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*Brief Note*

# Exact least-weight truss layouts for rectangular domains with various support conditions

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**Abstract** A comprehensive review of both previously known and newly derived exact optimal layouts for rectangular domains with point loads and either line or point supports is given in this note.

## 1 Introduction

In this brief note, a short but systematic survey of optimal truss topologies for rectangular domains with various support conditions is presented. Detailed derivations of new optimal layouts will be given in a full length paper. It is shown subsequently that the solutions become more complicated if a smaller part of the boundary is supported.

Research into optimal truss layouts gained a new impetus recently through the discovery that the *optimal topology of perforated plates in plane stress* tends at low volume fractions to that for *least-weight trusses*, if a compliance constraint is used (e.g. Allaire and Kohn 1992). A similar conclusion was reached earlier in relation to axisymmetric perforated plates in flexure and grillages (Rozvany, Olhoff, Bendsøe *et al.* 1987). All solutions discussed in this paper are valid for a stress constraint (Michell 1904) or a compliance constraint.

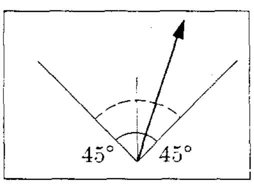


**Fig. 1.** Admissible range of load directions for optimal multibar layouts in Section 2

For simplicity, only *one point load parallel to one side* of the rectangular domain is considered in this note. However, the following *generalizations* can be made *within the same optimal layout* for all problems in Section 2, if the topology consists of more than one bar.

- The direction of the point load may be changed by an angle not exceeding 45°.
- Any number of point loads can be applied simultaneously if all of them enclose an angle not exceeding ±45° with one side of the rectangular domain (Fig. 1)

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## 2 A summary of known solutions

### *2.1 Line supports along two or more sides*

The optimal layout is relatively simple for the above support conditions. In the case of supports along *two adjacent sides* (Figs. 2a–c), the domain is to be divided into two regions by a straight boundary passing through the supported corner and having a slope of 2:1 to the direction of the sides parallel to the load. If the load is on one side of this line, then the optimal topology consists of a *single bar* (Fig. 2a). If it is on the other side, the topology consists of *two bars* (Fig. 2b), one in tension (continuous line) and the other one in compression (broken lines). If the load acts on the above boundary, then the solution may consist of *three bars*, but any statically admissible distribution of the forces in those bars gives the same structural weight, including a two-bar and a one-bar topology (Fig. 2c).

In the case of *three supported sides*, the region boundary consists of two lines starting from the corners formed by two supported sides and having a slope 2:1 with the direction of the point load (Fig. 2d). The solution again consists, in general, of one bar (Fig. 2d) or two bars (Fig. 2e), but it may contain three bars if the point acts on the region boundary. In special cases, the optimal topology may even consist of *four bars* (load at equal distance from two parallel line supports, Fig. 2f) or *five bars* (load at the intersection of boundary lines, Fig. 2g), but these solutions are again non-unique.

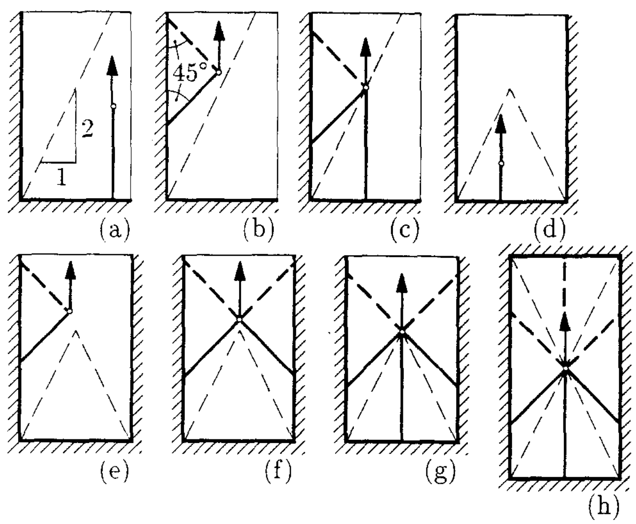


**Fig. 2.** Optimal layouts for line supports along two or more sides

Similar rules apply to *four supported sides.* In that case, however, even *six bars* are possible in the optimal topology if the side length ratio is 2:1 and the load acts at the centre of the domain (Fig. 2h). The above solutions were explained in detail elsewhere (Rozvany and Gollub 1990).

