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On the design of Beck's column

U.T. Ringertz

Department of Lightweight Structures, Royal Institute of Technology, S-100 44 Stockholm, Sweden

Abstract The optimization of a single column subject to a follower load is studied. The shape of the column that maximizes the critical load for a given amount of material is found using optimization. The objective function is nonsmooth and there are also multiple local optima. Improved solutions to the optimization problem are found and compared to previous columns suggested in the literature. The sensitivity to perturbations in shape and refined discretization is also investigated.

1 Introduction

Optimal design of structures subject to nonconservative forces is significantly more difficult compared to the design of structures subject to conservative forces. The main difference is that stability analysis in the former case involves analysis of unsymmetric eigenvalue problems, whereas the latter case involves eigenvalues of symmetric matrices, usually in the form of the matrix of second variations of a potential energy function. An overview of structural optimization subject to stability constraints can be found in the work of Haftka and Adelman (1993), Olhoff and Taylor (1983), Mróz (1993) and Zyczkowski (1982).

It is common that there are coalescing eigenvalues for the optimal design, see the paper by Cox and Overton (1992) for a typical example. The eigenvalues are not smooth functions of parameters such as shape and loading, when there are coalescing eigenvalues, causing significant difficulties in solving the optimization problem. Coalescing eigenvalues of unsymmetric matrices are also known to depend in a much more complicated way on parameters in the problem, compared to eigenvalues of symmetric matrices, see the paper by Seyranian (1991) for a thorough analysis.

A classical model problem for stability analysis of structures subject to nonconservative forces is shown in Fig. 1. The column, usually referred to as Beck's column, is clamped at one end and is subject to a follower force at the other end. There is no potential energy function for this mechanical system and one is therefore forced to consider the dynamics of the structure in order to analyse stability. The column is said to be in stable equilibrium if small perturbations of the deformations in the equilibrium state do not cause the deformations to become unbounded.

If the column is uniform in shape and material properties, it is possible to determine the critical load of the column using a semi-analytical approach (see Ziegler 1968). The critical load of the uniform column is approximately given by

$$P_{cr} = 20.05\, EI_0/L^2 \,, \tag{1}$$

where L denotes the length, E the modulus of elasticity, and I_0 the moment of inertia of the column.

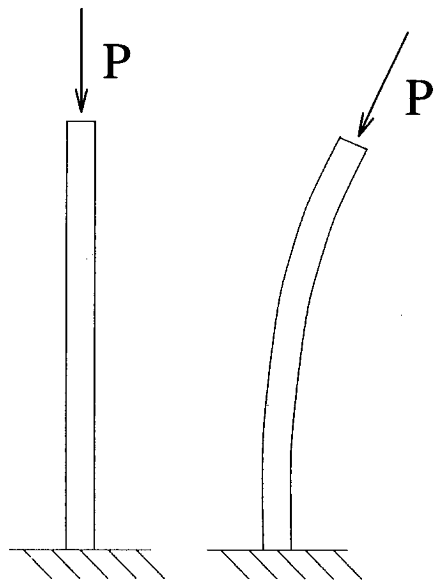

Fig. 1. Beck's column, initial and deformed state

The purpose of this paper is to derive improved solutions to the optimal design of Beck's column, that is, finding the shape of the column that maximizes the load for a given amount of material. The sensitivity of the critical load with respect to perturbations in geometry and refined discretization is also discussed.

2 Stability analysis

The equations of motion in discretized form are

$$\mathbf{M}\ddot{u} + \mathbf{K}u - \eta\mathbf{K}_g u - \eta\mathbf{F}u = 0 \,, \tag{2}$$

where the vector $\mathbf{u} \in I\!R^m$ denotes the transverse displacement of the column, $\mathbf{M}$ the mass matrix, $\mathbf{K}$ the stiffness matrix, $\mathbf{K}_g$ the geometric stiffness matrix, $\mathbf{F}$ the matrix defining the follower load, and η the scalar load parameter. The matrices $\mathbf{M}$ and $\mathbf{K}$ are symmetric positive definite, $\mathbf{K}_g$ symmetric, and $\mathbf{F}$ unsymmetric.

