{ in } \Gamma_u\,, \tag{16}$$

where ε_{ij} is defined as,

$$\varepsilon_{ij}(\mathbf{u}^{00})=\frac{1}{2}\left(\frac{\partial u_i^{00}}{\partial x_j}+\frac{\partial u_j^{00}}{\partial x_i}\right)\,, \tag{17}$$

and, denoting $\mathbf{u}^{10}=c^{te}\phi$, the pair (P_{cr},ϕ) solution of the eigenvalue problem

$$\int_{\Omega} E_{ijk\ell}^{H}\varepsilon_{ij}(\phi)\varepsilon_{k\ell}(\psi)\mathrm{d}\Omega+$$

$$P_{\mathrm{cr}}\int_{\Omega} E_{ijkm}^{H}\varepsilon_{km}(\mathbf{u}^{00})\left(\frac{\partial\phi_\ell}{\partial x_i}\frac{\partial\psi_\ell}{\partial x_j}\right)\mathrm{d}\Omega=0\,,$$

$$\forall\psi=0 \text{ in } \Gamma_u\,. \tag{18}$$

In the previous equations $E_{ijk\ell}^{H}$ denotes the homogenized (effective) coefficients of the porous material (see e.g. Guedes and Kikuchi 1990). If the cell material is homogeneous, $E_{ijk\ell}^{H}$ is defined as

$$E_{ijk\ell}^{H}=\mu E_{ijk\ell}-\left(\frac{1}{|Y|}\right)\int_{Y}\left(E_{ijpm}\frac{\partial X_p^{k\ell}}{\partial y_m}\right)\mathrm{d}Y\,, \tag{19}$$

where μ is the material "density" ($\mu=1-a^2$), and the correcting term $\mathbf{X}^{k\ell}(k,\ell=1,2,3)$ is a Y-periodic function, solution of the elastostatic problem

$$\int_{Y} E_{ijpm}\frac{\partial X_p^{k\ell}}{\partial y_m}\frac{\partial v_i}{\partial y_j}\mathrm{d}Y=\int_{Y} E_{ijk\ell}\frac{\partial v_i}{\partial y_j}\mathrm{d}Y\,,$$

$$\forall\, \mathbf{v}-Y\text{-periodic}\,. \tag{20}$$

A detailed derivation of (16)-(18) is presented by Neves (1994) based on the bifurcation model presented by Novozhilov (1953).

3 The topology design problem

The finite element approximations of the equilibrium equations (16) and (18) are obtained using nine-node 2D isoparametric finite elements. This choice is justified by the fact that the four-node element is very rigid to approximate instability modes of frame type structures in two-dimensional elasticity.

Using the finite element approximation, the problem is transformed to the generalized eigenvalue problem

$$\mathbf{K}(\boldsymbol{\mu})\boldsymbol{\phi}-P\mathbf{G}(\boldsymbol{\mu},\mathbf{u})\boldsymbol{\phi}=\mathbf{0}\,, \tag{21}$$

where $\boldsymbol{\mu}=\{\mu_e\}\in\mathbf{R}^M$ denotes the vector of design variables, $\mathbf{u}=\{u_i\}\in\mathbf{R}^N$ is the displacement vector solution of a finite element approximation of a linear elasto-static problem

$$\mathbf{K}(\boldsymbol{\mu})\mathbf{u}=\mathbf{f}\,, \tag{22}$$

and $\mathbf{K}$ and $\mathbf{G}$ denote the stiffness and geometric stiffness matrices, respectively.

The design problem is stated as an optimal reinforcement of a given structure so that its buckling critical load is maximized. To limit the amount of reinforcement material an upper bound on the reinforcement volume is introduced.

Let us assume that the buckling load factors are positive, different from zero and numbered such that $0<P_1\le P_2\le \ldots P_r\le P_{r+1}\ldots$, with the critical load factor defined by $P_{\mathrm{cr}}=P_1$.