In the case of line supports along *two opposite sides* with a load parallel to those sides, the solutions for narrow domains are similar to the ones in Figs. 2b and f. For longer spans, the optimal layout becomes more complicated (Rozvany, Lewiński *et al.* 1993).

### *2.2 Line support along one side with a point load parallel to that side*

For this support condition, the optimal topology consists of only *two bars* if the distance $d$ of the loaded point from the support is smaller than both its distances from the two sides normal to it (Fig. 3a).

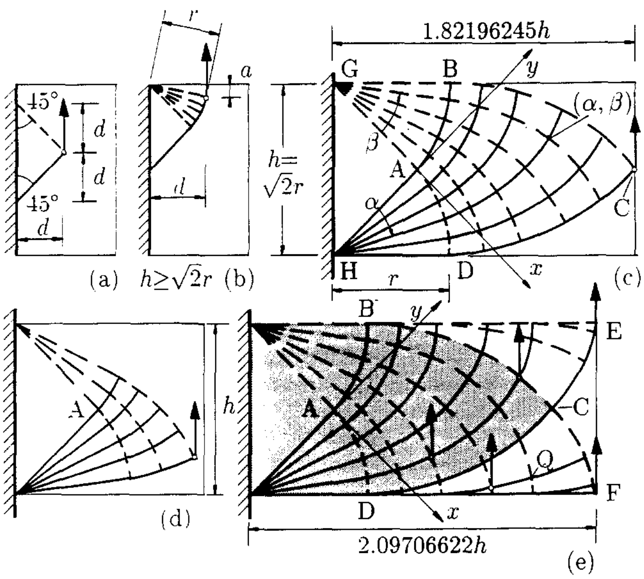


**Fig. 3.** Optimal layouts for line support along one side

The topology changes to one consisting of *two* "concentrated" *straight bars and a circular fan* if the distance $a$ from one normal free edge is smaller than the distance $d$ from the support (Fig. 3b). A further condition for this solution is that the length $h$ of the support is greater than or equal to $\sqrt{2}r$ where $r$ is the distance of the loaded point from the nearer end of the support.

If the loading falls anywhere within the area $ABCD$ in Fig. 3c, then the solution consists of *two circular fans* (e.g. $AGB$ and $ADH$) *and a Hencky-net* with curved members (e.g. $ABCD$), the members $GB$, $BC$, $HD$ and $DC$ being "concentrated" ones. The solution in Fig. 3c is for a load at the point $C$; for other load locations (e.g. Fig. 3d) the layout becomes a subset of the one in Fig. 3c. The members within the area $ABCD$ are given by (Chan 1967)

$$x(\alpha,\beta) = \bar{x}(\alpha,\beta)\cos\phi - \bar{y}(\alpha,\beta)\sin\phi\,,$$

$$y(\alpha,\beta) = \bar{x}(\alpha,\beta)\sin\phi + \bar{y}(\alpha,\beta)\cos\phi\,, \quad \phi = -\alpha+\beta\,, \qquad (1)$$

with

$$\bar{x}(\alpha,\beta)/r = F_0(\alpha,\beta) + F_1(\alpha,\beta) - \cos(\beta-\alpha)\,, \qquad (2)$$

$$\bar{y}(\alpha,\beta)/r = F_1(\alpha,\beta) + F_2(\alpha,\beta) + \sin(\beta-\alpha)\,, \qquad (3)$$

$$F_n(\alpha,\beta) = \sum_{m=0}^{\infty}(-1)^m(\alpha/\beta)^{m+\frac{n}{2}} I_{2m+n}\left(2\sqrt{\alpha\beta}\right)\,, \qquad (4)$$

in which $I_{2m+n}$ are modified Bessel functions, $(\alpha,\beta)$ are curvilinear coordinates (Fig. 3c) and $F_n(\alpha,\beta) = U_n(2\alpha, 2i\sqrt{\alpha\beta})$ where $U_n$ are Lommel functions of two variables (Watson 1966). The curves $BC$ and $DC$ in Fig. 3c are given by (1) with $\beta = \pi/4$ and $\alpha = \pi/4$, respectively.

