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Why multi-load topology designs based on orthogonal microstructures are in general non-optimal

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Abstract It is well established that for a compliance constraint, the optimal topology of perforated plates under plane stress tends to that for least-weight trusses (Michell structures) as the "volume fraction" (i.e. the ratio material volume/available volume) approaches zero. It is shown in this note that for two loading conditions the optimal bar orientations for Michell structures are in general non-orthogonal and hence the assumption of orthogonal microstructures in multi-load plate topology optimization must lead to erroneous results.

1 Introduction

We shall term a microstructure of a two-dimensional periodic structure *orthogonal* if the axes of intersecting ribs form a right angle, as in the case of least-weight trusses or "Michell structures" (Fig. 1a), rank-2 laminates (e.g. Lurie and Cherkaev 1986; Kohn and Strang 1986) and Vigdergauz's (1992) microstructure (Figs. 1b and c, both optimal for perforated plates with a compliance constraint). For a single load condition, the *optimal orientation* of the rib axes of these microstructures usually coincides with the principal stress directions (1 and 2 in Fig. 1a), which was also confirmed by Pedersen (1989). An example of a *non-orthogonal microstructure*, which will be shown to be optimal for trusses with two alternative loads, is shown in Fig. 1d.

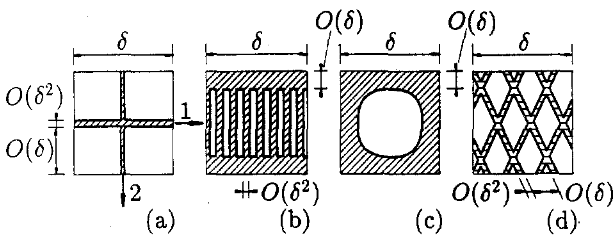

Fig. 1. Examples of orthogonal (a–c) and non-orthogonal (d) microstructures

2 Implications for optimal topologies with multiple loading

In most *real-life engineering problems*, the weight or cost of a structure is minimized subject to a number of load conditions and design constraints. Whilst the authors follow the above formulation, many mathematical studies consider an *inverse problem*, in which the weight of the structure is given and a single state (or behavioural) variable is minimized or maximized. A popular objective function is the so-called *compliance* (total amount of external or internal work), which represents some sort of measure of average stiffness of the structure and may also ensure a fully stressed design for some simple structures with one load condition and constant permissible stress. More recently, the compliance exercise was extended to multiple load conditions, often in terms of *artificial problems*, in which either a weighted combination of the compliances for various load conditions was used (e.g. Díaz and Bendsøe 1992), or for each element the maximum value of the compliance (out of several load conditions, e.g. Fukushima *et al.* 1993) was taken. The *most relevant multi-load formulation*, minimization of the maximum total compliance value (out of all loading conditions, e.g. Bendsøe *et al.* 1993) is the *inverse problem of weight minimization subject to a compliance inequality for each load condition*, which will be considered herein.

It was shown by Rozvany, Olhoff, Bendsøe *et al.* (1987), and later in greater detail by Allaire and Kohn (1992) as well as by Bendsøe and Haber (1993) that *the optimal topology of perforated plates* (Fig. 1b) for a compliance constraint tends to the *Michell layout for least-weight trusses* as the "volume fraction" (i.e. the ratio material volume/available volume) approaches zero. Whilst the above conclusion was obtained for a single load condition, it is intuitively obvious that it can be extended to multi-load problems. This is because the effect of rib intersections becomes negligible for both orthogonal and non-orthogonal grids (Figs. 1a and d), if the volume fraction (i.e. material volume/available volume) tends to zero. The implication of this conjecture is that most *topology designs for perforated plates* that are based on *orthogonal microstructures* (rank-2 laminates or rectangular holes) are clearly *non-optimal at low volume fractions*, and almost certainly at other volume fractions.