With this notation the reinforcement design problem is stated as

$$\max_{\mu_{\min}\le\mu\le 1}\left(\min_r P_r\right)\,, \tag{23}$$

subjected to the volume constraint

$$\sum_{e=1}^{M}\mu_e\int_{\Omega^e}\mathrm{d}\Omega=\overline{V}\,, \tag{24}$$

and to the density bound constraints

$$\mu_{\min}\le\mu_e\le 1\,,\quad e=1,2,\ldots M\,. \tag{25}$$

The problem defined by (23)-(25) is a nonsmooth optimization probelm if $P_{\mathrm{cr}}=P_1$ is a multiple eigenvalue.

To overcome this problem using, formally, the concept of the generalized gradient (see Clarke 1983), the problem is restated using the Rayleigh variational principle as

$$\min_{\boldsymbol{\mu},\mathbf{u}} \max_{\substack{\boldsymbol{\phi}\neq 0\\ \boldsymbol{\phi}\in\mathbf{R}^N}} \frac{\boldsymbol{\phi}^T\mathbf{G}(\boldsymbol{\mu},\mathbf{u})\boldsymbol{\phi}}{\boldsymbol{\phi}^T\mathbf{K}(\boldsymbol{\mu})\boldsymbol{\phi}}, \tag{26}$$

subject to the set of constraints (22), (24) and (25). Stating the problem in this way the buckling equation (21) is implicitly satisfied.

We should note that in the Rayleigh variational principle as stated here the roles of the stiffness and geometric stiffness matrices were reversed, implying that the inner maximum problem will give as solution the inverse of P_{cr}, i.e. $\lambda_{\mathrm{cr}} = 1/P_{\mathrm{cr}}$, assumed nonzero.

4 Optimal solution necessary conditions

To obtain the necessary conditions for the topology optimization problem let us introduce the Lagrangian of the problem, $L(.)$, where the prebifurcation equilibrium equation (22) is considered as an additional constraint and the volume constraint is relaxed with a penalty function,

$$L = \left[\max_{\boldsymbol{\phi}\neq 0} \frac{\boldsymbol{\phi}^T\mathbf{G}(\boldsymbol{\mu},\mathbf{u})\boldsymbol{\phi}}{\boldsymbol{\phi}^T\mathbf{K}(\boldsymbol{\mu})\boldsymbol{\phi}}\right] + \mathbf{v}^T[\mathbf{K}(\boldsymbol{\mu})\mathbf{u} - \mathbf{f}] +$$

$$\frac{1}{2\rho}\left\{\max\left[0; \left(\sum_{e=1}^{M} \mu_e \int_{\Omega^e} \mathrm{d}\Omega - \overline{V}\right)\right]\right\}^2 +$$

$$\sum_{e=1}^{M}\left[\eta_1^e(\mu_e - 1) - \eta_2^e(\mu_e - \underline{\mu}_e)\right] . \tag{27}$$

In the previous augmented functional the Lagrange multipliers satisfy the inequalities defined in Table 1, and $\rho > 0$ is the penalty factor for the volume constraint.

Table 1. Lagrange multipliers for equilibrium and bound constraints

$\eta_1^e \geq 0$	$e = 1, 2, \ldots, M$
$\eta_2^e \geq 0$	$e = 1, 2, \ldots, M$
$\mathbf{v} = 0$	on Γ_u

Based on this Lagrangian, the necessary conditions for minimum can be identified with the stationarity conditions at the optimal solution (Clarke 1983).

From stationarity with respect to the nodal displacements $\mathbf{u}$ and element "densities" $\boldsymbol{\mu}$, we obtain the stationarity condition [see the paper by Rodrigues *et al.* (1995) for a complete derivation of this condition]

$$0 \in \partial_\mu L\,, \tag{28}$$

with the generalized gradient set characterized by

$$\partial_\mu L = \mathrm{co}\left\{\mathbf{x}, x_e = \alpha_p\alpha_q\left(\boldsymbol{\phi}_p^T\left[\frac{\partial\mathbf{G}}{\partial\mu_e} - \frac{1}{P_{\mathrm{cr}}}\frac{\partial\mathbf{K}}{\partial\mu_e}\right]\boldsymbol{\phi}_q - \mathbf{v}^{pq}\frac{\partial\mathbf{K}}{\partial\mu_e}\hat{\mathbf{u}}\right) : \boldsymbol{\alpha} \in \mathbf{R}^m, ||\boldsymbol{\alpha}|| = 1\right\} +$$

$$\frac{1}{\rho}\max\left[0; \left(\sum_{e=1}^{M}\mu_e\int_{\Omega^e}\mathrm{d}\Omega - \overline{V}\right)\right]\int_{\Omega_e}\mathrm{d}\Omega + \eta_1^e - \eta_2^e\,, \tag{29}$$

where co denotes the convex hull of the set.