If some loads fall within the areas $BEC$ of $DCF$ in Fig. 3e, then in the above regions *further Hencky-nets* appear in the solution. For example, the coordinates of the point $Q(\alpha,\beta)$ in Fig. 3e is given by (1) and (4) with (Chan 1967)

$$\bar{x}(\alpha,\beta)/r = F_1(\beta,\alpha)+F_2(\beta,\alpha)-\sin(\beta-\alpha)-F_1(\alpha-\theta;\beta+\theta)-F_2(\alpha-\theta;\beta+\theta)\,, \quad \bar{y}(\alpha,\beta)/r = F_0(\beta,\alpha)+F_1(\beta,\alpha)-\cos(\beta-\alpha)-F_2(\alpha-\theta;\beta+\theta)-F_3(\alpha-\theta;\beta+\theta)\,, \quad \theta=\pi/4\,. \qquad (5)$$

The layout in Fig. 3e, or a subset of it, remains valid if several point loads act anywhere within the regions $ABCD$, $BEC$ or $DCF$.

## 3 New solutions: rectangular domains with simple supports

Figures 4a and c show two loading conditions for a simply supported truss and Figs. 4b and d the equivalent support conditions for a half-truss. A problem similar to that in Fig. 4a is often referred to as the "*MBB-beam*" (e.g. Olhoff *et al.* 1991). If the span length $L$ is relatively small in comparison to the height $h$ (i.e. $L \le 2h$ ) for the problem in Fig. 4c, then the optimal layout (Fig. 4e) consists of a circular fan (Hemp 1973, p. 82). For other aspect ratios, the approximate optimal topology was shown by numerical solutions (Figs. 5a and b, see also Zhou and Rozvany 1991) obtained by Zhou for perforated plates in plane stress by the SIMP formulation (Rozvany, Zhou and Birker 1992).

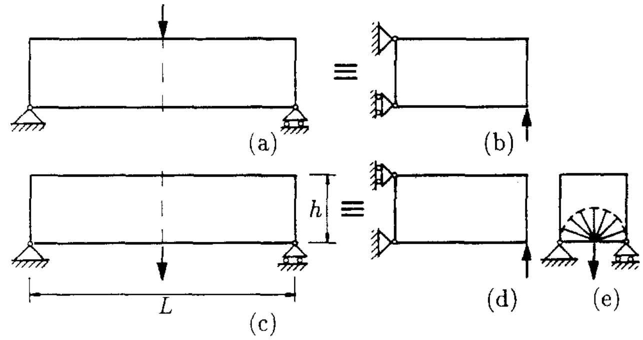


**Fig. 4.** Simple supports considered in this note

The correct optimal topology was then suggested by Zhou and its close relation to Chan's (1967) work was pointed out by Rozvany. The detailed mathematical proof of the new types of Michell-fields was obtained by Lewiński through rather lengthy derivations. The optimal layout for the problem in Fig. 4a is shown in Fig. 6a (numerical implementation by Zhou) and the corresponding regions in Fig. 6b. The layout of bars in Fig. 6 is given by the following equations.

*Region I* [extension of Chan's (1967) results]: (1), (4) and (5) with $\theta = 0$.

*Region II*: (1) and (4) with

$$\bar{x}(\alpha,\beta)/r = -\cos(\beta-\alpha)-F_2(\beta-\theta,\,\alpha+\theta)+F_4(\beta-\theta,\,\alpha+\theta)+$$

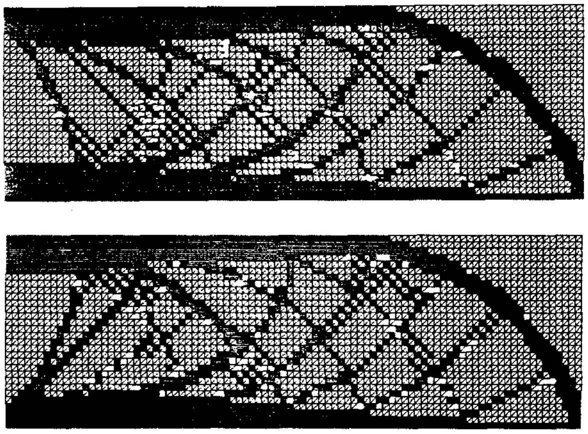

**Fig. 5.** Numerical solutions by the SIMP method for perforated plates in plane stress for the support and load conditions in Figs. 4b and d, respectively