Solutions to (2) are sought in the form

$$\mathbf{u} = \mathbf{q}\, e^{i\omega t} \,. \tag{3}$$

Introducing (3) in (2) gives the unsymmetric generalized linear eigenvalue problem

$$[\mathbf{A}(\eta) - \omega^2\mathbf{M}]\mathbf{q} = 0 \,, \tag{4}$$

where $\mathbf{A}(\eta) = \mathbf{K} - \eta\mathbf{K}_g - \eta\mathbf{F}$. The structure is stable whenever ω^2 is real and positive (Bažant and Cedolin 1991).

The stability analysis is to determine the largest value η_c of the load parameter η such that ω^2 is real and positive. This computation can be done numerically using bisection where the eigenvalue problem (4) is solved repeatedly for each trial value of η. However, the computation of η_c must be done with great care, since there may exist loads $\eta > \eta_c$ for which all ω_i^2 are real and positive. A proper definition of η_c is

$$\eta_c = \max_{\eta}\{\eta \mid \omega_i^2(\hat{\eta}) \geq 0\,,\ \ \mathrm{Im}[\omega_i^2(\hat{\eta})] = 0\,,$$

$$i = 1, \ldots, m, \text{ for all } \hat{\eta} \in (0, \eta_c)\}\,. \tag{5}$$

However, this is not a very useful definition, since it is impossible to check ω^2 for all $\hat{\eta} \in (0, \eta_c)$. A more practical definition is obtained by considering a discrete subset of the range $(0, \eta_c)$. In this study η_c is computed by first finding the smallest integer k such that the column is stable for $\underline{\eta} = k\,\Delta\eta$, stable for all $\eta = j\Delta\eta$, $j < k$, and unstable for $\overline{\eta} = (k+1)\Delta\eta$. A more precise value for η_c is then obtained using bisection with the initial interval defined by $(\underline{\eta}, \overline{\eta})$. The numerical value $\Delta\eta = 0.1$ is used in this study and the bisection algorithm is terminated when the length of the interval of uncertainty is less than 10^{-10}. The unsymmetric eigenvalue problem (5) is solved using LAPACK (Andersson *et al.* 1992).

3 Optimal design

Obviously, the critical load η_c depends on the shape of the column since both $\mathbf{A}(\eta)$ and $\mathbf{M}$ depend on the shape. The optimal design problem of finding the shape of the column that maximizes the critical load for a given amount of material can be posed as

$$\max_{x} \eta_c(\mathbf{x}), \quad \mathbf{x} \in I\!R^n\,, \tag{6}$$

$$\mathbf{a}^T\mathbf{x} = v_0\,, \tag{7}$$

$$\underline{x}_j \leq x_j \leq \overline{x}_j, \quad j = 1, \ldots, n\,, \tag{8}$$

where x_j denotes the cross-sectional area of the column in the j-th cross-section, the vector $\mathbf{a}$ defines the constraint on constant volume v_0, and $\underline{x}_j$ and $\overline{x}_j$ the lower and upper bounds on the cross-sectional area. The values $\underline{x} = 0$ and $\overline{x} = \infty$ are used in this study.

A somewhat standardized form (Gutkowski *et al.* 1993) of the problem assumes that the column is modelled using 10 beam finite elements, with piecewise cubic approximation of the displacements, and piecewise linear cross-sectional area giving 11 design variables. The cross-section is assumed solid and circular. It is also convenient to put the problem in dimensionless form such that the true critical load is $\eta_c EI_0/L^2$, the true cross-sectional area $x_j A_0$, and the square of the true eigenfrequency $\omega^2 EI_0/(\rho A_0 L^4)$, where L denotes the length, E the modulus of elasticity, ρ the mass density, A_0 the cross-sectional area of the initial uniform column, and I_0 the moment of inertia of the initial uniform column. Consequently, η_c, x_j, and ω^2 are all dimensionless quantities.

The best solution presented in the literature so far, has been obtained by Gutkowski *et al.* (1993). Using an optimality criteria method, they obtain a design with a buckling load of $\eta_c = 92.56$.