3 Optimality criteria for trusses with compliance constraints and multiple loading

The *analytical treatment of layout optimization of elastic trusses with multiple load conditions and several displacement constraints* was outlined briefly in recent contributions (Rozvany 1992; Rozvany *et al.* 1993). A *compliance problem* can be regarded as a *special case of a displacement problem*, in which the loads on the real structure are identical with the so-called adjoint loads associated with the corresponding displacement constraints. Assuming a compliance constraint for each load condition (k), more general optimality conditions (Rozvany 1992) reduce to the following for each member (i):

$$\varepsilon_{ik} = \frac{F_{ik}}{E_i A_i}, \quad \bar{\varepsilon}_{ik} = \frac{\nu_k F_{ik}}{E_i A_i}, \quad A_i = \sqrt{\sum_k (\nu_k F_{ik}^2 / E_i \rho_i)}, \quad (1)$$

$$(E_i/\rho_i) \sum_k \nu_k \varepsilon_{ik}^2 = 1 \quad (\text{for } A_i > 0), \quad (2)$$

$$(E_i/\rho_i) \sum_k \nu_k \varepsilon_{ik}^2 \leq 1 \quad (\text{for } A_i = 0), \quad (3)$$

where ε_{ik} and $\bar{\varepsilon}_{ik}$ are *kinematically admissible* real and adjoint strains in the member i under the load k, ν_k Lagrange multipliers, $F_{ik} = \overline{F}_{ik}$ the real and adjoint member forces, whilst A_i, E_i and ρ_i denote the cross-sectional area, Young's modulus and specific weight of material, respectively, for the member i.

4 Introductory example

In this note we consider a class of problems with the following features:

- the support conditions are symmetric;
- the two alternative loads are antisymmetric, each consisting of a unit point load; and
- the displacement in the direction of either load must not exceed unity.

A simple example of such a problem is given in Fig. 2, in which the two alternative loads are denoted by P_1 and P_2.

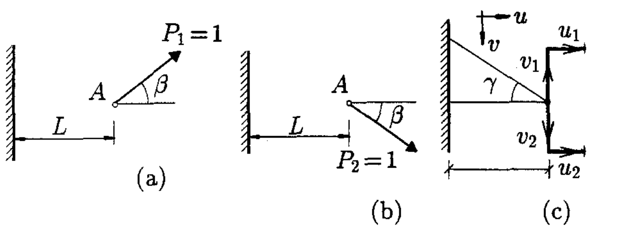

Fig. 2. Example: (a,b) support conditions and loading; (c) scaled displacement field

5 Proof of optimality for an assumed topology

Assuming a symmetric two-bar topology (see insert in Fig. 3), it can be shown easily that the *member forces* F_{ik} are given by

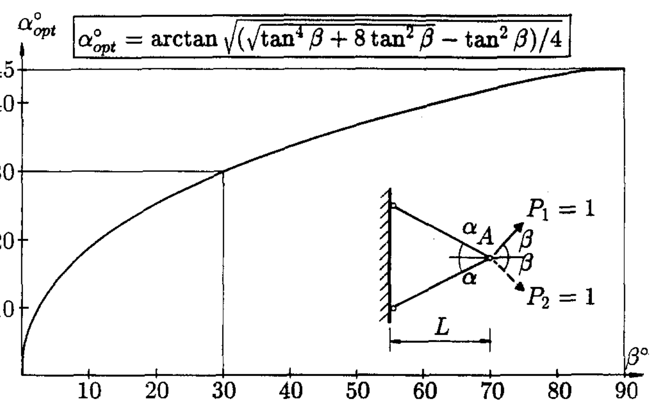

Fig. 3. Optimal bar orientations in dependence of force orientation

$$F_{11} = F_{22} = (\cos\beta/\cos\alpha - \sin\beta/\sin\alpha)/2,$$

$$F_{12} = F_{21} = (\cos\beta/\cos\alpha + \sin\beta/\sin\alpha)/2, \quad (4)$$

and the *total weight* Φ for a unit displacement in the direction of the forces (i.e. unit compliance for both load conditions) we have

$$\Phi = \frac{L^2\gamma}{E\cos^2\alpha}\left[\frac{\cos^2\beta}{\cos^2\alpha} + \frac{\sin^2\beta}{\sin^2\alpha}\right]. \quad (5)$$

Then the stationarity condition $\mathrm{d}\Phi/\mathrm{d}\alpha = 0$ readily yields the optimal orientation α of the bars:

$$\tan^3\alpha\tan(2\alpha) = \tan^2\beta, \quad (6)$$

$$\alpha_{\text{opt}} = \arctan\sqrt{\left(\sqrt{\tan^4\beta + 8\tan^2\beta} - \tan^2\beta\right)/4}. \quad (7)$$

The relation between α_{opt} and β is shown graphically in Fig. 3 and the weight variation in terms of α is given for $\beta = 10°$, $45°$ and $90°$ in Fig. 4.