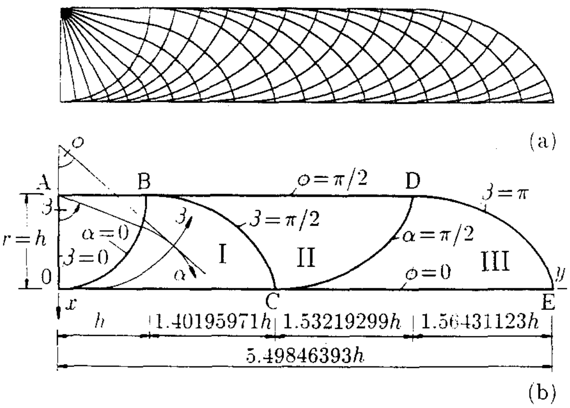


**Fig. 6.** Optimal layout for the problem in Fig. 4b

$F_0(\alpha,\beta) - F_2(\alpha,\beta)\,,$

$\overline{y}(\alpha,\beta)/r = \sin(\beta-\alpha) - F_1(\beta-\theta,\,\alpha+\theta) + F_3(\beta-\theta,\,\alpha+\theta) +$

$$F_1(\alpha,\beta) - F_3(\alpha,\beta)\,, \qquad \theta = \pi/2\,. \tag{6}$$

*Region III*: (1) and (4) with

$\overline{x}(\alpha,\beta)/r = F_0(\alpha,\beta) - F_0(\beta,\alpha) + F_4(\beta-\theta,\,\alpha+\theta) -$

$F_4(\alpha-\theta,\,\beta+\theta) + F_2(\alpha-\theta,\,\beta+\theta) - F_2(\beta-\theta,\,\alpha+\theta)\,,$

$\overline{y}(\alpha,\beta)/r = F_1(\alpha,\beta) - F_{-1}(\beta,\alpha) + F_3(\beta-\theta,\,\alpha+\theta) -$

$F_5(\alpha-\theta,\,\beta+\theta) + F_3(\alpha-\theta,\,\beta+\theta) - F_1(\beta-\theta,\,\alpha+\theta)\,,$

$$\theta = \pi/2\,. \tag{7}$$

It can be seen from Fig. 6b that the length of the regions increases as we move to the right from the supports (support conditions in Fig. 4b or points 0 and $A$ in Fig. 6b). At an infinite distance from the supports, the bar shapes are given by cycloids (Hemp 1973, pp. 94–95, with $x \to y$, $y \to -x$)

$x = -(h/2)\{1 - \cos[2(\alpha+\beta)]\}\,,$

$$y = (h/2)\{\sin[2(\alpha+\beta)] + 2(\alpha-\beta)\}\,, \tag{8}$$

The projection of each curved bar given by (8) onto the $y$-axis has a length of $h\pi/2 = 1.57079633h$ which is only 0.4% bigger then the length of the last segment in Fig. 6b. This convergence of the segment lengths to a known limiting value represents an indirect confirmation of the results in this note.

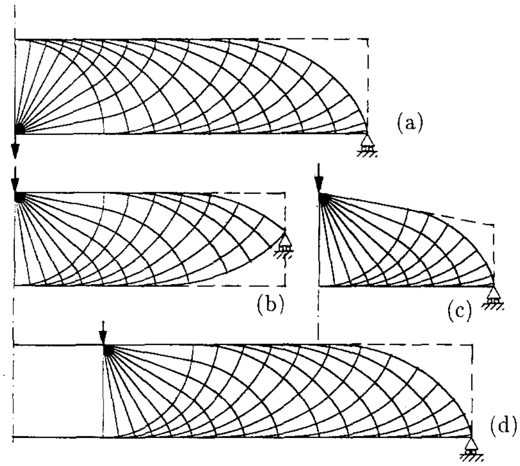


**Fig. 7.** Further solutions for simply supported trusses

The optimal layout for the problem in Fig. 4c and some further applications of the proposed new Michell fields are given in Fig. 7 which includes an optimal truss with *two point loads* (Fig. 7d).


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