In the generalized gradient definition, m is the multiplicity of the eigenvalue P_{crit}, $\boldsymbol{\phi}_p$, $(p = 1, \ldots m)$ is any set of m orthonormal (with respect to the stiffness matrix $\mathbf{K}$) eigenvectors associated with this eigenvalue, $\hat{\mathbf{u}}$ is the displacement vector solution of (22) and the $1/2m(m+1)$ adjoint displacement fields, $\mathbf{v}^{pq}$, $(p, q = 1, \ldots m)$, are the solution of the auxiliary problems

$$\mathbf{K}\mathbf{v}^{pq} = \left\{\boldsymbol{\phi}_p^T\frac{\partial\mathbf{G}}{\partial u_i}\boldsymbol{\phi}_q\right\} . \tag{30}$$

If the adjoint displacements are zero, for example in the case that the matrix $\mathbf{G}$ is independent of $\mathbf{u}$, the necessary condition defined by (28)-(29) is equivalent to the condition presented by Seyranian *et al.* (1994).

Finally, the necessary condition (28) is supplemented with the complementary slackness conditions

$$(\mu_e - 1)\eta_1^e = 0\,, \quad (\mu_e - 1) \leq 0\,, \quad \eta_1^e \geq 0\,, \tag{31}$$

$$(\mu_e - \mu_{\min})\eta_2^e = 0\,, \quad \mu_e \geq \mu_{\min}\,, \quad \eta_2^e \geq 0\,. \tag{32}$$

5 The computational model

The stationarity condition (28) is only necessary for a local minimum and does not give a more precise characterization of the candidate "optimal" point. The ideal approach would be to state the sufficient conditions, then one would have the necessary tools to verify if the candidate solution is indeed a minimum.

At this stage and due to the complexity of the problem this approach is not realizable. In optimization problems a practical way to overcome this problem "assuring" convergence, at least for a local minimum, is to solve the necessary conditions by an optimization algorithm with design updates based on a descent direction of some "cost" function (the objective function or of some weighted sum of the objective function and constraint violation). Such an algorithm is described in the following.

Let us consider a general iteration "k". At this iteration and once the design $\boldsymbol{\mu}^k$, the displacement $\mathbf{u}^k$, the critical load factor P_{cr} and the respective eigenvectors $\boldsymbol{\phi}_{p=1,\ldots,m}$ are known, the question is to define a design change decreasing the cost function value at least in a neighbourhood of the current design, i.e. to choose a direction of descent.

To characterize this direction let $\varepsilon > 0$ be a small number defined by the user and let m_ε, which we will call ε-multiplicity of λ_{cr}, be equal to the number of eigenvalues satisfying the inequality $(\lambda_i - \lambda_{\mathrm{cr}}) \leq \varepsilon\lambda_{\mathrm{cr}}$, $i = 1, \ldots N$.

Let also $\mathbf{d}^p$, $p = 1, 2, \ldots, m_\varepsilon$ be the vectors whose components are

$$d_e^p = \left(\boldsymbol{\phi}_p^T\left[\frac{\partial\mathbf{G}}{\partial\mu_e} - \frac{1}{P_{\mathrm{cr}}}\frac{\partial\mathbf{K}}{\partial\mu_e}\right]\boldsymbol{\phi}_p - \mathbf{v}^{pp}\frac{\partial\mathbf{K}}{\partial\mu_e}\hat{\mathbf{u}}\right) +$$

$$\frac{1}{\rho}\max\left\{0; \left(\sum_{e=1}^{M}\mu_e\int_{\Omega^e}\mathrm{d}\Omega - \overline{V}\right)\right\}\int_{\Omega_e}\mathrm{d}\Omega\,, \quad e = 1, M\,. \tag{33}$$

These vectors are obtained from the definition (29) without the Lagrange multipliers associated with the material density upper and lower bound constraints. These constraints are not considered in this phase, instead they will be strictly enforced at each design iteration.

