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Review Paper

Multiple eigenvalues in structural optimization problems*

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Abstract This paper discusses characteristic features and inherent difficulties pertaining to the lack of usual differentiability properties in problems of sensitivity analysis and optimum structural design with respect to multiple eigenvalues. Computational aspects are illustrated via a number of examples.

Based on a mathematical perturbation technique, a general multiparameter framework is developed for computation of design sensitivities of simple as well as multiple eigenvalues of complex structures. The method is exemplified by computation of changes of simple and multiple natural transverse vibration frequencies subject to changes of different design parameters of finite element modelled, stiffener reinforced thin elastic plates.

Problems of optimization are formulated as the maximization of the smallest (simple or multiple) eigenvalue subject to a global constraint of e.g. given total volume of material of the structure, and necessary optimality conditions are derived for an arbitrary degree of multiplicity of the smallest eigenvalue. The necessary optimality conditions express (i) linear dependence of a set of generalized gradient vectors of the multiple eigenvalue and the gradient vector of the constraint, and (ii) positive semi-definiteness of a matrix of the coefficients of the linear combination.

It is shown in the paper that the optimality condition (i) can be directly applied for the development of an efficient, iterative numerical method for the optimization of structural eigenvalues of arbitrary multiplicity, and that the satisfaction of the necessary optimality condition (ii) can be readily checked when the method has converged. Application of the method is illustrated by simple, multiparameter examples of optimizing single and bimodal buckling loads of columns on elastic foundations.

1 Introduction

Multiple eigenvalues in the form of buckling loads and natural frequencies of vibration very often occur in complex structures that depend on many design parameters and have many degrees of freedom. For example, stiffener-reinforced thin-walled plate and shell structures have a dense spectrum of eigenvalues, and multiple eigenvalues are found very often. Also, symmetry of structural systems may lead to the appearance of several linearly independent buckling modes and vibration modes with multiple eigenvalues.

In 1977, Olhoff and Rasmussen discovered that the optimum eigenvalue of a clamped-clamped column of given volume is bimodal. This optimization problem was first considered by Lagrange, and its interesting history was presented in a recent paper by Cox (1992). Olhoff and Rasmussen (1977) showed that the bimodality of the optimum eigenvalue must be taken into account in the mathematical formulation of the problem in order to obtain the correct optimum solution. They first demonstrated that an analytical solution obtained earlier by Tadjbakhsh and Keller (1962) under the tacit assumption of a simple buckling load is not optimal, then presented a bimodal formulation of the problem, solved it numerically, and obtained the correct optimum design. The optimum bimodal buckling load obtained was later confirmed to be correct to within a slight deviation of the sixth digit# by analytical solutions obtained independently by Seyranian (1983, 1984) and Masur (1984). The discovery in 1977 of multiple optimum eigenvalues in structural optimization problems, and the necessity of applying a bi- or multimodal formulation in such cases, opened a new field for theoretical investigations and development of methods of numerical analysis and solution.

Prager and Prager (1979) and Choi and Haug (1981) presented unimodal and bimodal optimum solutions for systems with few degrees of freedom, confirming the appearance of multiple eigenvalues in optimization problems. A wealth of references on multimodal optimization problems and specific results for columns, arches, plates and shells can be found in comprehensive surveys by Olhoff and Taylor (1983), Gajewski and Zyczkowski (1988), Zyczkowski (1989) and Gajewski (1990). A survey of other problems of optimum design with respect to structural eigenvalues was earlier published by Olhoff (1980).

One of the main problems related to multiple eigenvalues is their non-differentiability in the common (Fréchet) sense. This was revealed by Masur and Mroz (1979, 1980) and Haug and Rousselet (1980). The non-differentiability creates difficulties in finding sensitivities of multiple eigenvalues with respect to design changes and derivation of necessary optimality conditions in optimization problems. Choi and Haug (1981) used a Lagrange multiplier method for bimodal problems and showed that this method, which is very useful for differentiable criteria and constraints, may yield incorrect results.

Haug and Rousselet (1980) proved the existence of directional derivatives of multiple eigenvalues and obtained explicit formulae for derivatives. Bratus and Seyranian (1983) and Seyranian (1987) presented sensitivity analysis of mul-

*Dedicated to the memory of Ernest F. Masur

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#This accuracy of the numerical result in the paper by Olhoff and Rasmussen (1977) was obtained by solving the problem for different values of the mesh length d, and extrapolating the result to $d = 0$ by means of Newton's formula.

tiple eigenvalues based on a perturbation technique and derived necessary optimality conditions. The main advantage of these necessary optimality conditions is that, when compared with those obtained by previous researchers, they do not contain variations of design variables. Similar developments were presented by Masur (1984, 1985). It was with the use of these necessary optimality conditions that Seyranian (1983, 1984) and Masur (1984) independently of each other obtained the analytical solution to the bimodal optimum clamped-clamped column problem mentioned above.

Overton (1988) considered the minimization of the maximum eigenvalue of a symmetric matrix. This problem is similar to the problem of maximizing the minimum eigenvalue. Derivation of necessary optimality conditions using a so-called bound formulation of such problems had earlier been presented by Bendsøe *et al.* (1983) and Taylor and Bendsøe (1984). In a recent paper Cox and Overton (1992) presented new mathematical results for optimization problems of columns against buckling. They derived necessary optimality conditions using advanced nonsmooth optimization methods. Their results for optimum columns are in good agreement with the results obtained earlier by Olhoff and Rasmussen (1977), Seyranian (1983, 1984), and Masur (1984).

Numerical algorithms for the solution of structural optimization problems with multiple eigenvalues have been suggested and discussed by, among others, Olhoff and Rasmussen (1977), Choi *et al.* (1982), Olhoff and Plaut (1983), Bendsøe *et al.* (1983), Myslinski and Sokolowski (1985), Zhong and Cheng (1986), Plaut *et al.* (1986), Gajewski and Zyczkowski (1988), Overton (1988), and Cox and Overton (1992).

The present paper is devoted to the development of efficient methods for design sensitivity analysis and optimization of simple as well as multiple eigenvalues of complex structures. Section 2 presents the mathematical basis of the development which is a perturbation technique that allows us to obtain sensitivities of both multiple eigenvalues and corresponding eigenvectors. We first consider the case where only a single design parameter is changed and then present a method for efficient calculation of design sensitivities of simple and multiple eigenvalues when all design parameters are changed simultaneously.

It is shown that the design sensitivities of multiple eigenvalues of finite element modelled structures can be computed within the framework of the semi-analytical approach for sensitivity analysis. This approach can be easily augmented to yield "exact" numerical sensitivities by a new technique published by Olhoff *et al.* (1993) and Lund and Olhoff (1993). By this technique, which is computationally inexpensive and very easy to implement as an integral part of the finite element analysis, we completely avoid the often severely erroneous shape design sensitivities of beam, plate, and shell structures that may result from application of the traditional method of semi-analytical sensitivity analysis (see e.g. Barthelemy and Haftka 1988).

The efficiency of the new method of sensitivity analysis of multiple eigenvalues is demonstrated and illustrated by studies of problems of changes of different types of design parameters for free, transversely vibrating stiffener reinforced elastic plates. It is found that the method is very reliable and accurate, and that it constitutes an excellent basis for the solution of structural optimization problems.

Sections 3, 4 and 5 are devoted to the optimization of multiple eigenvalues. Section 3 contains three simple examples that illustrate the main ideas. Then, in Section 4, we formulate the structural eigenvalue optimization problem as the maximization of the smallest eigenvalue and derive the necessary optimality conditions, first for bimodal optimization, and then for the general case where the smallest eigenvalue has arbitrary multiplicity. The optimality conditions express (i) linear dependence between a set of generalized gradient vectors of the multiple eigenvalue and the gradient vector of the constraint condition (given total volume of structural material), and (ii) positive semi-definiteness of a matrix of the coefficients of the linear combination.

Section 5 shows that these optimality conditions are very useful for the construction of an iterative numerical procedure for optimization of structural eigenvalues of arbitrary multiplicity. Thus, we develop a numerical method of solution in which the condition (i) is directly used to determine an ascent direction in the design space for the smallest (simple or multiple) eigenvalue, and in which the satisfaction of (ii) can be readily checked when the method has converged. The application and efficiency of this method of optimization of simple or multiple fundamental eigenvalues is demonstrated for examples of the optimum design of the columns on elastic foundations.

2 Sensitivity analysis for eigenvalue problems

In this section we seek general formulations for sensitivity analysis of finite element discretized structural eigenvalue problems. These problems appear in structural vibration and stability analyses. For conservative systems without damping these problems lead to real eigenvalues representing free vibration frequencies or buckling loads. We will confine the analysis to seeking sensitivities of eigenvalues only.

2.1 Introduction

The finite element formulation for the real, symmetric, structural eigenvalue problem is

$$\mathbf{K}\phi_j = \lambda_j \mathbf{M}\phi_j, \quad j = 1, \dots, n, \tag{1}$$

where $\mathbf{K}$ and $\mathbf{M}$ are symmetric positive definite matrices, λ_j is the eigenvalue and ϕ_j is the corresponding eigenvector. The dimension of the problem is denoted by n, so (1) has n solutions consisting of eigenvalues λ_j and corresponding eigenvectors ϕ_j. The eigenvalues are all real and can be ordered in the following manner after magnitude:

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \leq \dots \leq \lambda_n . \tag{2}$$

In the following it is assumed that the eigenvectors have been $\mathbf{M}$-orthonormalized, i.e.

$$\phi_j^T \mathbf{M} \phi_k = \delta_{jk} , \quad j, k = 1, \dots, n , \tag{3}$$

where δ_{jk} denotes Kronecker's delta.

If we premultiply (1) by ϕ_j^T we obtain

$$\phi_j^T \mathbf{K} \phi_k = \lambda_j \delta_{jk} , \quad j, k = 1, \dots, n , \tag{4}$$

meaning that the eigenvectors are also $\mathbf{K}$-orthogonal.

So far we have not mentioned multiple eigenvalues and eigenvectors. In this case the eigenvectors are not unique. In fact, an infinite number of linear combinations of the eigenvectors corresponding to the repeated eigenvalue will satisfy (1) and (3). However, we can choose a set of M-orthonormal eigenvectors which span the subspace that corresponds to a multiple eigenvalue. In other words, if we assume that λ_j has multiplicity N (i.e. $\lambda_j = \lambda_{j+1} = \ldots = \lambda_{j+N-1}$), then we can choose N eigenvectors $\boldsymbol{\phi}_j$ which span the N-dimensional subspace corresponding to the eigenvalues of magnitude λ_j and satisfy the orthogonality conditions in (3) and (4).

2.2 *Design sensitivity analysis of simple eigenvalues*

We assume that the shape and size of the structure are governed by a set of design variables a_i, $i = 1, \ldots, I$ and the goal is to obtain expressions for eigenvalue sensitivities with respect to these design variables. It is also assumed that the components of the $\mathbf{K}$ and $\mathbf{M}$ matrices are smooth functions of design variables a_i.