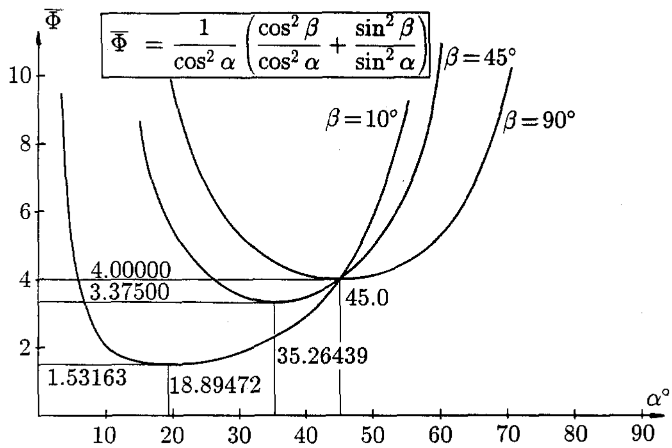

Fig. 4. Weight variation in dependence of bar orientation

6 Proof of optimality of the two-bar topology

For this proof, the strains for the two bars in Fig. 3 must be imbedded in a *kinematically admissible plane strain field* satisfying the optimality conditions in (2) and (3). Since this strain field can be scaled arbitrarily, this means only that along any line segment of the half span the quantity $(\varepsilon_{11}^2 + \varepsilon_{12}^2)$ must be smaller than or equal to its value along the non-vanishing bars.

It is easy to show that at the intersection of the two bars in Fig. 3 the horizontal and vertical displacements (u_A, v_A) are

$$u_A = \frac{L\cos\beta}{2EA\cos^3\alpha}, \quad v_A = \frac{L\sin\beta}{2EA\sin^2\alpha\cos\alpha}, \quad (8)$$

and then by (6) and (8) we have

$$t^2 = \frac{v_A^2}{u_A^2} = \tan^2\beta\cot^4\alpha = \tan(2\alpha)/\tan\alpha. \quad (9)$$

Then a *scaled version* of the real (and adjoint) displacement fields can be represented by (Fig. 2c)

$$u_1 = x, \quad v_1 = -tx, \quad u_2 = x, \quad v_2 = tx, \quad (10)$$

which with (9) gives the correct (scaled) strain values for the non-vanishing bars having an orientation α with (7). Taking the projections of these displacements onto a line of arbitrary orientation (γ), we find the following strains along that line:

$$\varepsilon_{11} = \varepsilon_{22} = (-t \sin\gamma + \cos\gamma)\cos\gamma\,,$$

$$\varepsilon_{12} = \varepsilon_{21} = (t \sin\gamma + \cos\gamma)\cos\gamma\,, \tag{11}$$

$$\varepsilon_{11}^2 + \varepsilon_{12}^2 = 2\left[t^2 \sin^2(2\gamma)/4 + \cos^4\gamma\right]. \tag{12}$$

Then the stationarity condition $\mathrm{d}(\varepsilon_{11}^2 + \varepsilon_{12}^2)/\mathrm{d}\gamma$ with (9) and (12) and substitution of α for γ yields

$$\sin(4\alpha)\tan(2\alpha)/\tan\alpha = 8\cos^3\alpha\sin\alpha\,, \tag{13}$$

which can be shown to be an identity. This shows that a necessary condition for the maximum of $\left[\varepsilon_{11}^2(\gamma) + \varepsilon_{12}^2(\gamma)\right]$ along $\gamma = \alpha$ is satisfied. The above maximality was also checked by plotting the variation of the above scaled quantity in dependence of γ for various force orientations (see Fig. 5). The results obtained in Section 5 and in this section are in complete agreement.