To investigate the optimality of $\eta_c = 92.56$, a linear interpolation of this design $\hat{\mathbf{x}}$ and the initial uniform design $\mathbf{x}^0$ is made, such that

$$\mathbf{x}(\xi) = (1 - \xi)\mathbf{x}^0 + \xi\hat{\mathbf{x}}\,, \tag{9}$$

where ξ is a scalar parameter. The value $\xi = 0$ gives the uniform column, and $\xi = 1$ gives the column of Gutkowski *et al.* (1993). The vector $\mathbf{x}(\xi)$ is feasible with respect to the constraints (7) and (8) for ξ in the interval $(-1.408, 1.318)$. The critical load is plotted as a function of ξ in Fig. 2. Clearly, the design of Gutkowski *et al.* (1993) is not optimal. It is not even a local optimum, since increasing ξ from $\xi = 1.0$ gives higher critical load. It is also obvious from the figure that there are several local optima, and that the function η_c is nonsmooth. The best design along this direction is obtained for $\xi_1 = 1.261$ where the critical load is $\eta_c = 110.8$, which compares favourably with the results of Gutkowski *et al.* (1993). The values of x for the design of Gutkowski *et al.* (1993) is given in the first column of Table 1, and the design $x(\xi_1)$ in the second column.

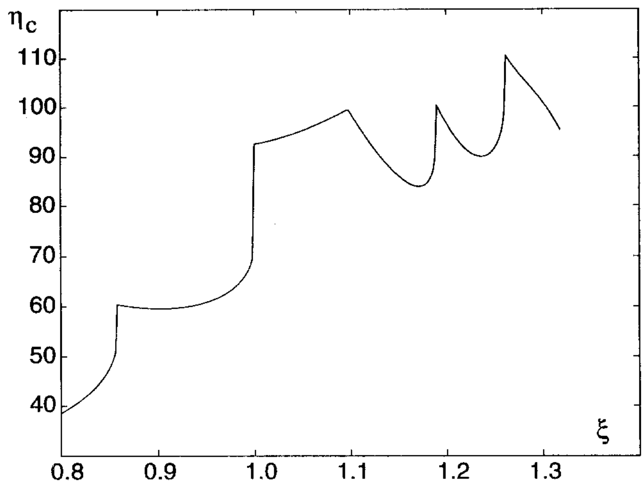

Fig. 2. The critical load η_c versus the design parameter ξ

Table 1. The cross-sectional area of the columns

	$\hat{\mathbf{x}}$	$\mathbf{x}(\xi_1)$	$\mathbf{x}(\xi_2)$	$\mathbf{x}^*$
η_c	92.56	110.8	105.8	188.1
x_1	1.080722	1.101791	1.103325	1.760551
x_2	0.777793	0.719798	0.715576	0.174082
x_3	1.152735	1.192599	1.195501	2.237730
x_4	1.709925	1.895215	1.908704	1.557436
x_5	1.572900	1.722427	1.733312	1.746399
x_6	1.308235	1.388684	1.394541	0.965645
x_7	0.982000	0.977302	0.976960	0.959646
x_8	0.749315	0.683887	0.679124	0.691606
x_9	0.673849	0.588723	0.582527	0.502391
x_{10}	0.412146	0.258716	0.247547	0.284706
x_{11}	0.241472	0.043496	0.029084	0.000169

The variation of the real part of the smallest eigenfrequencies ω_i^2, with the load η for the design $\hat{\mathbf{x}}$, is shown in Fig. 3. The eigenvalues are all real and positive for small loads η, but for increased load, the eigenvalues merge and form a complex pair with the same real part. There is not a complete set of eigenvectors for the point where the eigenvalues coalesce, which is known as strong interaction of eigenvalues (Seyranian 1991). Obviously, this design is very sensitive to small perturbations since eigenvalue ω_2^2 almost coalesce with ω_3^2 at a load significantly lower than the critical load $\eta_c = 92.56$. The jump in Fig. 2 for $\xi = 1$ occurs when eigenvalues ω_2^2 and ω_3^2 coalesce. For $\xi < 1$, ω_2^2 and ω_3^2 form a complex pair such that the critical load is $\eta_c \approx 67$. However, for $\xi > 1$, ω_2^2 and ω_3^2 are real and distinct giving a significantly higher critical load.

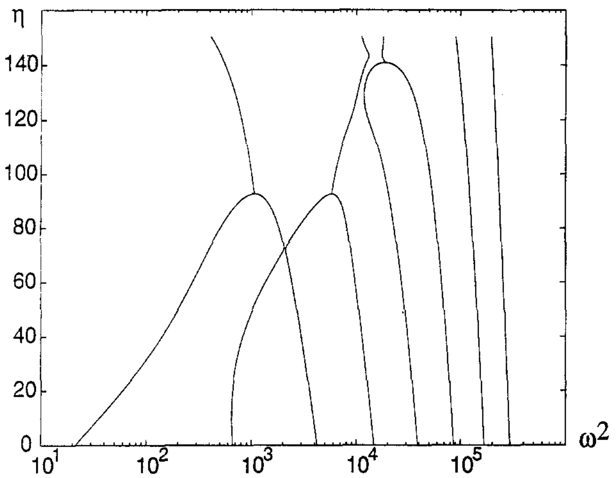

Fig. 3. The load η versus ω^2 of the design by Gutkowski *et al.* (1993)