The direct approach to obtaining the eigenvalue sensitivities is to differentiate (1) with respect to a design variable a_i. Assuming that λ_j is simple, we have

$$\frac{\partial \mathbf{K}}{\partial a_i}\boldsymbol{\phi}_j + (\mathbf{K} - \lambda_j \mathbf{M})\frac{\partial \boldsymbol{\phi}_j}{\partial a_i} = \frac{\partial \lambda_j}{\partial a_i}\mathbf{M}\boldsymbol{\phi}_j + \lambda_j \frac{\partial \mathbf{M}}{\partial a_i}\boldsymbol{\phi}_j\,,$$
$$i = 1, \ldots, I\,. \tag{5}$$

By premultiplying (5) by $\boldsymbol{\phi}_j^T$ and making use of (1) and the normalization of (3), the following expression is obtained for the eigenvalue sensitivity in the case of simple eigenvalues λ_j (see e.g. Courant and Hilbert 1953; Wittrick 1962):

$$\frac{\partial \lambda_j}{\partial a_i} = \boldsymbol{\phi}_j^T \left(\frac{\partial \mathbf{K}}{\partial a_i} - \lambda_j \frac{\partial \mathbf{M}}{\partial a_i}\right)\boldsymbol{\phi}_j\,, \quad i = 1, \ldots, I\,. \tag{6}$$

The most important point here is that the only unknown quantities in (6) are the derivatives of the $\mathbf{K}$ and $\mathbf{M}$ matrices. These derivatives are normally calculated at the element level, i.e.

$$\frac{\partial \mathbf{K}}{\partial a_i} = \sum_{n_e} \frac{\partial \mathbf{k}}{\partial a_i}\,, \quad i = 1, \ldots, I\,, \tag{7}$$

$$\frac{\partial \mathbf{M}}{\partial a_i} = \sum_{n_e} \frac{\partial \mathbf{m}}{\partial a_i}\,, \quad i = 1, \ldots, I\,, \tag{8}$$

where $\mathbf{k}$ and $\mathbf{m}$ are element matrices and n_e is the number of finite elements.

These element derivatives can be either calculated analytically, if possible, or by using first-order finite difference approximations as known from the method of semi-analytical sensitivity analysis of finite element discretized structures with linearly elastic, static response, i.e.

$$\frac{\partial \mathbf{k}(\mathbf{a})}{\partial a_i} \simeq \frac{\Delta \mathbf{k}(a_1, \ldots, a_I)}{\Delta a_i} =$$
$$\frac{\mathbf{k}(a_1, \ldots, a_i + \Delta a_i, \ldots, a_I) - \mathbf{k}(a_1, \ldots, a_i, \ldots, a_I)}{\Delta a_i}\,, \tag{9}$$

$$\frac{\partial \mathbf{m}(\mathbf{a})}{\partial a_i} \simeq \frac{\Delta \mathbf{m}(a_1, \ldots, a_I)}{\Delta a_i} =$$
$$\frac{\mathbf{m}(a_1, \ldots, a_i + \Delta a_i, \ldots, a_I) - \mathbf{m}(a_1, \ldots, a_i, \ldots, a_I)}{\Delta a_i}\,. \tag{10}$$

These finite difference approximations can be upgraded to "exact" numerical derivatives (exact except for round-off errors) by using the method of exact semi-analytical sensitivity analysis (see Olhoff *et al.* 1993; Lund and Olhoff 1993).

If all the design variables a_i are changed simultaneously then we can find the linear increment of the simple eigenvalue λ_j in the form

$$\Delta \lambda_j = \sum_{i=1}^{I} \boldsymbol{\phi}_j^T \left[\frac{\partial \mathbf{K}}{\partial a_i} - \lambda_j \frac{\partial \mathbf{M}}{\partial a_i}\right]\boldsymbol{\phi}_j \Delta a_i\,. \tag{11}$$

This equation is valid due to the differentiability of simple eigenvalues with respect to design variables.

The expression obtained in (11) can be written in the form of the scalar product

$$\Delta \lambda_j = \boldsymbol{\nabla}^T \lambda_j \boldsymbol{\Delta}\mathbf{a}\,, \tag{12}$$

where $\boldsymbol{\nabla}\lambda_j$ denotes the gradient vector of λ_j and $\boldsymbol{\Delta}\mathbf{a}$ is the vector of changes of the design variables a_i

$$\boldsymbol{\nabla}\lambda_j = \left(\frac{\partial \lambda_j}{\partial a_1}, \ldots, \frac{\partial \lambda_j}{\partial a_I}\right)\,, \quad \boldsymbol{\Delta}\mathbf{a} = (\Delta a_1, \ldots, \Delta a_I)\,. \tag{13}$$

These notations are useful for parametric studies of eigenvalues as well as for formulation of optimization problems.

2.3 *Design sensitivity analysis of multiple eigenvalues*

When the solution of the generalized eigenvalue problem in (1) yields an N-fold multiple eigenvalue

$$\tilde{\lambda} = \lambda_j\,, \quad j = 1, \ldots, N\,, \tag{14}$$

where, for convenience, the repeated eigenvalues have been numbered from 1 to N, the computation of the sensitivities of this eigenvalue is not straightforward. This is due to the fact that the eigenvectors $\boldsymbol{\phi}_j$, $j = 1, \ldots, N$ of the repeated eigenvalues are not unique. Thus, any linear combination of the eigenvectors will satisfy the original eigenvalue problem (1). We assume that the eigenvectors $\boldsymbol{\phi}_j$ have been M-orthonormalized, i.e.

$$\boldsymbol{\phi}_j^T \mathbf{M}\boldsymbol{\phi}_k = \delta_{jk}\,, \quad j, k = 1, \ldots, N\,. \tag{15}$$

In the following sensitivity analysis we shall use such eigenvectors $\tilde{\boldsymbol{\phi}}_j$ that remain continuous with design changes, see Courant and Hilbert (1953). For this purpose we introduce linear combinations of eigenvectors $\boldsymbol{\phi}_k$

$$\tilde{\boldsymbol{\phi}}_j = \sum_{k=1}^{N} \beta_{jk}\boldsymbol{\phi}_k\,, \quad j = 1, \ldots, N\,, \tag{16}$$

where β_{jk} are unknown coefficients to be determined.

Works by Courant and Hilbert (1953), Wittrick (1962), and Lancaster (1964) have provided a basis for calculating the sensitivities of multiple eigenvalues. It is shown that the design sensitivities of multiple eigenvalues can be found by formulation and solution of a subeigenvalue problem.

Let us first consider a change Δa_i of one arbitrarily chosen design parameter a_i. Due to the change of this parameter the $\mathbf{K}$ and $\mathbf{M}$ matrices will be incremented, i.e. the new matrices become

$$\mathbf{K} + \frac{\partial \mathbf{K}}{\partial a_i}\Delta a_i \quad \text{and} \quad \mathbf{M} + \frac{\partial \mathbf{M}}{\partial a_i}\Delta a_i\,, \quad i = 1, \ldots, I\,. \tag{17}$$

Then multiple eigenvalues and corresponding eigenvectors for the perturbed design can be written as

$$\lambda_j(a_i + \varepsilon \Delta a_i) = \tilde{\lambda} + \varepsilon \mu_j(a_i, \Delta a_i) + o(\varepsilon)\,, \quad j = 1, \ldots, N\,,$$

$$\phi_j(a_i+\varepsilon\Delta a_i) = \tilde{\phi}_j+\varepsilon\nu_j(a_i,\Delta a_i)+o(\varepsilon), \quad j=1,\ldots,N, \tag{18}$$

where μ_j and ν_j are unknown eigenvalue and eigenvector sensitivities, respectively.

Substituting (17) and (18) into the main eigenvalue problem in (1), we obtain in the first approximation

$$\left(\frac{\partial \mathbf{K}}{\partial a_i} - \tilde{\lambda}\frac{\partial \mathbf{M}}{\partial a_i}\right)\tilde{\phi}_j + (\mathbf{K}-\tilde{\lambda}\mathbf{M})\nu_j = \mu_j \mathbf{M}\tilde{\phi}_j . \tag{19}$$

Premultiplying this equation by ϕ_s^T gives

$$\phi_s^T = \left(\frac{\partial \mathbf{K}}{\partial a_i} - \tilde{\lambda}\frac{\partial \mathbf{M}}{\partial a_i}\right)\tilde{\phi}_j = \mu_j \phi_s^T \mathbf{M}\tilde{\phi}_j ,$$

$$s=1,\ldots,N. \tag{20}$$

Here the term $\phi_s^T(\mathbf{K}-\tilde{\lambda}\mathbf{M})\nu_j = \nu_j^T(\mathbf{K}-\tilde{\lambda}\mathbf{M})\phi_s$ drops out because ϕ_s is an eigenvector corresponding to $\tilde{\lambda}$.

Recalling that $\tilde{\phi}_j$ is the linear combination in (16) of the original eigenvectors ϕ_k, we obtain from (20) the system of linear algebraic equations of unknown coefficients β_{jk}

$$\sum_{k=1}^{N} \beta_{jk}\left[\phi_s^T\left(\frac{\partial \mathbf{K}}{\partial a_i} - \tilde{\lambda}\frac{\partial \mathbf{M}}{\partial a_i}\right)\phi_k - \mu_j\delta_{sk}\right] = 0,$$

$$s=1,\ldots,N, \tag{21}$$

where the **M**-orthonormalization (15) has been used.

A nontrivial solution to these equations only exists if the determinant of the system is equal to zero

$$\det\left[\phi_s^T\left(\frac{\partial \mathbf{K}}{\partial a_i} - \tilde{\lambda}\frac{\partial \mathbf{M}}{\partial a_i}\right)\phi_k - \mu\delta_{sk}\right] = 0,$$

$$s,k=1,\ldots,N, \quad i=1,\ldots,I. \tag{22}$$

This is the main equation for determining the coefficients μ_j, $j=1,\ldots,N$ in (18) which represent the sensitivities of the multiple eigenvalue $\tilde{\lambda}$ with respect to changes Δa_i of a single design parameter a_i. As in the case of simple eigenvalues, the derivatives of the **K** and **M** matrices, respectively, must be calculated first, and then the eigenvalue problem of (22) is easily formulated and solved.

If the off-diagonal terms in the quadratic matrix of dimension N in (22) are equal to zero, the eigenvalues of this matrix, i.e. the directional derivatives of the multiple eigenvalue $\tilde{\lambda}$, are equal to the traditional Fréchet derivatives obtained by using (6).

Let us consider the general case when all the design variables a_i, $i=1,\ldots,I$ are changed simultaneously. It should be noted that multiple eigenvalues are not differentiable in the common sense, i.e. not Fréchet-differentiable (see e.g. Haug *et al.* 1986). This means that the expressions for the eigenvalue increments in (11) and (12) are no longer valid. Thus, to find the sensitivities of multiple eigenvalues we must use directional derivatives in the design space.

For this purpose, for the vector of design variables $\mathbf{a} = (a_1,\ldots,a_I)$, we consider a variation in the form $\mathbf{a}+\varepsilon\mathbf{e}$, where $\mathbf{e}$ is an arbitrary vector of variation $\mathbf{e}=(e_1,\ldots,e_I)$ with the unit norm $\| e \| = \sqrt{e_1^2+\ldots+e_I^2} = 1$ and ε is a small positive parameter. The vector $\mathbf{e}$ represents a direction in the design space along which the design variables a_i are changed, and ε represents the magnitude of the perturbation in this direction.