Using these m_ε vectors, let D be the set obtained by convex combinations of the different $\mathbf{d}^p$, i.e.

$$D = \left\{ \sum_{p=1}^{m_\varepsilon} \omega^p \mathbf{d}^p : \omega^p \geq 0 \text{ and } \sum_{p=1}^{m_\varepsilon} \omega^p = 1 \right\} . \tag{34}$$

A "candidate" direction of descent $\mathbf{d}^*$ can now be defined as the negative of the vector in D with minimum norm, i.e. $\mathbf{d}^*$ solves the minimization problem (see e.g. Demyanov and Molozemov 1990; Kiwiel 1986)

$$||\mathbf{d}^*||^2 = \min_{d \in D} ||\mathbf{d}||^2 . \tag{35}$$

Note that if $\mathbf{0} \in D$ one has $\mathbf{d}^* = \mathbf{0}$ and the necessary condition is satisfied. However, if $\mathbf{d}^* \neq \mathbf{0}$ it does not imply that we are not at a stationary point since D is an approximation of the generalized gradient set and, in general, strictly contained in it.

In the case of multiplicity $m_\varepsilon = 1$, the set D has only one element and $\mathbf{d}^* = -\nabla_\mu L$.

Based on the descent direction $\mathbf{d}^*$ and introducing the upper and lower bound constraint thickness parameter ζ (defined by the user), the iterative procedure is

$$\mu_e^{k+1} = \begin{cases} \max\left\{(1-\zeta)\,\mu_e^k \,, \mu_{\min}\right\} & \text{if } \mu_e^k + \alpha \mathrm{d}_e^* \leq \max\left\{(1-\zeta)\,\mu_e^k \,, \mu_{\min}\right\} \\ \mu_e^k + \alpha \mathrm{d}_e^* & \text{if } \max\left\{(1-\zeta)\,\mu_e^k \,, \mu_{\min}\right\} \leq \mu_e^k + \alpha \mathrm{d}_e^* \\ & \text{and } \mu_e^k + \alpha \mathrm{d}_e^* \leq \min\left\{(1+\zeta)\,\mu_e^k \,, 1\right\} \\ \min\left\{(1+\zeta)\,\mu_e^k \,, 1\right\} & \text{if } \mu_e^k + \alpha \mathrm{d}_e^* \geq \min\left\{(1+\zeta)\,\mu_e^k \,, 1\right\} \end{cases} , \tag{36}$$

where the step length factor α is a positive number defined by

$$\alpha = \frac{\alpha^*}{\max_e |\mathrm{d}_e^*|} , \tag{37}$$

with the parameter $\alpha^* > 0$ selected by a line search procedure.

6 Example problems

In this section two example problems are presented to show the feasibility of the model developed. In the example the cell base material (see Fig. 1) is isotropic with Young's modulus $E = 210$ GPa and Poisson's coefficient $\nu = 0.3$.

The generalized eigenvalue problem (21) is solved by the subspace iteration method using the Householder method to solve the resultant reduced problem and the eigenvalue multiplicity parameter ε is equal to 0.05.

6.1 Example A

This example considers the reinforcement of a portal frame clamped at the base. All members in the initial structure have the same geometry (see Fig. 2).

The computational first critical load factor (first eigenvalue) of the structure without reinforcement material is $P_{\mathrm{cr}} = 14600$, using a nine-node isoparametric finite element model, and $P_{\mathrm{cr}} = 12851$ using an exact beam model.