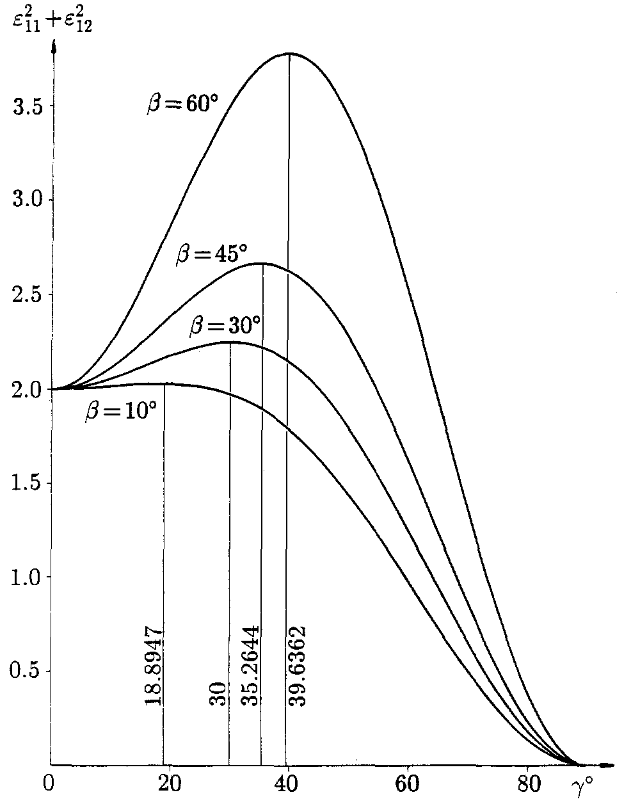

Fig. 5. Variation of a scaled value of the quantity $(\varepsilon_{11}^2 + \varepsilon_{12}^2)$ in dependence of the orientation (γ) of the truss element

7 Confirmation of the analytical results by a discretized truss topology

Although several authors developed discretized truss topology optimization methods later (for a review, see Bendsøe *et al.* 1993), a very powerful algorithm for a number of design conditions has been available for discretized truss solutions since 1988 (e.g. Rozvany *et al.* 1989). Using an extension of this algorithm (DCOC, Zhou and Rozvany 1993), Zhou employed the structural universe (ground structure) shown in Fig. 6a and obtained the solution in Fig. 6b. Owing to the limited number of bar directions, this solution consists of four bars (instead of two), but the correctly weighted mean value of the bar orientations shows an excellent agreement with the analytical result [$\alpha_{\mathrm{opt}} = 13.76131906°$ for $\beta = 5°$ by (7)]. Moreover, the exact weight value given by (5) for $L = 3$ is $\Phi = 11.315835$, whilst the discretized solution has a weight of $\Phi = 11.316581$, representing an error of only 0.007%.