As a result of perturbation of the vector $\mathbf{a}$, the matrices **K** and **M** are incremented and become

$$\mathbf{K}+\varepsilon\sum_{i=1}^{I}\frac{\partial \mathbf{K}}{\partial a_i}e_i, \quad \mathbf{M}+\varepsilon\sum_{i=1}^{I}\frac{\partial \mathbf{M}}{\partial a_i}e_i . \tag{23}$$

Using expansions for λ_j and ϕ_j in the form

$$\lambda_j = \tilde{\lambda} + \varepsilon\mu_j + o(\varepsilon),$$

$$\phi_j = \tilde{\phi}_j + \varepsilon\nu_j + o(\varepsilon), \quad j=1,\ldots,N, \tag{24}$$

and performing the same manipulations as earlier, instead of (22) we obtain the following N-th order equation for determining the sensitivities $\mu=\mu_j$, $j=1,\ldots,N$ of the eigenvalues λ_j:

$$\det\left[\sum_{i=1}^{I}\phi_s^T\left(\frac{\partial \mathbf{K}}{\partial a_i} - \tilde{\lambda}\frac{\partial \mathbf{M}}{\partial a_i}\right)\phi_k e_i - \mu\delta_{sk}\right] = 0,$$

$$s,k=1,\ldots,N. \tag{25}$$

If we introduce the *generalized gradient* vectors $\mathbf{f}_{sk}$ of dimension I

$$\mathbf{f}_{sk} =$$

$$\left(\phi_s^T\left[\frac{\partial \mathbf{K}}{\partial a_1} - \tilde{\lambda}\frac{\partial \mathbf{M}}{\partial a_1}\right]\phi_k,\ldots,\phi_s^T\left[\frac{\partial \mathbf{K}}{\partial a_I} - \tilde{\lambda}\frac{\partial \mathbf{M}}{\partial a_I}\right]\phi_k\right), \tag{26}$$

then (25) can be written in the form

$$\det\left[\mathbf{f}_{sk}^T\mathbf{e} - \mu\delta_{sk}\right] = 0, \quad s,k=1,\ldots,N. \tag{27}$$

Note that $\mathbf{f}_{sk} = \mathbf{f}_{ks}$ due to the symmetry of the matrices **K** and **M**. Note also that although equipped with two subscripts, the generalized gradients $\mathbf{f}_{sk}$ are vectors (of dimension I); *the two subscripts refer to the modes from which the generalized gradient vector is calculated.* Thus, in (27) $\mathbf{f}_{sk}^T\mathbf{e}$, $s,k=1,\ldots,N$, denotes scalar products.

Thus, knowing the eigenvectors ϕ_k, $k=1,\ldots,N$ corresponding to the multiple eigenvalue $\tilde{\lambda}$, we can construct the generalized gradient vectors $\mathbf{f}_{sk}$ and determine the sensitivities $\mu=\mu_j$, $j=1,\ldots,N$ for any vector of variation $\mathbf{e}$, i.e. for any direction in the space of the design variables. The quantities $\mu_j = \Delta\lambda_j/\Delta\varepsilon$ depend on $\mathbf{e}$ and constitute the directional derivatives of the multiple eigenvalue $\tilde{\lambda}$, cf. (24). In this form (27) was obtained by Bratus and Seyranian (1983), and Seyranian (1987), see also Haug and Rousselet (1980), Masur (1984, 1985), and Haug *et al.* (1986).

In many cases it is expedient to eliminate the unit vector $\mathbf{e}$ from (27) and establish a formula for determining the increments $\Delta\lambda_j$, $j=1,\ldots,N$ of the N-fold eigenvalue $\tilde{\lambda}$ subject to a given vector $\Delta\mathbf{a} = (\Delta a_1,\ldots,\Delta a_I)$ of actual increments of the design variables a_i, $i=1,\ldots,I$. To this end, we multiply each of the components in (27) by ε, note from the foregoing that $\varepsilon\mathbf{e} = \Delta\mathbf{a}$ and $\varepsilon\mu_j = \Delta\lambda_j$, $j=1,\ldots,N$, and obtain

$$\det\left[\mathbf{f}_{sk}^T\Delta a - \delta_{sk}\Delta\lambda\right] = 0, \quad s,k=1,\ldots,N. \tag{28}$$

If we solve this N-th order algebraic equation for $\Delta\lambda$, we obtain the increments $\Delta\lambda = \Delta\lambda_j$, $j=1,\ldots,N$ of the N-fold eigenvalue corresponding to the vector $\Delta\mathbf{a}$ of increments of the design variables.

2.4 Numerical example 1: design sensitivity analysis of a vibrating plate with ribs

To illustrate how the results of Sections 2.2 and 2.3 can be used in eigenfrequency design sensitivity analysis, we consider a square plate as shown in Fig. 1.

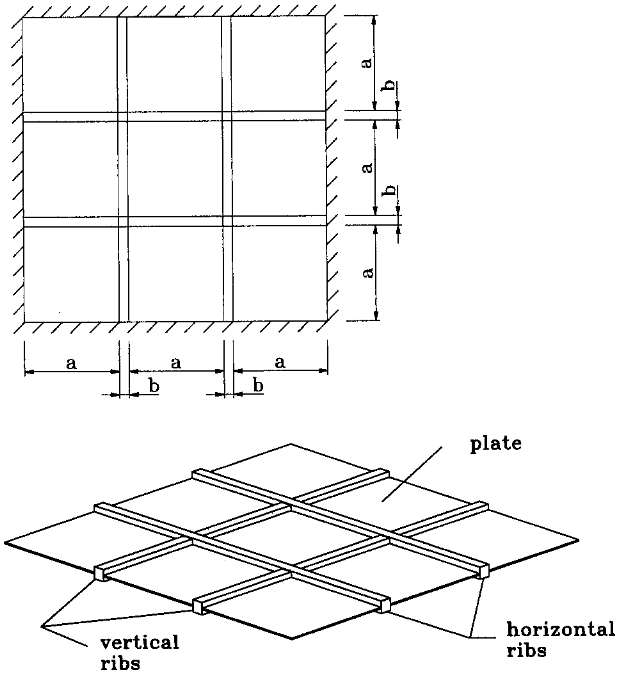

Fig. 1. Square plate with ribs

The plate is clamped at all edges and reinforced by two horizontal and two vertical ribs. The length $a = 0.5$ m, $b = 0.05$ m, the thickness of the ribs is 0.05 m, and the thickness of the plate is 0.005 m. Both the plate and the ribs are made of steel with the following material properties:

Young's modulus = 210000 MPa
Poisson's ratio = 0.3
mass density = 7800 kg/m^3

The structure is symmetric and therefore multiple eigenfrequencies are expected. We want to find sensitivities of both the simple and the multiple eigenfrequencies with respect to 6 different design variables.

The finite element model consists of 1156 4-node isoparametric Mindlin plate elements and the model has 5445 d.o.f. The lowest eigenfrequencies f_j are found by the subspace iteration method (see Bathe 1982). All eigenvectors are **M**-orthonormalized, see (3), and we will consider the 4 lowest eigenfrequencies. From the analysis it is found that the lowest eigenfrequency is simple and has the value 92.20 Hz, but the second and third eigenfrequencies are identical and have the value 161.71 Hz. In this example, the eigenfrequencies are considered to be identical when the relative difference between the values is $\leq 10^{-4}$. The fourth eigenfrequency is simple and equal to 175.03 Hz. The eigenmodes corresponding to the 4 eigenfrequencies can be seen in Figs. 2-5, and the influence of the ribs is very clear.

As the second eigenfrequency is multiple, an infinite number of linear combinations of the eigenvectors shown in Figs. 3 and 4 corresponding to the multiple eigenfrequency will

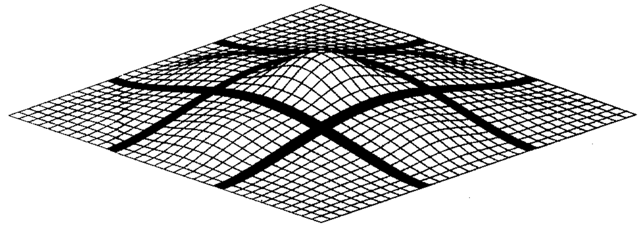

Fig. 2. First eigenmode, $f_1 = 92.20$ Hz

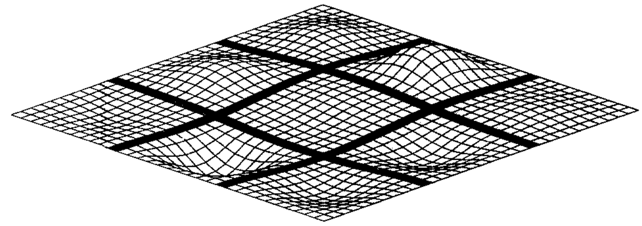

Fig. 3. Second eigenmode, $f_2 = 161.71$ Hz

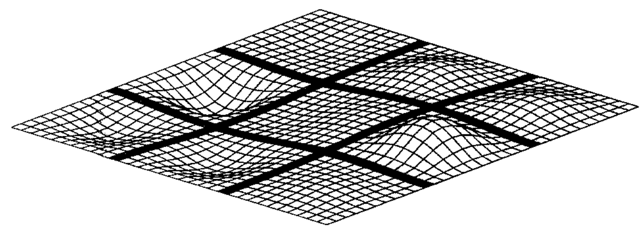

Fig. 4. Third eigenmode, $f_3 = 161.71$ Hz

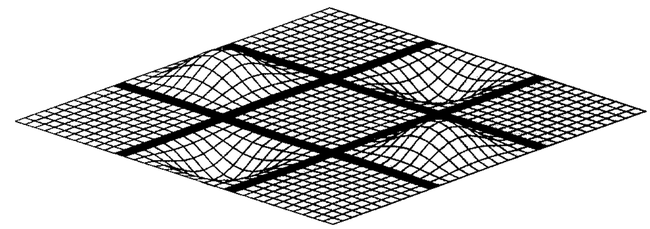

Fig. 5. Fourth eigenmode, $f_4 = 175.03$ Hz

satisfy the general eigenvalue problem in (1) and the **M**-orthonormalization condition in (3).

The design sensitivities with respect to changes of single design variables will be computed by two different methods, namely the overall finite difference (OFD) method and the semi-analytical (SA) method using (6) and (22). The OFD method implies that the design is perturbed, a new eigenfrequency analysis performed, and that the eigenfrequency sensitivities μ_j are then found by

$$\mu_j \simeq \frac{\Delta f_j(a_1, \ldots, a_I)}{\Delta a_i} = \frac{f_j(a_1, \ldots, a_i + \Delta a_i, \ldots, a_I) - f_j(a_1, \ldots, a_i, \ldots, a_I)}{\Delta a_i} .$$

The OFD method is used as a reference method whose limit with regard to design sensitivity accuracy is known to be set only by the solution procedure, the discretization and the usual accuracy capabilities of the applied finite element, when the design perturbation Δa_i is sufficiently small.

In the usual SA method, the derivatives of the element stiffness and mass matrices [see (7), (8), (9) and (10)] are calculated by simple finite difference approximations. It has been shown by Barthelemy and Haftka (1988) that the usual SA method is prone to severe inaccuracy problems when used for calculation of design sensitivities of beams, plates and shells, so we apply a new, recently developed version of the SA method (see Olhoff *et al.* 1993; Lund and Olhoff 1993). In this new method, the element matrix derivatives are upgraded to "exact" numerical derivatives (exact up to round-off errors). Then the eigenfrequency sensitivities μ_j are determined by using (6) for the simple eigenfrequencies f_1, f_4 and using (22) for the multiple eigenfrequency $f_2 = f_3$ as we consider the case of change of single design variables. Furthermore, we shall calculate the sensitivities of all four eigenfrequencies regarding them as simple in order to see the consequences of this erroneous assumption.

The first design sensitivity analysis is with respect to the thickness of the plate as shown in Fig. 6.

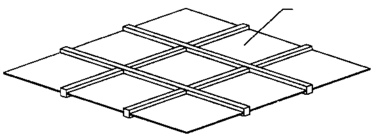

Fig. 6. Design variable 1: plate thickness

The sensitivities are shown in Table 1 and it is seen that the same results are obtained by the OFD method and the SA method using (6) and (22).

Table 1. Eigenfrequency sensitivities with respect to design variable 1: plate thickness

Frequency j	OFD method $\frac{\Delta f_j}{\Delta a_1}$	(6) and (22) μ_j	(6) $\frac{\partial f_j}{\partial a_1}$
1	-1551.6	-1551.6	-1551.6
2	14093.2	14094.1	14094.1
3	14093.2	14094.1	14094.1
4	31407.9	31406.8	31406.8

Increasing the plate thickness is a symmetric design change and therefore we may expect that the multiple eigenfrequency remains multiple. Note that while the sensitivities of the other eigenfrequencies are positive, the sensitivity of the lowest eigenfrequency is negative, i.e. the first eigenfrequency will decrease if the thickness of the plate is increased. These results require some explanation.