However, the design $\mathbf{x}(\xi_1)$ is not much better in the sense that eigenvalue ω_6^2 almost coalesce with ω_7^2 (see Fig. 4) at a load $\eta \approx 98.7$ which is much lower than the critical load $\eta_c = 110.8$.

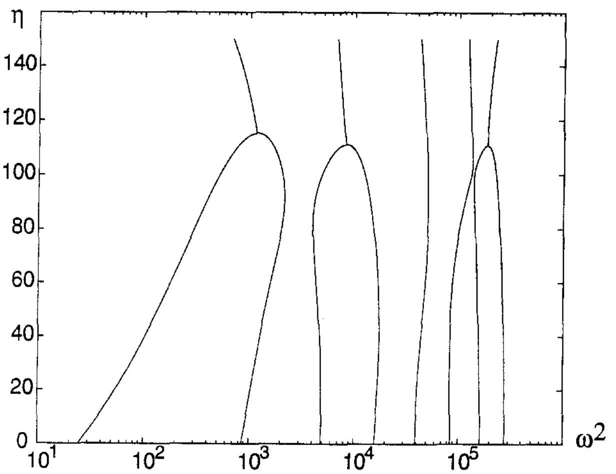

Fig. 4. The load η versus ω^2 for $\mathbf{x}(\xi_1)$

In an attempt to find even better solutions than $\mathbf{x}(\xi_1)$, an initial point for further optimization is chosen as $\mathbf{x}(\xi_2)$, where $\xi_2 = 1.28$. This column has eigenvalues that are well separated (see Fig. 5) and a critical load of $\eta_c = 105.8$.

4 A barrier algorithm

It may be possible to devise an algorithm for solving (6)–(8), but it is convenient to consider a somewhat different formulation. The volume of the column is minimized subject to stability constraints for a fixed magnitude of the load η_c, giving

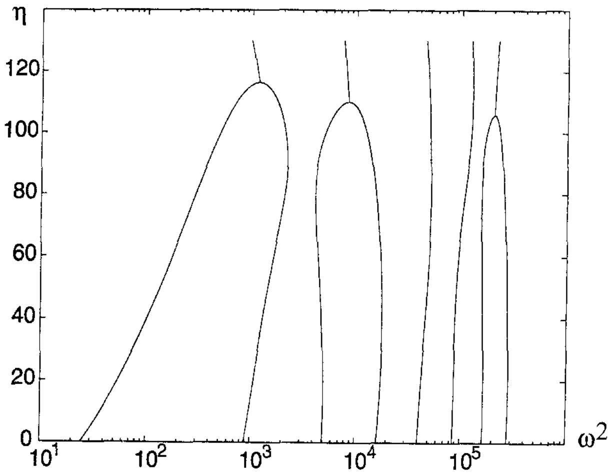

Fig. 5. The load η versus ω^2 for $\mathbf{x}(\xi_2)$

$$\min_x \mathbf{a}^T\mathbf{x}\,, \quad \mathbf{x} \in \mathbb{R}^n\,, \tag{10}$$

$$\omega_1^2(\eta_k, \mathbf{x}) \geq 0, \quad k = 1, \ldots, n_\eta\,, \tag{11}$$

$$\omega_{i+1}^2(\eta_k, \mathbf{x}) - \omega_i^2(\eta_k, \mathbf{x}) \geq 0\,,$$
$$i = 1, \ldots, m-1, \; k = 1, \ldots, n_\eta\,, \tag{12}$$

$$\underline{x}_j \leq x_j \leq \overline{x}_j, \quad j = 1, \ldots, n\,. \tag{13}$$

The constraints on the eigenvalues are imposed for a large number of different load levels η_k in the interval $(0, \eta_c)$, giving a total of $m \times n_\eta$ nonlinear inequality constraints.