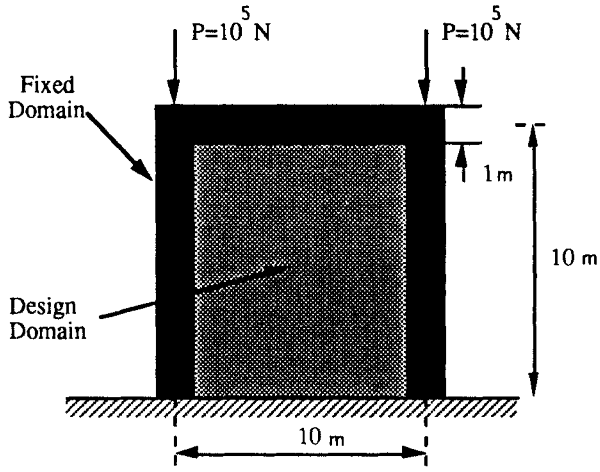
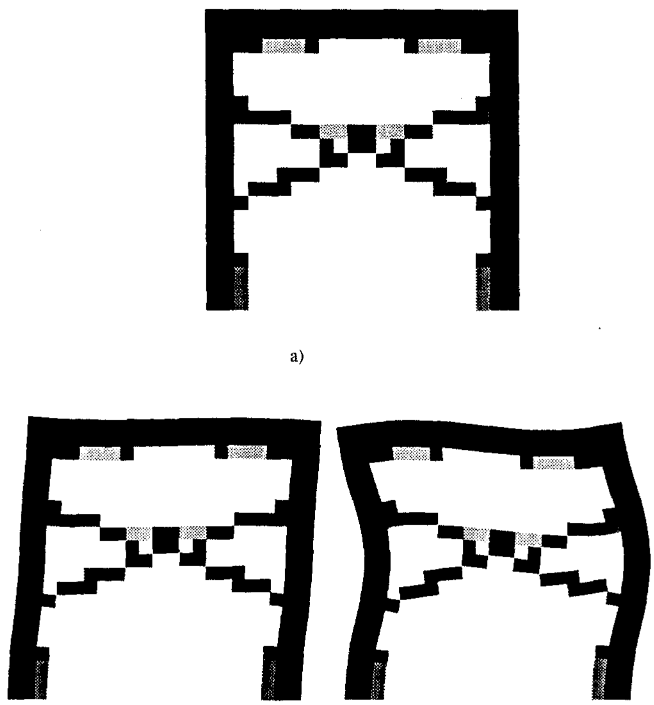


Fig. 2. Initial structure

The problem is modelled computationally using a finite element mesh with 22×21 nine-node isoparametric finite elements. The admissible volume of the reinforcement material is 13 m^3.

Considering an initial homogeneous distribution of the reinforcement material satisfying the volume constraint, we obtain $P_{\mathrm{cr}} = 24580$. The final design (see Fig. 3) was obtained after 80 iterations, the first critical load factor is $P_1 = 89320$ and the second is $P_2 = 92170$.



b) c)

Fig. 3. Example A. Final solution: (a) final design; (b) first eigenmode, $P_1 = 89320$; (c) second eigenmode, $P_2 = 92170$

Using the final topology to define an equivalent frame type structure, its buckling load factor is 81898 (see Fig. 5). Here we can observe a good agreement between the load factors of the final design and "equivalent" frame. This is not the case for the eigenmodes where the first mode of the equivalent structure agrees with the final structure second mode. This fact is not surprising due to the proximity of the respective load factors. The results obtained are summarized in Table 2.

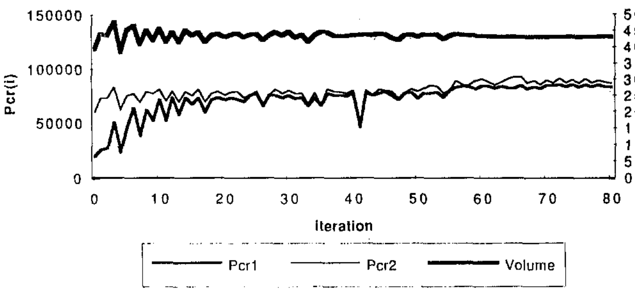


Fig. 4. Iteration history

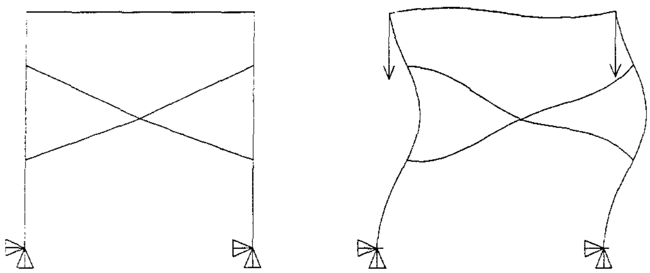


Fig. 5. "Equivalent" structure

Table 2. Critical load factors – Example A

	Critical load factor		Volume
	Q9	Beam element	
Initial structure	14600	12851	30 m^3
+ Uniform material distribution	24580	—	43 m^3
+ Final solution	89320	81898	43 m^3

One important aspect that should be discussed is the model numerical instability due to ill-conditioning of the stiffness and geometric stiffness matrices.