The eigenmode corresponding to the lowest eigenfrequency can be characterized as a global mode while all higher order eigenmodes can be regarded as nearly-local modes for each of the nine subdomains of the plate, see Figs. 2-5. Thus, increasing the plate thickness will have little effect on the overall stiffness of the structure because it is mainly governed by the ribs, whereas the thickness of the plate has a large influence on the total mass, i.e. the inertia forces. Therefore, increasing the plate thickness will decrease the lowest eigenfrequency which mainly depends on the overall stiffness and total mass of the structure.

The higher order eigenmodes, on the other hand, mainly depend on the local stiffness and mass of each of the nine subdomains, and increase of the plate thickness has a larger effect on the local stiffness than on the local mass of each subdomain. This is the reason why the higher eigenfrequencies will increase with increasing plate thickness.

For this design change, the erroneous assumption of using (6) to the double eigenfrequency (see last column in Table 1) in fact gives the same sensitivities as obtained by (22). This shows that the influence of the off-diagonal terms $\mathbf{f}_{12}^T\mathbf{e} = \mathbf{f}_{21}^T\mathbf{e}$ in the sensitivity matrix in (22) is very weak for this symmetric design change.

Next, the eigenfrequency sensitivities are found with respect to the thickness of the ribs as shown in Fig. 7. The results are shown in Table 2.

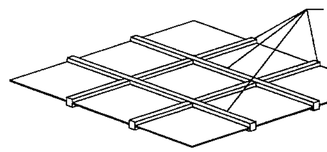

Fig. 7. Design variable 2: rib thickness

Table 2. Eigenfrequency sensitivities with respect to design variable 2: rib thickness

Frequency j	OFD method $\frac{\Delta f_j}{\Delta a_2}$	(6) and (22) μ_j	(6) $\frac{\partial f_j}{\partial a_2}$
1	1878.6	1878.6	1878.6
2	1714.2	1714.2	1714.2
3	1714.2	1714.2	1714.2
4	304.7	305.1	305.1

It is seen in Table 2 that again the same results are obtained by the OFD method and the SA method using (6) and (22). All the sensitivities are positive, and again the multiple eigenfrequency remains multiple with this design change.

Next we want to determine sensitivities of the eigenfrequencies when the position of the horizontal ribs is changed. The design variable is the distance between the horizontal ribs, see Fig. 8. The results are shown in Table 3.

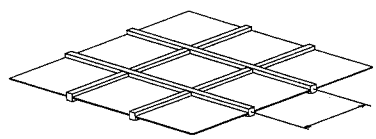

Fig. 8. Design variable 3: position of horizontal ribs

Table 3. Eigenfrequency sensitivities with respect to design variable 3: position of horizontal ribs

Frequency j	OFD method $\frac{\Delta f_j}{\Delta a_3}$	(6) and (22) μ_j	(6) $\frac{\partial f_j}{\partial a_3}$
1	-40.9	-40.9	-40.9
2	-380.6	-380.6	-287.5
3	186.9	186.9	93.7
4	-169.1	-168.6	-168.6

It is seen from Table 3 that the multiple eigenfrequency $\tilde{f} = f_2 = f_3$ will split when the distance between the horizontal ribs is increased. Now we can also see differences between the two last columns in Table 3 showing the influence of off-diagonal terms in (22), which implies that calculation of sensitivities of the double eigenfrequency by means of the single-modal formula in (6) gives erroneous results.

Next the distance between the vertical ribs is used as a design variable as shown in Fig. 9, and the results are presented in Table 4.

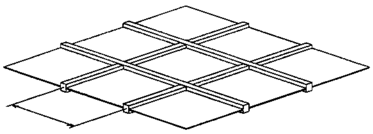

Fig. 9. Design variable 4: position of vertical ribs

Table 4. Eigenfrequency sensitivities with respect to design variable 4: position of vertical ribs

Frequency j	OFD method $\frac{\Delta f_j}{\Delta a_4}$	(6) and (22) μ_j	(6) $\frac{\partial f_j}{\partial a_4}$
1	-40.9	-40.9	-40.9
2	-380.6	186.9	93.7
3	186.9	-380.6	-287.5
4	-166.0	-167.0	-167.0

The sensitivities for this design variable should be the same as those obtained for design variable 3, the distance between the horizontal ribs, except that the sensitivities for the multiple eigenfrequency $f_2 = f_3$ should be interchanged when using (22). It is seen that the OFD method does not display this situation because the eigenfrequencies are ordered by magnitude in each analysis when using the OFD method. More importantly, we again note that application of the single-modal equation (6) yields erroneous sensitivities of the multiple eigenfrequency $f_2 = f_3$.

The next design variable is the width of the horizontal ribs, see Fig. 10, and the results are shown in Table 5.

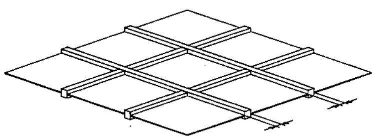

Fig. 10. Design variable 5: width of horizontal ribs

Table 5. Eigenfrequency sensitivities with respect to design variable 5: width of horizontal ribs

Frequency j	OFD method $\frac{\Delta f_j}{\Delta a_5}$	(6) and (22) μ_j	(6) $\frac{\partial f_j}{\partial a_5}$
1	-273.4	-273.4	-273.4
2	-866.2	-866.2	-719.7
3	26.4	26.4	-120.1
4	-401.4	-400.9	-400.9

Again the results obtained by the OFD method and the SA method using (6) and (22) are very similar, and it is seen that the multiple eigenfrequency splits with this design change. Furthermore, the sensitivities of the double eigenfrequency obtained by using (6) even have a wrong sign, so using the erroneous assumption of regarding the eigenfrequencies as simple leads to completely wrong results for this design change.

The last design variable is the width of the vertical ribs as shown in Fig. 11.

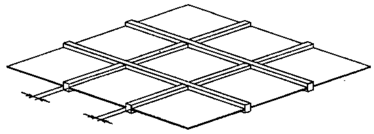

Fig. 11. Design variable 6: width of vertical ribs

Table 6. Eigenfrequency sensitivities with respect to design variable 6: width of vertical ribs

Frequency j	OFD method $\frac{\Delta f_j}{\Delta a_6}$	(6) and (22) μ_j	(6) $\frac{\partial f_j}{\partial a_6}$
1	-273.4	-273.4	-273.4
2	-866.2	26.4	-120.1
3	26.4	-866.2	-719.7
4	-399.5	-399.8	-399.8

As before, the sensitivities with respect to this design variable should be the same as those obtained for design variable 5, the width of the horizontal ribs, except that the sensitivities for the multiple eigenfrequency $f_2 = f_3$ should be interchanged when using the SA method and (6) and (22).

Again it is seen that very similar results are obtained by the OFD and SA methods when (6) and (22) are used properly, whereas the last column again witnesses a shortcoming of (6) when applied to the double eigenfrequency $f_2 = f_3$.

Up to now the eigenfrequency sensitivities have been found with respect to single design changes of each of the 6 design variables: thickness of plate, thickness of ribs, position of horizontal ribs, position of vertical ribs, width of horizontal ribs, and width of vertical ribs.

Let us finally show that it is possible to determine sensitivities for any direction in the space of the 6 design variables when some of them are changed simultaneously. The position of both the horizontal and vertical ribs, see Figs. 8 and 9, will be changed simultaneously and again the OFD method is used as a reference. The two design variables will be given unit increments.

All the generalized gradient vectors $\mathbf{f}_{sk}$ in (27) have been calculated and (28) is used for determining the increments $\Delta f = \Delta f_2$ and $\Delta f = \Delta f_3$ of the multiple eigenfrequency $f_2 = f_3$. Equation (12) is used for determining increments of the simple eigenfrequencies f_1 and f_4. The sensitivities for this simultaneous design change are shown in Table 7.

Table 7. Eigenfrequency sensitivities for unit increments of design variable 3: position of horizontal ribs and design variable 4: position of vertical ribs

Frequency j	OFD method Δf_j	(12) and (28) Δf_j
1	-81.7	-81.7
2	-193.8	-193.8
3	-193.8	-193.8
4	-335.5	-335.5

It is seen that very accurate results are obtained by using (12) and (28) for determining sensitivities of single and bimodal eigenfrequencies, respectively, in any direction in the space of design parameters.

2.5 Numerical example 2: ribbed plate with a cluster of eigenfrequencies

Next we shall illustrate how important it is for sensitivity analysis to decide correctly from the numerical results whether some eigenvalues coalesce and become multiple or remain distinct.

We consider the same example as before, i.e. the square plate with ribs shown in Fig. 1, but now the thickness of the plate is 2 1/2 times less, i.e. 0.0020 m. This causes the 9 lowest eigenfrequencies to become very close because their corresponding eigenmodes can be regarded as local modes for each of the nine subdomains of the plate, see Figs. 12–21.

The first eigenfrequency is 70.404 Hz and the second through the ninth eigenfrequency are close to 72.2 Hz. These 9 eigenfrequencies are so close that it is difficult to decide which of them are multiple.

If we use as a criterion for identical eigenfrequencies that the relative difference between the frequencies must be $\leq 10^{-3}$, then the 8 eigenfrequencies from the second to the ninth should be considered as multiple, but if we use a tighter criterion such as 10^{-4} then the second and third should be considered as a double eigenfrequency and the sixth, seventh, eighth, and ninth should be considered as a 4-fold multiple eigenfrequency. If the criterion 10^{-5} is used, the second and third should be considered as a double eigenfrequency, and similarly with the seventh and eighth.

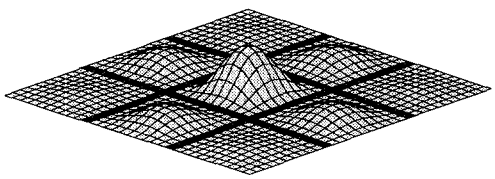

Fig. 12. First eigenmode, $f_1 = 70.40406$ Hz

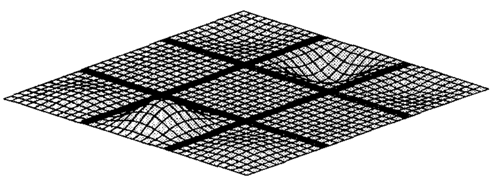

Fig. 13. Second eigenmode, $f_2 = 72.17485$ Hz

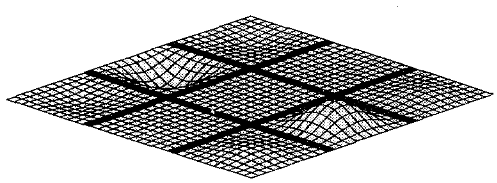

Fig. 14. Third eigenmode, $f_3 = 72.17485$ Hz

We will determine sensitivities of the eigenfrequencies when the width of the horizontal ribs is changed as shown in Fig. 10. The sensitivities are calculated (i) by the OFD method which is used as reference, (ii) using (6) and (22) and assuming two double eigenfrequencies, (iii) using (6) and (22) and assuming one double and one 4-fold multiple eigenfrequency, (iv) using (6) and (22) and assuming one 8-fold multiple eigenfrequency, and (v) only using (6) which is only valid in cases of simple eigenvalues. The results are shown in Table 8.