Using the logarithmic barrier function, (10)–(13) is transformed to the unconstrained problem

$$\min_x \mathbf{a}^T\mathbf{x} - \mu \sum_{k=1}^{n_\eta} \log \omega_1^2(\eta_k, \mathbf{x}) -$$
$$\mu \sum_{k=1}^{n_\eta} \sum_{i=1}^{m} \log[\omega_{i+1}^2(\eta_k, \mathbf{x}) - \omega_i^2(\eta_k, \mathbf{x})] -$$
$$\mu \sum_{j=1}^{n} \log(x_j - \underline{x}_j) - \mu \sum_{j=1}^{n} \log(\overline{x}_j - x_j)\,, \tag{14}$$

where μ is a positive scalar barrier parameter. The barrier function is minimized using a modified Newton method where the first derivatives are explicitly computed and second-derivatives obtained by finite difference approximations.

A significant feature of the barrier formulation is that the iterates $\mathbf{x}^k$ are all strictly feasible with respect to the constraints. Consequently, the eigenvalues cannot merge and form a complex pair if they are initially real and distinct.

The barrier subproblem (14) is usually solved for a decreasing sequence of barrier parameters μ. However, in this case it is desirable to use a good initial approximation of the solution making it necessary to use a small initial value for μ. The drawback is that the barrier function is usually nonconvex and illconditioned for small μ.

The barrier function is minimized with a constant value of $\mu = 0.001$ using $\mathbf{x}(\xi_2)$ as initial point. The bounds on x are chosen as $\underline{x} = 0$ and $\overline{x} = 10$. The fixed load is chosen to be $\eta_c = 105$ ensuring that the vector $\mathbf{x}(\xi_2)$ is strictly feasible to (10)–(13). The eigenvalue constraints (11) and (12) are enforced for the loads $\eta_k = 1.0, 2.0, \ldots, 105.0$, giving a total of 2100 nonlinear constraints.

The minimization is terminated after 17 iterations when the change in volume is less than 0.01. The volume is initially 1.0 and is reduced to 0.75. In order to make comparison with the previous columns possible, the cross-sectional areas are scaled uniformly to a column with unit volume. The cross-sectional areas x_j^* of this improved column are given in column 4 of Table 1. The critical load of the column is 188.1, a significantly higher critical load compared to the other columns in Table 1.

Unfortunately, this column is also sensitive to small imperfections. Eigenvalue ω_2^2 almost coalesces with ω_3^2 (see Fig. 6) for the load $\eta \approx 150$, which is much less than the critical load. Consequently, a small imperfection could give a drastic reduction in the load carrying capacity of the column. Minimizing the barrier function for smaller values of μ than 0.001, gives columns with marginally improved critical load at the expense of increased imperfection sensitivity. In theory, a higher value of μ would give a column where the eigenvalues are further separated, but the distance between eigenvalues will only increase on average, there is no guarantee that the column actually is less sensitive to imperfections.

Finally, the shapes of the different columns discussed in this section are shown in Fig. 7. The columns appear in the same order as in Table 1.

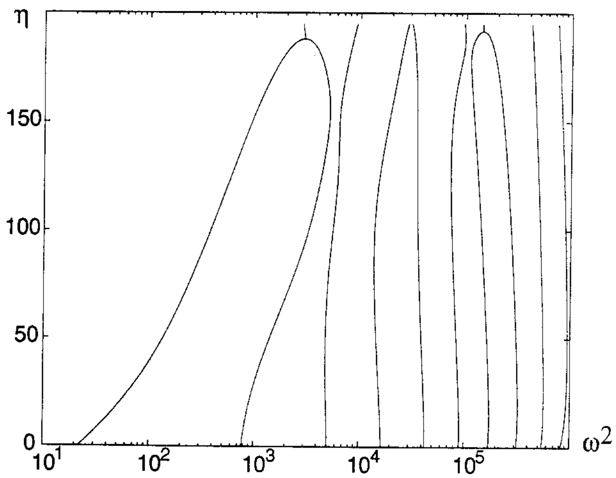

Fig. 6. The load η versus ω^2 of the improved design $\mathbf{x}^*$

5 Influence of a refined discretization

Beck's column has been frequently used as a model problem for optimal design of structures subject to nonconservative loading. However, the discretization of the column is usually rather crude. The standard form (Gutkowski *et al.* 1991) of the problem uses only ten finite elements and a piecewise linear cross-sectional area. It is reasonable to assume that the optimal design is dependent on the quality of the discretization since the critical load of the column is in many cases sensitive to small variations in shape.