The material distribution model substitutes voids by a very weak (low density) material. This has the big advantage of maintaining the same finite mesh during the design process, however, it originates stiffness and/or geometric stiffness matrices with very small entries for the degrees of freedom belonging to elements with low density value. This fact, which is not so critical in elastostatic problems, can originate localized modes in the low density regions (see Fig. 6), that are not present in the real structure since in this case these subdomains are not considered.

To prevent this, elements with "low" density and "low" normalized stress ($\leq X$lim) do not contribute to the global geometric stiffness matrix. This is an artificial way of imposing that the stiffness geometric effects go to zero faster than the structural stiffness, thus removing the modes associated with these low density regions. In Fig. 7 the influence of this parameter in the final solutions obtained is shown.

6.2 Example B

This example considers the reinforcement of a five-storey frame clamped at the base. All members in the initial structure have the same geometric properties (see Fig. 8). A similar example, but for natural frequency optimization, was presented by Díaz and Kikuchi (1993).

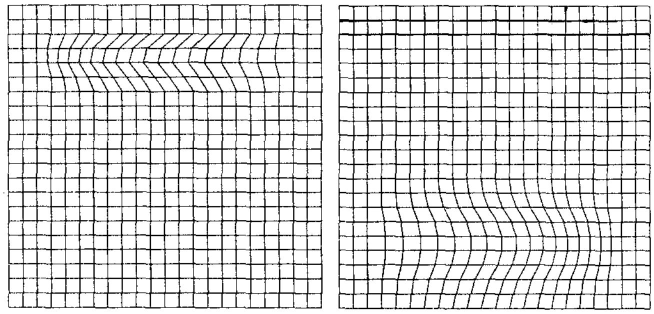


Fig. 6. Eigenmodes in low density regions

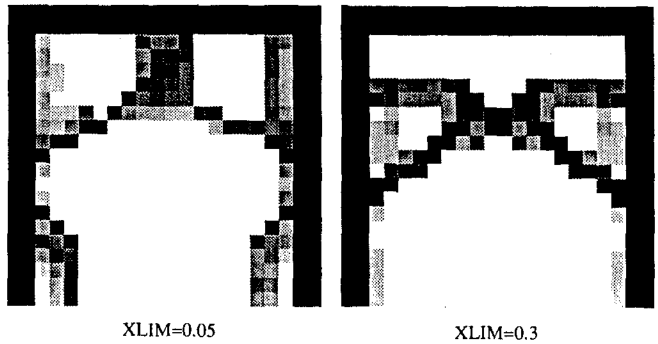


Fig. 7. Effect of the Xlim parameter on the final design

The first critical load of the structure without reinforcement is $P_{cr} = 805$ (nine-node finite element model) and $P_{cr} = 722$ (beam model).

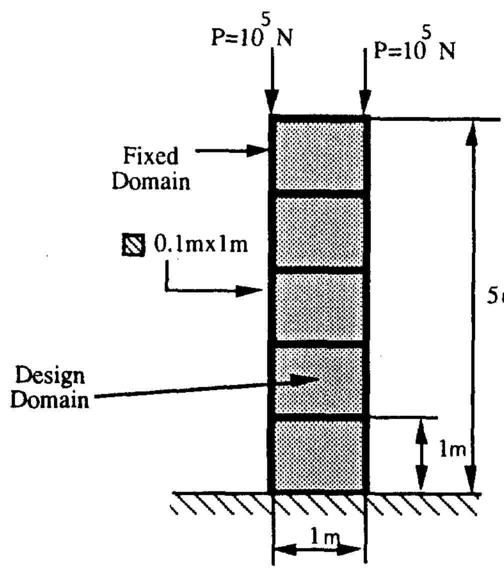


Fig. 8. Initial structure

The problem is modelled computationally using a mesh with 101 × 22 nine-node finite elements. The total volume of the reinforcement material is 0.96 m^3, for an initial structural volume of 1.5 m^3. For the final solution the first buckling load factor is $P_1 = 6472$ and the second is $P_1 = 6780$ (see Fig. 9). Figure 10 shows the iteration history for this example.