It is seen that we obtain the same results by the SA method assuming either two double eigenfrequencies $f_2 = f_3$ and $f_7 = f_8$, or one double eigenfrequency $f_2 = f_3$ and one 4-fold multiple eigenfrequency $f_6 = f_7 = f_8 = f_9$. These two columns are in excellent agreement with the OFD method,

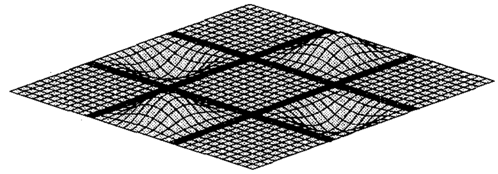

Fig. 15. Fourth eigenmode, $f_4 = 72.24689$ Hz

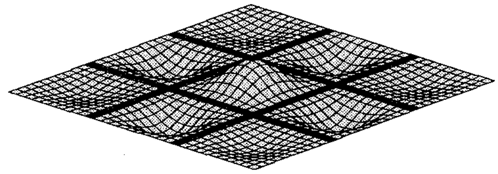

Fig. 16. Fifth eigenmode, $f_5 = 72.27588$ Hz

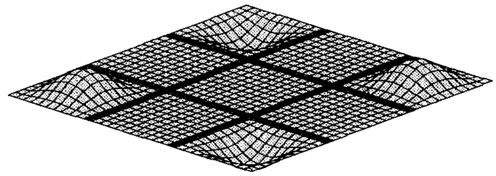

Fig. 17. Sixth eigenmode, $f_6 = 72.31243$ Hz

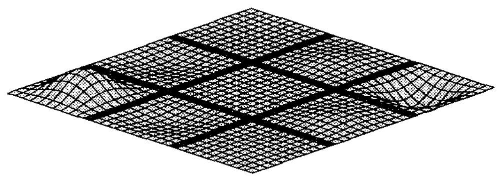

Fig. 18. Seventh eigenmode, $f_7 = 72.31890$ Hz

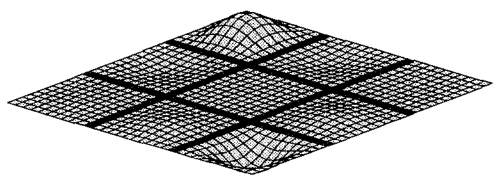

Fig. 19. Eighth eigenmode, $f_8 = 72.31890$ Hz

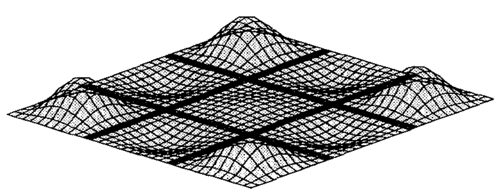

Fig. 20. Ninth eigenmode, $f_9 = 72.32089$ Hz

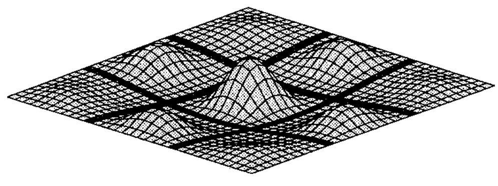

Fig. 21. Tenth eigenmode, $f_{10} = 109.96799$ Hz

Table 8. Eigenfrequency sensitivities with respect to design variable 5: width of horizontal ribs, see Fig. 10

Freq. j	OFD method $\frac{\Delta f_j}{\Delta a_5}$	(6) and (22) $f_2 = f_3$ and $f_7 = f_8$ μ_j	(6) and (22) $f_2 = f_3$ and $f_6 = f_7 = f_8 = f_9$ μ_j	(6) and (22) and $f_2 = f_3 = f_4 = f_5 = f_6 = f_7 = f_8 = f_9$ μ_j	(6) $\frac{\partial f_j}{\partial a_5}$
1	-247.7	-247.7	-247.7	-247.7	-247.7
2	-270.6	-270.6	-270.6	-290.2	-270.2
3	-1.1	-1.1	-1.1	-1.1	-1.5
4	-145.5	-145.1	-145.1	-44.6	-145.1
5	-174.7	-175.1	-175.1	-289.0	-175.1
6	-0.9	-0.9	-0.9	-0.9	-0.9
7	-20.6	-20.6	-20.6	-1.1	-11.4
8	-0.8	-0.8	-0.8	-0.8	-10.0
9	-14.4	-14.5	-14.5	-0.9	-14.5
10	-146.5	-146.5	-146.5	-146.5	-146.5

but it is not a general situation that different choices of multiplicity lead to the same results. This is also illustrated by the next column in Table 8 because if we assume having an 8-fold multiple eigenfrequency $f_2 = f_3 = f_4 = f_5 = f_6 = f_7 = f_8 = f_9$, wrong sensitivities are obtained for several of the eigenfrequencies, e.g. f_5 and f_9. Finally, if we consider all eigenfrequencies as simple, i.e. use (6), then the results for the multiple eigenfrequencies $f_2 = f_3$ and $f_7 = f_8$ are erroneous while the other sensitivities are correct.

This illustrates that the influence of the off-diagonal terms in the sensitivity matrix in (22) is quite small for this design change. This is also the reason why the same sensitivities are obtained for f_6, f_7, f_8, and f_9 using (22) independently of whether the assumption $f_7 = f_8$ or the assumption $f_6 = f_7 = f_8 = f_9$ is used. In both cases, the off-diagonal terms in the sensitivity matrix in (22) are small compared to the diagonal terms for this design change, and therefore the same sensitivities are obtained for both assumptions.

These results show the *importance for the design sensitivity analysis of deciding correctly whether an eigenfrequency is multiple or simple. Thus, wrong assumptions concerning the multiplicity of an eigenfrequency may lead to erroneous sensitivity results.*

3 Optimization problems for eigenvalues

Before formulating optimization problems for eigenvalues, we consider three simple examples that illustrate the main ideas. Only examples involving 2 × 2 matrices depending on two design variables x and y will be considered.

3.1 Example 1

The first example is described by the following **K** and **M** matrices:

$$\mathbf{K} = \begin{bmatrix} 1+x & y \\ y & 1-x \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{29}$$

The characteristic equation for this system is

$$\lambda^2 - 2\lambda + 1 - x^2 - y^2 = 0,$$

and

$$\lambda_{1,2} = 1 \pm \sqrt{x^2 + y^2}. \tag{30}$$

The level curves $\lambda = c$, where c is a constant, for this function are described by the equations

$$c = 1 \pm \sqrt{x^2 + y^2},$$

and

$$x^2 + y^2 = (c-1)^2.$$

This surface is a circular cone, see Fig. 22.

Bimodality occurs at $x = 0$, $y = 0$, for which we have $\lambda_1 = \lambda_2 = 1$.

It can be seen immediately from (30) and Fig. 22 that the eigenvalues λ are not differentiable at the bimodal point in the usual (Fréchet) sense. Indeed,

$$\frac{\partial \lambda_{1,2}}{\partial x} = \pm \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial \lambda_{1,2}}{\partial y} = \pm \frac{y}{\sqrt{x^2 + y^2}}.$$

As $x \to 0$ and $y \to 0$, the two right-hand side expressions become undefined and have no limit. Furthermore, L'Hôpital's rule cannot help either because the derivatives of the denominators also tend to zero.

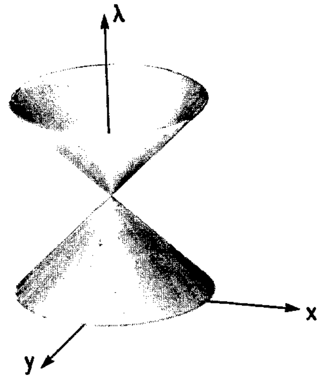

Fig. 22. Circular cone surface for eigenvalue λ

Now we proceed to sensitivity analysis of the double eigenvalue $\tilde{\lambda} = 1$ at $x = 0$, $y = 0$, using the results of Section 2. Taking the variation $\varepsilon \mathbf{e}$ we have

$$x = \varepsilon e_1, \quad y = \varepsilon e_2, \quad \sqrt{e_1 + e_2} = 1.$$

For the sake of simplicity, we can introduce the angle α and write the directional vector $\mathbf{e}$ in the form

$$e_1 = \cos\alpha, \quad e_2 = \sin\alpha. \tag{31}$$

Let us determine the directional derivatives μ_1, μ_2 for the double eigenvalue $\tilde{\lambda} = 1$. The orthonormalized eigenvectors corresponding to $\tilde{\lambda}$ are

$$\phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{32}$$

Using the expressions in (29) and (32), we obtain the generalized gradient vectors $\mathbf{f}_{sk}$ according to (26)

$$\mathbf{f}_{11}^T = \left((1\ 0) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (1\ 0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = (1, 0),$$

$$\mathbf{f}_{12}^T = \left((1\ 0)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}, (1\ 0)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right) = (0,1),$$

$$\mathbf{f}_{22}^T = \left((0\ 1)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}, (0\ 1)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right) = (-1,0).$$

Thus, (27) takes the form

$$\det\begin{bmatrix}\cos\alpha - \mu & \sin\alpha\\ \sin\alpha & -\cos\alpha - \mu\end{bmatrix} = 0, \quad (33)$$

or

$$\mu_{1,2}^2 = \sin^2\alpha + \cos^2\alpha = 1. \quad (34)$$

So, $\mu_1 = 1$ and $\mu_2 = -1$ for any direction $\mathbf{e} = (\cos\alpha, \sin\alpha)$. Hence, the double eigenvalue $\tilde{\lambda}$ splits into $\lambda_{1,2} = 1 \pm \varepsilon$ for any variation $\varepsilon\mathbf{e}$. This means that the bimodal solution $x = 0$, $y = 0$ is the optimum solution to the problem

$$\max_{x,y}\ \min_{j=1,2}\ \lambda_j. \quad (35)$$

This result, of course, can be seen immediately from Fig. 22.

3.2 Example 2

Consider another example with the **K** and **M** matrices given by

$$\mathbf{K} = \begin{bmatrix}1+2x & y\\ y & 1+x\end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \quad (36)$$

The characteristic equation for this system is

$$\lambda^2 - \lambda(2+3x) + (1 + 3x + 2x^2 - y^2) = 0, \quad (37)$$

and

$$\lambda_{1,2} = \frac{2 + 3x \pm \sqrt{x^2 + 4y^2}}{2}. \quad (38)$$

The bimodal solution $\lambda_1 = \lambda_2 = 1$ occurs at the point $x = 0$, $y = 0$.

Let us study the level curves. Inserting $\lambda = c$, where c is a constant, into (37) we obtain

$$\left(x - \frac{3(c-1)}{4}\right)^2 = \left(\frac{c-1}{4}\right)^2 + \frac{y^2}{2}.$$

If we denote $b = \frac{c-1}{4}$ this equation takes the simple form

$$(x - 3b)^2 = b^2 + \frac{y^2}{2}, \quad (39)$$

which defines a hyperbola with the asymptotes $y = \pm\sqrt{2}(x - 3b)$, see Fig. 23.

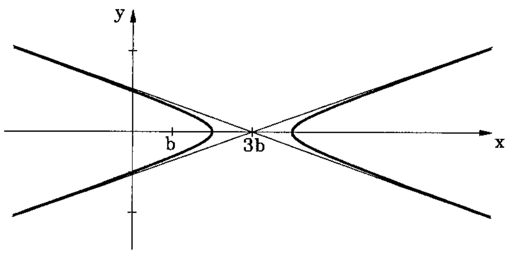

Fig. 23. Level curves for eigenvalue λ

If $c = 1$ then $b = 0$, and we obtain from (39)

$$y = \pm\sqrt{2}x. \quad (40)$$

Near the bimodal point $x = 0$, $y = 0$, according to (38) we have

$$\lambda_{1,2} = 1 \pm y \quad \text{when} \quad x = 0,$$

and

$$\left.\begin{aligned}\lambda_1 &= 1 + 2x\\ \lambda_2 &= 1 + x\end{aligned}\right\} \text{when} \quad y = 0. \quad (41)$$

The last equation implies that along the direction $y = 0$, $x > 0$ we have $\lambda_1 > \tilde{\lambda}$, $\lambda_2 > \tilde{\lambda}$, where $\tilde{\lambda} = 1$. Based on (38)–(41) we can plot surfaces of the eigenvalues $\lambda_{1,2}$, see Fig. 24.