To investigate the influence of a refined discretization for the displacements u, the last column discussed in the previous section is analysed using a refined mesh. The cross-sectional areas are the same (column 4 of Table 1), but the number

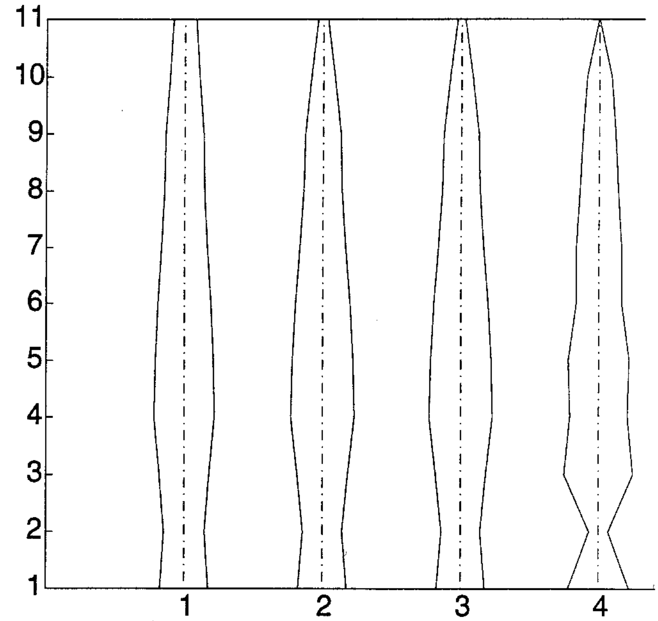

Fig. 7. The geometry of the four columns given in Table 1

of finite elements N is increased. Consequently, the design is fixed but the accuracy of the analysis is refined.

Table 2. The critical load for several different numbers of finite elements

N	η_c
10	188.07
20	141.94
30	143.01
40	143.52
50	143.58
60	143.57
70	143.57
80	143.57
90	143.58
100	143.59

Simply increasing the number of finite elements to $N = 20$ causes a significant drop in the critical load η_c, see Table 2. For $N = 20$ and more, interaction of eigenvalues ω_4^2 and ω_5^2 causes a significant drop in the critical load. As expected, the variation of the smallest eigenvalues is hardly affected by the refined discretization, whereas the larger eigenvalues significantly changes behaviour. The dependence of the eigenvalues on η for $N = 50$ is shown in Fig. 8. The variation of the two smallest eigenvalues is hardly different in Figs. 6 and 8, but the difference is quite apparent for the larger eigenvalues. Note that the scale on the x-axis is slightly different in Figs. 6 and 8, in order to show all the interacting eigenvalues.

6 Discussion

Although Beck's column is a rather academic optimal design problem, it illustrates many of the difficulties with optimal design considering nonconservative forces. The basic equations are the same for simple problems of aeroelasticity, and one can expect similar sensitivity to changes in design and loading. In particular, the sensitivity to variation in load-

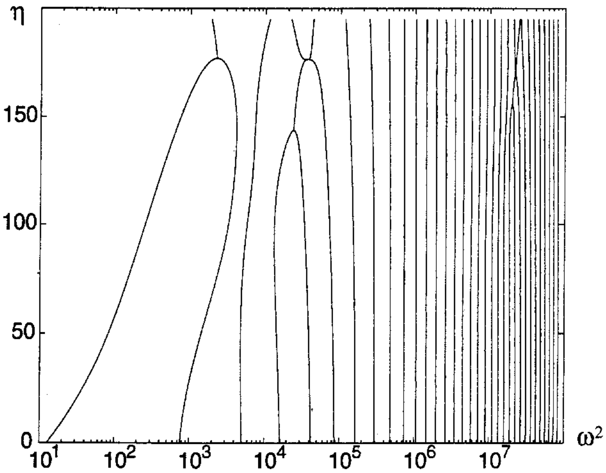

Fig. 8. The eigenvalues of the design $\mathbf{x}^*$ for $N = 50$

ing would be very important for problems in aeroelasticity since the load is usually derived as an approximate numerical solution to the differential equations modelling the flow field.

It is possible to obtain dramatic improvement in performance using optimization. In the case of Beck's column, an increase of the critical load by a factor of nine over the uniform column is obtained. The drawback is that the optimized structure tends to be extremely sensitive to imperfections in geometry, boundary conditions, and loading. Some strategy for considering imperfections in the optimization process would be highly desirable.

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