Using the resultant topology to define an equivalent frame

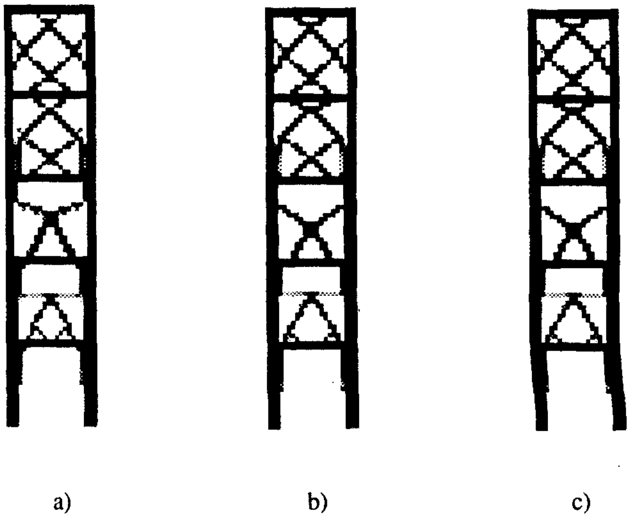


Fig. 9. Example B. Final solution: (a) final design; (b) first eigenmode, $P_1 = 6472$; (c) second eigenmode, $P_2 = 6780$

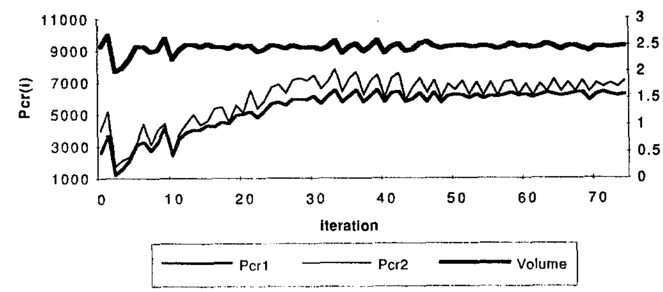


Fig. 10. Iteration history

model the critical load factor is 6187 (see Fig. 11). Table 3 summarizes the results obtained.

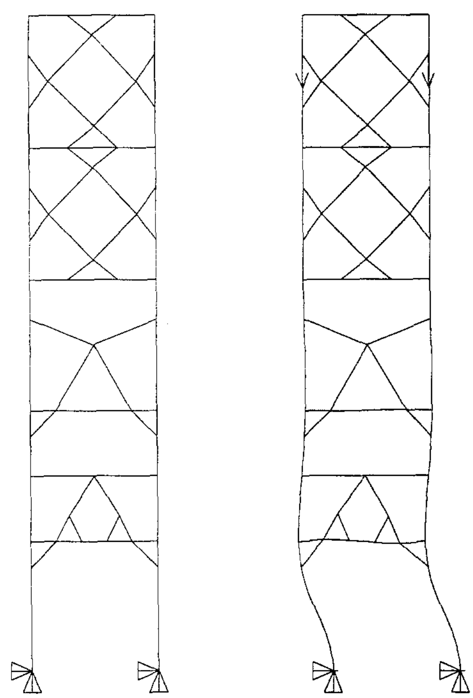


Fig. 11. "Equivalent" structure

Table 3. Critical load factors – Example B

	Critical load factor		Volume
	Q9	Beam element	
Initial structure	805	722	1.50 m^3
+ Uniform material	4400	—	2.46 m^3
+ Final solution	6472	6187	2.46 m^3

7 Concluding remarks

The development presented in this work extends the material based topology optimization model to include a critical load criterion.

The problem is solved through the use of finite element modelling and a mathematical programming method based on descent directions.

The feasibility of the approach presented was substantiated through the resolution of two numerical examples. From these examples a substantial increase in the critical load of the structures is observed. Also, the topologies obtained led to the identification of reinforced frame type structures with much higher critical loads.