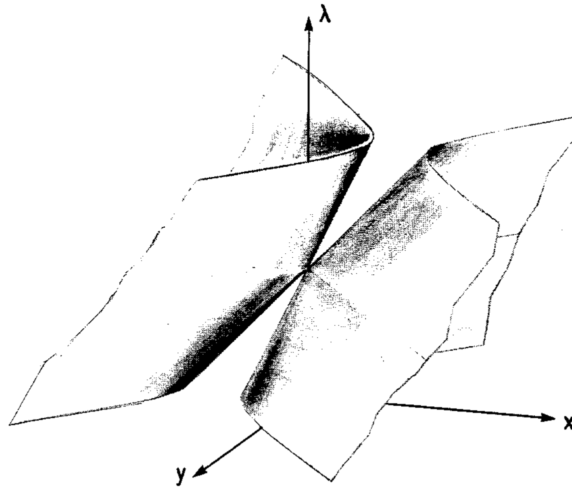

Fig. 24. Surface plot of eigenvalue λ

Let us proceed to sensitivity analysis near the bimodal point $x = 0$, $y = 0$ on the basis of the results of Section 2. The orthonormalized eigenvectors corresponding to $\lambda_1 = \lambda_2 = 1$ are

$$\phi_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad \phi_2 = \begin{pmatrix}0\\1\end{pmatrix}. \quad (42)$$

According to (26) and with use of (36) and (42), we can find the vectors $\mathbf{f}_{sk}$

$$\mathbf{f}_{11}^T = \left((1\ 0)\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}, (1\ 0)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right) = (2,0),$$

$$\mathbf{f}_{12}^T = \left((1\ 0)\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}, (1\ 0)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right) = (0,1),$$

$$\mathbf{f}_{22}^T = \left((0\ 1)\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}, (0\ 1)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right) = (1,0).$$

(43)

Taking the directional vector $\mathbf{e}$ in the form $\mathbf{e} = (\cos\alpha, \sin\alpha)$ and using (43), we obtain from (27) the characteristic equation

$$\det\begin{bmatrix}2\cos\alpha - \mu & \sin\alpha\\ \sin\alpha & \cos\alpha - \mu\end{bmatrix} = 0.$$

From this equation we obtain

$$\mu^2 - 3\mu\cos\alpha + 3\cos^2\alpha - 1 = 0, \quad (44)$$

or

$$\mu_{1,2} = \frac{3\cos\alpha \pm \sqrt{1 + 3\sin^2\alpha}}{2}. \quad (45)$$

Let us consider the term $3\cos^2\alpha$-1 in (44). According to the Vieta theorem, it is equal to the product of the roots of (44)

$$\mu_1\mu_2 = 3\cos^2\alpha - 1. \quad (46)$$

This quantity can be positive as well as negative depending on the value of α. If $\alpha = 0$ then $\mu_1\mu_2 = 2 > 0$. This means that μ_1 and μ_2 are of the same sign for the directions e = (1, 0) and e = (-1, 0), see also Fig. 24.

For $\alpha = 0$ we obtain from (45)

$$\mu_1 = 1, \quad \mu_2 = 2, \tag{47}$$

which is in good agreement with (41).

If $\alpha = \pm\pi/2$ then it follows from 46 that $\mu_1\mu_2 = -1<0$. This means that for the directions e = (0, 1) and e = (0, -1) the derivatives μ_1 and μ_2 are of opposite signs, see also Fig. 24.

This implies that in contrast to Example 1 the bimodal solution $x = 0$, $y = 0$ does not constitute the optimum solution to the problem

$$\max_{x,y} \min_{j=1,2} \lambda_j, \tag{48}$$

because there exists an improving variation e = (1, 0) for which $\mu_1 > 0$, $\mu_2 > 0$.

3.3 Example 3

Let us consider another simple example in which the double eigenvalue does not split along some direction.

$$\mathbf{K} = \begin{bmatrix} 1+x & y \\ y & 1+x \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{49}$$

The characteristic equation for this system leads to

$$\lambda_{1,2} = 1 + x \pm y. \tag{50}$$

Bimodality takes place at $y = 0$, where the eigenvalues remain double for arbitrary x.

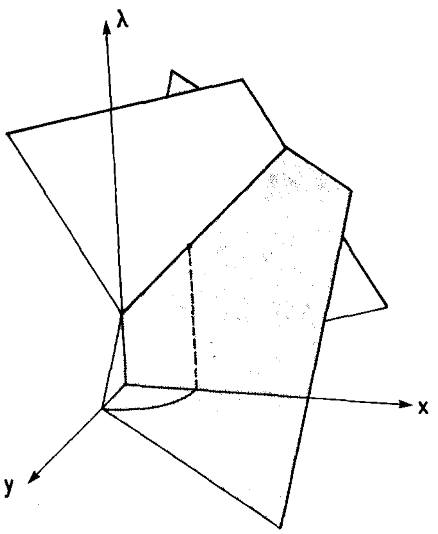

Fig. 25. Surface plot of eigenvalue λ

The surfaces for λ are plotted in Fig. 25. They represent two intersecting planes.

If we put a constraint as illustrated in Fig. 25

$$x^2 + y^2 = 1, \tag{51}$$

then the bimodal solution $x = 1$, $y = 0$ will constitute the optimum one for the problem

$$\max_{x,y} \min_{j=1,2} \lambda_j, \quad \text{subject to} \quad x^2 + y^2 = 1. \tag{52}$$

4 Formulation of the optimization problem

Consider again the eigenvalue problem

$$\mathbf{K}\phi_j = \lambda_j \mathbf{M}\phi_j, \quad j = 1, \ldots, n, \tag{53}$$

where the components of $\mathbf{K}$ and $\mathbf{M}$ are smooth functions of the design variables a_i, $i = 1, \ldots, I$.

The optimization problem is formulated as follows:

$$\max_{a_1,\ldots,a_I} \min_{j=1,\ldots,n} \lambda_j, \tag{54}$$

under the constraint

$$F(a_1, \ldots, a_I) = 0, \tag{55}$$

where F is a smooth scalar function of the design variables a_i, $i = 1, \ldots, I$.
In mechanical problems this constraint usually reflects a constant volume restriction.

4.1 Simple optimum fundamental eigenvalue

If the optimum is achieved at the simple lowest eigenvalue λ_1 with $\lambda_1 < \lambda_2 \le \lambda_3 \le \ldots$, then the necessary optimality condition implies linear dependence of the gradient vectors of λ_1 and F

$$\nabla\lambda_1 - \gamma_0 \mathbf{f}_0 = \mathbf{0}, \tag{56}$$

where

$$\nabla\lambda_1 = \left(\phi_1^T \left(\frac{\partial \mathbf{K}}{\partial a_1} - \lambda_1 \frac{\partial \mathbf{M}}{\partial a_1}\right)\phi_1, \ldots, \phi_1^T \left(\frac{\partial \mathbf{K}}{\partial a_I} - \lambda_1 \frac{\partial \mathbf{M}}{\partial a_I}\right)\phi_1\right),$$

$$\mathbf{f}_0 = \nabla F = \left(\frac{\partial F}{\partial a_1}, \ldots, \frac{\partial F}{\partial a_I}\right), \tag{57}$$

and γ_0 is a (Lagrangian) multiplier to be determined from (55).

4.2 Example

In this optimization example we consider the numerical example from Section 2.4 where the position of the ribs is taken as a design variable, see Fig. 26. We want to maximize the lowest eigenfrequency with a constant volume constraint, see (54) and (55).

All data for the problem are given in Section 2.4. The iteration history for the 4 lowest eigenfrequencies is shown in Fig. 27 and the final design is shown in Fig. 28.

The distance between the ribs has been reduced from 0.50 m to 0.2835 m. It turns out that the optimum solution is characterized by a distinct lowest eigenvalue. This is natural to expect because one-parameter symmetric eigenvalue problems usually possess simple eigenvalues (see Arnold 1989).

4.3 Double optimum fundamental eigenvalue

Consider now the case when the optimum is achieved at the double lowest eigenvalue $\lambda_1 = \lambda_2$, where $\lambda_1 = \lambda_2 < \lambda_3 \le \ldots$ This is the nondifferentiable case and we must use directional derivatives.

Taking the vector of varied design variables in the form $\mathbf{a} + \varepsilon\mathbf{e}$, $\| \mathbf{e} \| = 1$, according to (27), we obtain the directional derivatives μ_1 and μ_2 from

$$\det \begin{bmatrix} \mathbf{f}_{11}^T\mathbf{e} - \mu & \mathbf{f}_{12}^T\mathbf{e} \\ \mathbf{f}_{12}^T\mathbf{e} & \mathbf{f}_{22}^T\mathbf{e} - \mu \end{bmatrix} = 0. \tag{58}$$

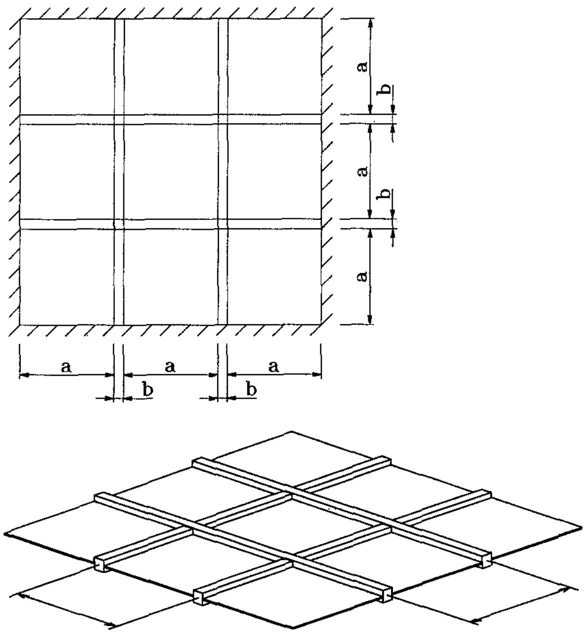

Fig. 26. Design variable for square plate with ribs: position of horizontal and vertical ribs

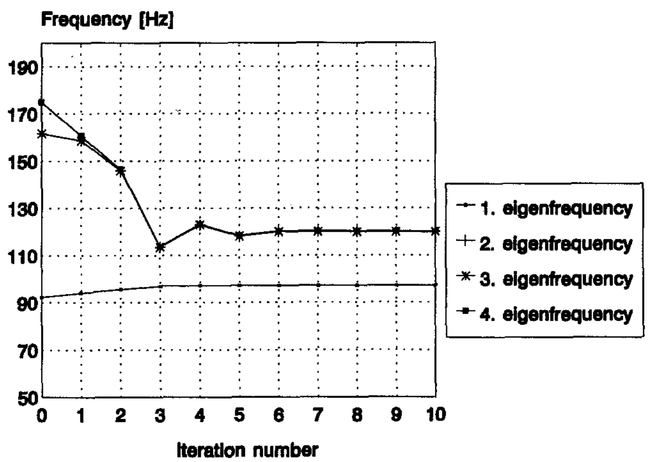

Fig. 27. Iteration history

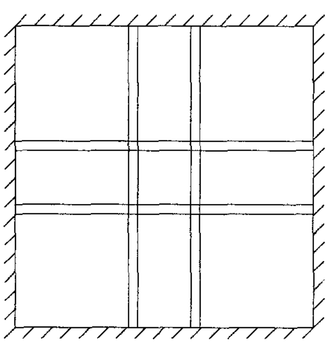

Fig. 28. Optimum design

This is a quadratic equation in μ. Solving it we obtain for any direction e

$$\mu_{1,2} = \frac{\mathbf{f}_{11}^T\mathbf{e} + \mathbf{f}_{22}^T\mathbf{e} \pm \sqrt{(\mathbf{f}_{11}^T\mathbf{e} - \mathbf{f}_{22}^T\mathbf{e})^2 + 4(\mathbf{f}_{12}^T\mathbf{e})^2}}{2}\,. \tag{59}$$

The necessary optimality condition for a maximum is

$$\min(\mu_1, \mu_2) \leq 0\,, \tag{60}$$

for any direction e satisfying the condition $\mathbf{f}_0^T\mathbf{e} = 0$.