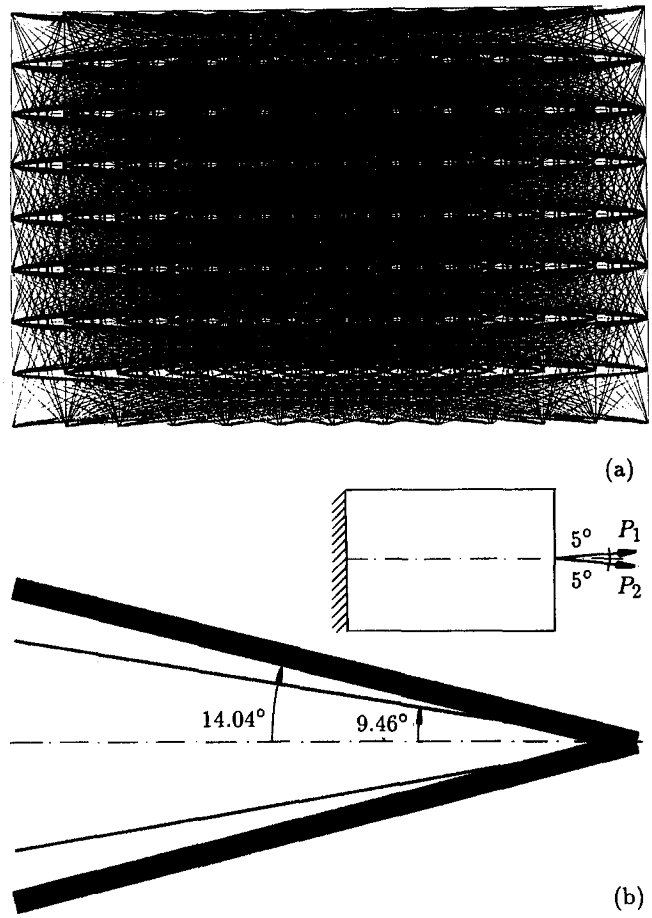

Fig. 6. Discretized optimal truss topology by DCOC

8 Confirmation of the analytical results by a discretized perforated plate topology

The above results were also confirmed by Birker who optimized the generalized shape of perforated plates using the SIMP procedure (Rozvany *et al.* 1992) and the DCOC algorithm (Zhou and Rozvany 1993). Figure 7a shows the initial design of uniform thickness and Figs. 7b and c the discretized plate solutions for $\beta = 10°$ and $\beta = 45°$, which show a very good agreement with the analytical truss solutions.

The black elements represent the full thickness (10), white areas denote the minimum thickness (10^{-6}) and most elements shown in grey have a thickness close to 10^{-6}. Hence the latter two have an insignificant effect on the total weight.

9 Extensions to solutions with periodic microstructures

The proofs given in Sections 5 and 6 are also valid for distributed loads along an edge (Figs. 8a and b), for which the solutions consist of a dense system of bars with infinitesimal

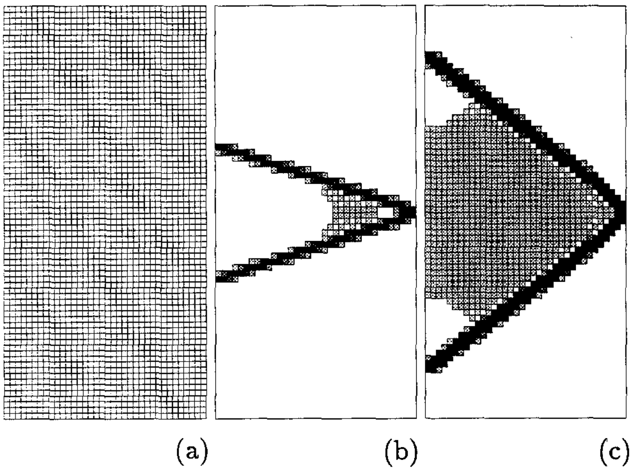

Fig. 7. Discretized optimal plate topology by DCOC and SIMP

spacing (Fig. 8c). The latter shows the relevance of the results in this paper to structures with periodic micro-topology subject to several load conditions. The possibility of constructing non-orthogonal Hencky-nets for trusses with several loading conditions is also being considered.

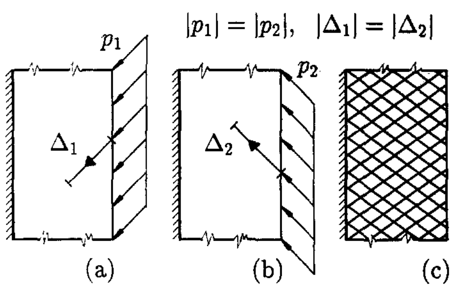

Fig. 8. Optimal truss layout for alternate distributed loads

10 Certain conceptual differences between Michell's theory and recent mathematical studies

If we apply the optimality criteria in (2) and (3) to a *single load condition*, we have

$$(\nu E_i/\rho_i)\varepsilon_i^2 = 1 \quad (\text{for } A_i > 0)\,, \tag{14}$$

$$(\nu E_i/\rho_i)\varepsilon_i^2 \leq 1 \quad (\text{for } A_i = 0)\,. \tag{15}$$