However, some important issues remain to be analysed. The optimization iterative procedure used is based on a method proposed in 1972 by Demyanov and Malozemov to find ε-stationarity points for minimax problems (see Demyanov and Malozemov 1990). Recently, an intensive research effort has been devoted to the study and development of efficient algorithms for nonsmooth optimization problems (see e.g. Kiwiel 1985), so it is expected that, in the near future, these new developments can lead to more efficient and reliable algorithms that will drastically reduce the computational time required to solve nonsmooth optimization problems such as the one presented.

Another very important issue is the optimal structure sensitivity with respect to geometric imperfections and the domain of validity of the mechanical model.

The hypotheses behind the linearized buckling model limits its range of applicability. To overcome these limitations a mechanical nonlinear model should be used. Mróz and Haftka (1993) present sensitivity expressions for such a model assuming simple eigenvalues and shape and size design variables.

However, in spite of its limitations one should note that the results obtained with the linearized model give important information for optimization purposes as an upper bound of the load capacity of the structure. Also if the nonlinear analysis model is solved iteratively by a set of linearized subproblems, the respective optimization model can be based on the developments presented here.

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References

Bendsøe, M.P.; Díaz, A.; Kikuchi, N. 1993: Topology and generalized layout optimization of elastic structures. In: Bendsøe, M.P.; Mota Soares, C.A. (eds.) *Topology design of structures*, pp. 159-206. Dordrecht: Kluwer

Bendsøe, M.P.; Kikuchi, N. 1988: Generating optimal topologies in structural design using a homogenization method. *Comp. Meth. Appl. Mech. Eng.* **71**, 197-224

Bendsøe, M.P.; Mota Soares, C.A. (eds.) 1993: *Topology design of structures*. Dordrecht: Kluwer

Clarke, F.H. 1983: *Optimization and non-smooth analysis*. New York: John Wiley & Sons

Demyanov, V.F.; Malozemov, V.N. 1990: *Introduction to minimax*. New York: Dover Publications Inc.

Díaz, A.; Kikuchi, N. 1993: Solutions to shape and topology eigenvalue optimization problems using a homogenization method. *Int. J. Num. Meth. Engrg.* **35**, 1487-1502

Guedes, J.M.; Kikuchi, N. 1990: Preprocessing and postprocessing for materials based on the homogenization method with adaptive finite elements methods. *Comp. Meth. Appl. Mech. Engrg.* **83**, 143-198

Kiwiel, K.C. 1985: Methods of descent for nondifferentiable optimization. *Lecture Notes in Mathematics* **1133**. Berlin, Heidelberg, New York: Springer

Mlejnek, H.P. 1993: Some explorations in the genesis of structures. In: Bendsøe, M.P.; Mota Soares, C.A. (eds.) *Topology design of structures*, pp. 287-300. Dordrecht: Kluwer

Mróz, Z.; Haftka, R.T. 1993: Design sensitivity analysis of nonlinear structures in regular and critical states. In: Haslinger, J. (ed.) *Mathematical methods in computer aided optimal design*. Prague: Faculty of Mathematics and Physics, Charles University

Neves, M.M. 1994: *Topology optimization of structures with stability constraints*. M.Sc. Thesis, Mechanical Engineering Dept. Instituto Superior Técnico, Lisbon, Portugal

Novozhilov, V.V. 1953: *Foundations on the non-linear theory of elasticity*. Rochester, New York: Graylock Press

Rodrigues, H.; Guedes, J.M.; Bendsøe, M.P. 1995: Necessary conditions for optimal design of structures with a non-smooth eigenvalue based criterion. *Struct. Optim.* **9**, 52-56

Rozvany, G.; Zhou, M.; Birker, T.; Sigmund, O. 1993: Topology optimization using iterative continuum type optimality criteria (COC) methods for discretized systems. In: Bendsøe, M.P.; Mota Soares, C.A. (eds.) *Topology design of structures*, pp. 273-286. Dordrecht: Kluwer

Seyranian, A.P.; Lund, E.; Olhoff, N. 1994: Multiple eigenvalues in structural optimization problems. *Struct. Optim.* **8**, 207-227

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