From (58) we see that if we take the direction as $-\mathbf{e}$, then both μ_1 and μ_2 will change their signs to the opposite ones. This means that if for some direction e both derivatives μ_1, μ_2 are negative, then the design point is not a maximum, since a change in sign of the direction e leads to $\mu_1 > 0$, $\mu_2 > 0$, i.e. a better design. This means that the necessary optimality condition in the bimodal case is

$$\mu_1\mu_2 \leq 0\,, \tag{61}$$

for any admissible variation e, i.e. a variation that satisfies the condition

$$\mathbf{f}_0^T\mathbf{e} = 0\,. \tag{62}$$

The optimality condition in (61) is fundamentally different from that of the differentiable case due to its nonlinear nature. The condition was first formulated by Masur and Mroz (1979, 1980).

Using (58) and (59) we can express the necessary optimality condition of (61) in the form

$$(\mathbf{f}_{11}^T\mathbf{e})(\mathbf{f}_{22}^T\mathbf{e}) - (\mathbf{f}_{12}^T\mathbf{e})^2 \leq 0\,, \tag{63}$$

for any arbitrary direction e satisfying the condition in (62).

Let us formulate the following lemma.

Lemma 1. If the vectors $\mathbf{f}_{11}$, $\mathbf{f}_{12}$, $\mathbf{f}_{22}$, $\mathbf{f}_0$ are linearly independent, then there exists an improving variation e for which $\mu_1 > 0$, $\mu_2 > 0$.

Proof. Consider the system of linear algebraic equations of the variables $e_1, \ldots, e_I$

$$\mathbf{f}_{11}^T\mathbf{e} = \nu_1^0 > 0, \quad \mathbf{f}_{12}^T\mathbf{e} = 0\,,$$

$$\mathbf{f}_{22}^T\mathbf{e} = \nu_2^0 > 0, \quad \mathbf{f}_0^T\mathbf{e} = 0\,. \tag{64}$$

If the vectors $\mathbf{f}_{11}$, $\mathbf{f}_{12}$, $\mathbf{f}_{22}$, $\mathbf{f}_0$ are linearly independent, then a solution e to the system in (64) exists for arbitrary values of ν_1^0 and ν_2^0. The vector $\tilde{\mathbf{e}} = \frac{\mathbf{e}}{\|\mathbf{e}\|}$ is then an improving variation since from (58) and (64) we have

$$\mu_1 = \mathbf{f}_{11}^T\tilde{\mathbf{e}} = \frac{\nu_1^0}{\|\mathbf{e}\|} > 0, \quad \mu_2 = \mathbf{f}_{22}^T\tilde{\mathbf{e}} = \frac{\nu_2^0}{\|\mathbf{e}\|} > 0\,, \tag{65}$$

which proves the lemma.

Now let us formulate the necessary optimality conditions for the bimodal case.

Theorem 1. If the vector of design variables a constitutes the solution of the optimization problem, (54) and (55), with the double eigenvalue $\lambda_1 = \lambda_2 < \lambda_3 \leq \ldots$, then the vectors $\mathbf{f}_{11}$, $\mathbf{f}_{12}$, $\mathbf{f}_{22}$, $\mathbf{f}_0$ are linearly dependent

$$\gamma_{11}\mathbf{f}_{11} + 2\gamma_{12}\mathbf{f}_{12} + \gamma_{22}\mathbf{f}_{22} - \gamma_0\mathbf{f}_0 = 0\,, \tag{66}$$

with the coefficients γ_{sk} satisfying the inequality

$$\gamma_{11}\gamma_{22} \geq \gamma_{12}^2\,. \tag{67}$$

Here it is assumed that the rank of the matrix consisting of the vectors $\mathbf{f}_{11}$, $\mathbf{f}_{12}$, $\mathbf{f}_{22}$, $\mathbf{f}_0$ is equal to 3. Note that the linear independence of the four vectors mentioned above is possible only when the dimension I of the vector of design variables a is greater than 3.

Proof. Linear dependence of vectors $\mathbf{f}_{sk}$, $\mathbf{f}_0$ in (66) is a consequence of Lemma 1. To prove (67) we express, for example, $\mathbf{f}_{22}$ from (66)

$$\mathbf{f}_{22} = -\frac{\gamma_{11}}{\gamma_{22}}\mathbf{f}_{11} - 2\frac{\gamma_{12}}{\gamma_{22}}\mathbf{f}_{12} + \frac{\gamma_0}{\gamma_{22}}\mathbf{f}_0\,, \tag{68}$$

and substitute this expression into (63). Then we obtain

$$\frac{\gamma_{11}}{\gamma_{22}}(\mathbf{f}_{11}^T\mathbf{e})^2 + 2\frac{\gamma_{12}}{\gamma_{22}}\mathbf{f}_{12}^T\mathbf{e}\,\mathbf{f}_{11}^T\mathbf{e} + (\mathbf{f}_{12}^T\mathbf{e})^2 \geq 0\,. \tag{69}$$

This quadratic form of $\mathbf{f}_{11}^T\mathbf{e}$ and $\mathbf{f}_{12}^T\mathbf{e}$ is positive semidefinite only if its coefficients satisfy the inequality in (67).

Lemma 1 and Theorem 1 were formulated and proved for the first time by Bratus and Seyranian (1983).

4.4 N-fold optimum fundamental eigenvalue

Consider next the general case when in the optimization problem, (54) and (55), the maximum is attained at an N-fold multiple lowest eigenvalue $\lambda_1 = \lambda_2 = \ldots = \lambda_N < \lambda_{N+1} \leq \ldots$.

In this case, for any admissible direction $\mathbf{e}$, i.e. direction satisfying the condition in (62), we find directional derivatives μ_j from (27)

$$\det[\mathbf{f}_{sk}^T\mathbf{e} - \mu\delta_{sk}] = 0, \quad s,k = 1,\ldots,N\,. \tag{70}$$

If the maximum is attained then there must be no admissible direction $\mathbf{e}$ for which all $\mu = \mu_j$, $j = 1,\ldots,N$ are of the same sign. This is an obvious generalization of the necessary optimality condition in (61).

The lemma and the theorem for the general case can be formulated as follows.

Lemma 2. If the vectors $\mathbf{f}_0$, $\mathbf{f}_{sk}$, $s,k = 1,\ldots,N$, $k \geq s$ [the total number of these vectors is equal to $(N+1)N/2+1$] are linearly independent, then there exists an improving variation $\mathbf{e}$ for which $\mu_j > 0$, $j = 1,\ldots,N$.

Note that the linear independence of the vectors is only possible if $I \geq (N+1)N/2+1$, where I is the dimension of the vector $\mathbf{a}$ of design variables.

Proof. Consider the system of linear equations in $e_1,\ldots,e_I$

$$\mathbf{f}_0^T\mathbf{e} = 0\,,$$

$$\mathbf{f}_{sk}^T\mathbf{e} = \delta_{sk}\nu_s^0\,, \quad s,k = 1,\ldots,N\,, \quad k \geq s\,, \tag{71}$$

where ν_s^0 are given positive constants. If the vectors $\mathbf{f}_0$, $\mathbf{f}_{sk}$ are linearly independent, a solution to (71) exists for any ν_s^0, in particular when $\nu_s^0 > 0$. Suppose the vector $\mathbf{e}$ is a solution to the system in (71) and let us normalize this vector as $\tilde{\mathbf{e}} = \frac{\mathbf{e}}{\|\mathbf{e}\|}$. Then we obtain from (71) and (70)

$$\mu_j = \mathbf{f}_{jj}^T\tilde{\mathbf{e}} = \frac{\nu_j^0}{\|\mathbf{e}\|} > 0\,, \quad j = 1,\ldots,N\,, \tag{72}$$

which implies the existence of an improving variation. This proves the lemma (see also Seyranian 1987).

When $I < (N+1)N/2+1$, the vectors $\mathbf{f}_0$, $\mathbf{f}_{sk}$ are always linearly dependent and hence an improving variation may not exist.

Let us formulate the theorem for the necessary optimality conditions.

Theorem 2. If the vector of design variables $\mathbf{a}$ renders a lowest N-fold eigenvalue $\lambda_1 = \lambda_2 = \ldots = \lambda_N$ a maximum, it is necessary that the vectors $\mathbf{f}_0$, $\mathbf{f}_{sk}$, $s,k = 1,\ldots,N$, $k \geq s$ are linearly dependent

$$\sum_{s,k=1}^{N} \gamma_{sk}\mathbf{f}_{sk} - \gamma_0\mathbf{f}_0 = \mathbf{0}\,, \tag{73}$$

with the coefficients γ_0, γ_{sk} satisfying conditions of positive semidefiniteness of the symmetric matrix γ_{sk}, $s,k = 1,\ldots,N$.

Here it is assumed that the rank of the matrix consisting of the vectors $\mathbf{f}_0$ and $\mathbf{f}_{sk}$, $s,k = 1,\ldots,N$, $k \geq s$, is equal to $N(N+1)/2$, and that this rank is less than I.

Note that due to the symmetry $\mathbf{f}_{sk} = \mathbf{f}_{ks}$ we delimit ourselves to symmetric coefficients $\gamma_{sk} = \gamma_{ks}$ such that the sum in (73) can be written in the form

$$\sum_{s,k=1}^{N} \gamma_{sk}\mathbf{f}_{sk} = \sum_{s=1}^{N} \gamma_{ss}\mathbf{f}_{ss} + 2\sum_{\substack{s,k=1\\ s>k}}^{N} \gamma_{sk}\mathbf{f}_{sk}\,. \tag{74}$$

Nevertheless, we prefer the form of (73) due to its convenience.

Proof. Linear dependence of the vectors $\mathbf{f}_0$, $\mathbf{f}_{sk}$, $s,k = 1,\ldots,N$, $k \geq s$, is an obvious consequence of Lemma 2. Note that we have just assumed the rank of the matrix of these vectors to be $N(N+1)/2$, i.e. one less than the total number of the vectors. Let us prove the necessity of positive semidefiniteness of the matrix γ_{sk}. To this end, we choose a new basis of eigenvectors $\tilde{\boldsymbol{\phi}}_1,\ldots,\tilde{\boldsymbol{\phi}}_N$ for which the matrix γ_{sk} is diagonal, and show that if an optimum is attained then all $\tilde{\gamma}_{ss} \geq 0$, $s = 1,\ldots,N$.

Let us transform the eigenvectors

$$\tilde{\boldsymbol{\phi}}_s = \sum_{k=1}^{N} g_{ks}\boldsymbol{\phi}_k\,, \quad s = 1,\ldots,N\,. \tag{75}$$

Here $\tilde{\boldsymbol{\phi}}_s$ are transformed eigenvectors satisfying the orthonormality condition in (15), and g_{sk} is the transformation matrix.