Relations (14) and (15) imply that at any point P (Fig. 9a), the direction of non-vanishing members must coincide with the *directionally maximum* absolute value of the strains, which are known to be the *principal directions*. Since (14) and (15) also require *kinematic admissibility* of the strains in all directions, the adjoint strain field for a truss with bars running in all directions can be replaced by a plane kinematically admissible strain field (satisfying all kinematic boundary and continuity conditions). This important *mechanical analogy* of the Michell-Prager-Rozvany layout theory (e.g. Rozvany 1989, Chapt. 8) enables us to determine the optimal layout of trusses, as well as of other structures.

Michell's original problem was not a compliance problem but a truss layout problem for a single load condition and a *given permissible stress* σ_p. Assuming a different value of σ_p for each member i, Michell's optimality criteria would read

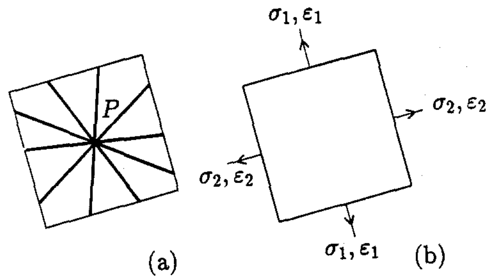

Fig. 9. Difference between Michell's problem and more recent mathematical studies

$$(1/\sigma_{pi})|\varepsilon| = 1 \quad (\text{for } A_i > 0)\,, \tag{16}$$

$$(1/\sigma_{pi})|\varepsilon| \leq 1 \quad (\text{for } A_i = 0)\,, \tag{17}$$

which *happen to give the same solution as the compliance problem* in (14) and (15) within a constant multiplier (if either $E_i/\varphi_i = 1/\sigma_{pi}$ for all i, or E_i, φ_i and σ_{pi} have the same value for all i).
for constant values of E_i, ρ_i, σ_{pi}).

The conceptual difference between the above (earlier) formulation and some highly rigorous recent mathematical studies (e.g. Strang and Kohn 1983) is that the latter define the problem in terms of an *orthotropic plane continuum* with a minimality condition either on the compliance or on the quantity

$$\min \Phi = \int_{\Omega} (|\sigma_1| + |\sigma_2|)\,\mathrm{d}\Omega\,, \tag{18}$$

without a proof that for the original problem (Fig. 9a) *only bars in the principal directions can be optimal.* As shown in this note, the above coaxiality is not self-evident and in fact not valid for most design conditions, such as multiple loads, but also deflection constraints for a single load condition (see Rozvany *et al.* 1993).

An *indirect mathematical proof* of the optimality of principal dirctions for a Michell truss for a single-load compliance constraint has been established by first showing that for perforated plates the optimal orientation of rank-2 cells is in the principal directions and then proving (Allaire and Kohn 1992; Bendsøe and Haber 1993) that the latter reduce to a Michell truss when the volume fraction tends to zero.

11 Conclusions

- The optimal truss layout for a vertical supporting line and antisymmetric alternate point loads consists of a symmetric two-bar system.
- For alternate vertical loads the bars enclose 45° with the horizontal (as for a single load) and this angle decreases with a decrease in the slope of the loads (Fig. 3).
- At a load orientation of 30° the bar orientation is also 30° and at smaller slopes of the load the bars enclose a wider angle with the horizontal than the loads. Naturally, at a zero slope of the loads the two bars also merge into a single horizontal bar.
- The results can be extended to "distributed" bar systems (Fig. 8) which imply that also for some perforated plates with several load conditions the optimal microstructure must be non-orthotropic. Hence, at least at low volume

fractions, solutions based on orthogonal microstructures are clearly uneconomical.

- Comparisons with discretized methods confirm (i) the validity of the proposed analytical layout theory for several load conditions, as well as (ii) the power and versatility of the DCOC and SIMP algorithms in topology optimization.

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The analytical results in this paper and the evaluation of their consequences are due to the first author, the discretized truss solution in Fig. 6 was obtained by the second author and the plate topology solution in Fig. 7 by the third author.

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