Using (75) in (15), we obtain

$$\tilde{\boldsymbol{\phi}}_i^T\mathbf{M}\tilde{\boldsymbol{\phi}}_j = \left(\sum_{s=1}^{N} g_{si}\boldsymbol{\phi}_s^T\right)\mathbf{M}\left(\sum_{k=1}^{N} g_{kj}\boldsymbol{\phi}_k\right) =$$

$$\sum_{s,k=1}^{N} g_{si}g_{kj}\boldsymbol{\phi}_s^T\mathbf{M}\boldsymbol{\phi}_k = \sum_{s,k=1}^{N} g_{si}g_{kj}\delta_{sk} =$$

$$\sum_{s=1}^{N} g_{si}g_{sj} = \delta_{ij}\,, \quad i,j = 1,\ldots,N\,. \tag{76}$$

In matrix form this equation is equivalent to

$$\mathbf{g}^T\mathbf{g} = \mathbf{I} \text{ and } \mathbf{g}^T = \mathbf{g}^{-1}\,, \tag{77}$$

where $\mathbf{I}$ is the unit matrix. The last equation means that the transformation matrix $\mathbf{g}$ is an orthogonal matrix. Now let us express vectors $\boldsymbol{\phi}_k$ from (75) by $\tilde{\boldsymbol{\phi}}_s$. Due to (77) we have

$$\boldsymbol{\phi}_s = \sum_{k=1}^{N} g_{sk}\tilde{\boldsymbol{\phi}}_k\,, \quad s = 1,\ldots,N\,. \tag{78}$$

Using the notation

$$\nabla \mathbf{K} - \tilde{\lambda}\nabla \mathbf{M} = \left(\frac{\partial \mathbf{K}}{\partial a_1} - \tilde{\lambda}\frac{\partial \mathbf{M}}{\partial a_1}, \dots, \frac{\partial \mathbf{K}}{\partial a_I} - \tilde{\lambda}\frac{\partial \mathbf{M}}{\partial a_I}\right),$$

and (26), we obtain

$$\sum_{k,s=1}^{N} \gamma_{ks}\mathbf{f}_{ks} = \sum_{k,s=1}^{N} \gamma_{ks}\phi_k^T(\nabla \mathbf{K} - \tilde{\lambda}\nabla \mathbf{M})\phi_s =$$

$$\sum_{k,s=1}^{N} \gamma_{ks}\left(\sum_{t=1}^{N} g_{kt}\tilde{\phi}_t^T\right)(\nabla \mathbf{K} - \tilde{\lambda}\nabla \mathbf{M})\left(\sum_{m=1}^{N} g_{sm}\tilde{\phi}_m\right) =$$

$$\sum_{t,m=1}^{N}\left(\sum_{k,s=1}^{N} g_{kt}\gamma_{ks}g_{sm}\right)\tilde{\phi}_t^T(\nabla \mathbf{K} - \tilde{\lambda}\nabla \mathbf{M})\tilde{\phi}_m =$$

$$\sum_{t,m=1}^{N} \tilde{\gamma}_{tm}\tilde{\mathbf{f}}_{tm}\,. \tag{79}$$

So, in the new basis the matrix γ_{ks} takes the form

$$\tilde{\gamma}_{tm} = \sum_{k,s=1}^{N} g_{kt}\gamma_{ks}g_{sm}\,. \tag{80}$$

In matrix form we have

$$\tilde{\gamma} = \mathbf{g}^T\gamma\mathbf{g} = \mathbf{g}^{-1}\gamma\mathbf{g}\,. \tag{81}$$

This means that there exists a basis in which the matrix $\tilde{\gamma}$ is diagonal. Then the optimality condition in (73) takes the form

$$\sum_{s=1}^{N} \tilde{\gamma}_{ss}\tilde{\mathbf{f}}_{ss} - \gamma_0\mathbf{f}_0 = \mathbf{0}\,. \tag{82}$$

The rank of the matrix of the vectors $\mathbf{f}_0$ and $\tilde{\mathbf{f}}_{sk}$, $s,k = 1,\dots,N$, $k \geq s$, is equal to the rank of the matrix of the vectors $\mathbf{f}_0$ and $\mathbf{f}_{sk}$, $s,k = 1,\dots,N$, $k \geq s$, since the two sets of vectors are equivalent and can be expressed linearly in terms of one another (see Kurosh 1962),

$$\mathbf{f}_{sk} = \sum_{i,j=1}^{N} g_{si}g_{kj}\tilde{\mathbf{f}}_{ij}\,, \quad \tilde{\mathbf{f}}_{sk} = \sum_{i,j=1}^{N} g_{is}g_{jk}\mathbf{f}_{ij}\,. \tag{83}$$

To show that the condition $\tilde{\gamma}_{ss} \geq 0$, $s = 1,\dots,N$ is the necessary condition for optimality, let us consider an admissible variation $\mathbf{e}$, i.e. a variation satisfying the condition (62). Multiplying (82) by $\mathbf{e}$, we obtain

$$\sum_{s=1}^{N} \tilde{\gamma}_{ss}\tilde{\mathbf{f}}_{ss}^T\mathbf{e} = 0\,. \tag{84}$$

Suppose that in (82) the j-th coefficient $\tilde{\gamma}_{jj} \neq 0$. Let us take the admissible variation $\mathbf{e}$ such that

$$\mathbf{f}_0^T\mathbf{e} = 0\,,$$

$$\tilde{\mathbf{f}}_{sk}^T\mathbf{e} = 0\,, \quad s,k = 1,\dots,N\,, \quad k > s\,,$$

$$\tilde{\mathbf{f}}_{tt}^T\mathbf{e} = \nu_t^0\,, \quad t = 1,\dots,N\,, \quad t \neq j\,, \tag{85}$$

where ν_t^0 are arbitrary positive constants.

Such a variation $\mathbf{e}$ exists for arbitrary ν_t^0 because the vectors $\mathbf{f}_0$, $\tilde{\mathbf{f}}_{sk}$, $s,k = 1,\dots,N$, $k > s$, and the vectors $\tilde{\mathbf{f}}_{tt}$, $t = 1,\dots,N$, $t \neq j$, are linearly independent. This is true because due to the assumption made earlier, and the equivalence of the set of vectors $\mathbf{f}_0$, $\mathbf{f}_{sk}$ and the set $\mathbf{f}_0$, $\tilde{\mathbf{f}}_{sk}$, $s,k = 1,\dots,N$, $k \geq s$, the rank of the matrix of the latter set of vectors is equal to $N(N+1)/2$, i.e. one less than the number of vectors in the set. The vector $\tilde{\mathbf{f}}_{jj}$ can be expressed as a linear combination of the remaining vectors $\mathbf{f}_0$, $\tilde{\mathbf{f}}_{ss}$ in (82) since $\tilde{\gamma}_{jj} \neq 0$. Then, if the vector $\tilde{\mathbf{f}}_{jj}$ is removed from the set $\mathbf{f}_0$, $\tilde{\mathbf{f}}_{sk}$, $s,k = 1,\dots,N$, $k \geq s$, the remaining vectors are linearly independent.

Now, determining a vector of variation $\mathbf{e}$ from (85) and normalizing this vector, $\tilde{\mathbf{e}} = \dfrac{\mathbf{e}}{\|\mathbf{e}\|}$, we obtain

$$\mu_s = \tilde{\mathbf{f}}_{ss}^T\tilde{\mathbf{e}} = \frac{\nu_s^0}{\|\mathbf{e}\|} > 0\,, \quad s = 1,\dots,N\,, \quad s \neq j\,,$$

$$\mu_j = \tilde{\mathbf{f}}_{jj}^T\tilde{\mathbf{e}}\,. \tag{86}$$

Thus we can find μ_j from (84) taking the variation $\tilde{\mathbf{e}}$

$$\mu_j = \tilde{\mathbf{f}}_{jj}^T\tilde{\mathbf{e}} = -\frac{1}{\tilde{\gamma}_{jj}}\sum_{\substack{s=1\\s\neq j}}^{N} \tilde{\gamma}_{ss}\tilde{\mathbf{f}}_{ss}^T\tilde{\mathbf{e}} =$$

$$-\frac{1}{\|\mathbf{e}\|}\sum_{\substack{s=1\\s\neq j}}^{N}\left(\frac{\tilde{\gamma}_{ss}}{\tilde{\gamma}_{jj}}\right)\nu_s^0\,. \tag{87}$$

Here we have used (86).

If the maximum of the lowest N-fold eigenvalue is achieved, then for any admissible variation $\mathbf{e}$ the sensitivities μ_k, $k = 1,\dots,N$ must not be of the same sign. Since we have chosen $\tilde{\mathbf{e}}$ such that all μ_s, $s = 1,\dots,N$, $s \neq j$, are positive, μ_j must be less than or equal to zero, i.e. $\mu_j \leq 0$. Using (87) we obtain

$$\sum_{\substack{s=1\\s\neq j}}^{N}\left(\frac{\tilde{\gamma}_{ss}}{\tilde{\gamma}_{jj}}\right)\nu_s^0 \geq 0\,, \tag{88}$$

for an arbitrary choice of the positive constants ν_s^0, $s = 1,\dots,N$, $s \neq j$.

The inequality in (88) can be satisfied only if

$$\frac{\tilde{\gamma}_{ss}}{\tilde{\gamma}_{jj}} \geq 0\,, \quad s = 1,\dots,N\,, \quad s \neq j\,, \tag{89}$$

since, otherwise, the constants ν_s^0 can be chosen such that the inequality in (88) is violated. This means that all $\tilde{\gamma}_{ss}$, $s = 1,\dots,N$ must be of the same sign. Without loss of generality, all $\tilde{\gamma}_{ss}$ can be regarded as nonnegative quantities. So, we have proved that $\tilde{\gamma}_{ss} \geq 0$, $s = 1,\dots,N$, which implies positive semidefiniteness of the matrix of coefficients γ_{sk}, $s,k = 1,\dots,N$, and this constitutes the necessary optimality condition.

Note that Masur (1984, 1985) formulated the condition of positive semidefiniteness of the matrix γ as a necessary optimality condition. Unfortunately, the proof presented by Masur (1985) is true only for the bimodal case $N = 2$. The admissible variation, used by Masur, possesses two nonzero values $\mu_1 \neq 0$, $\mu_2 \neq 0$ and $N-2$ zero values $\mu_s = 0$, $s = 3,\dots,N$. Masur assumed that at the optimum for this variation μ_1 and μ_2 must be of opposite signs. This is not true when $N > 2$, since the necessary optimality condition $\min(\mu_1,\mu_2,\dots,\mu_N) \leq 0$ is satisfied for arbitrary μ_1 and μ_2. Above, we have proved the necessity of this condition in the general case.

Similar results for minimizing the maximum eigenvalue were obtained by Overton (1988). Recently, Cox and Overton (1992) derived the necessary optimality conditions for discrete and distributed eigenvalue problems using Clarke's generalized gradient (see Clarke 1990). They also considered lower and upper bounds on design variables $\underline{a}_i \le a_i \le \overline{a}_i$. These results confirm the optimality conditions suggested by Olhoff and Rasmussen (1977), and also used by many others (see e.g. Gajewski and Zyczkowski 1988).

4.5 Example

Let us consider the case of a diagonal matrix $\mathbf{K}$

$$K = \begin{bmatrix} K_{11}(\mathbf{a}) & & & 0 \\ & K_{22}(\mathbf{a}) & & \\ & & \cdots & \\ 0 & & & K_{nn}(\mathbf{a}) \end{bmatrix} \tag{90}$$

and suppose that $\mathbf{M}$ is the unit matrix. For these matrices we have

$$\lambda_s = K_{ss}(a_1, \ldots, a_I), \quad s = 1, \ldots n. \tag{91}$$

This is the case of n separate differentiable functions $\Delta\lambda_s = \nabla^T K_{ss} \Delta a$, $s = 1, \ldots, n$.

Considering the optimization problem in (54) and (55), we have

$$\max_{\mathbf{a}} \min[K_{11}(\mathbf{a}), \ldots, K_{nn}(\mathbf{a})], \quad F(\mathbf{a}) = 0. \tag{92}$$

Suppose that the maximum is achieved at the N-fold multiple eigenvalue $\lambda_1 = \ldots = \lambda_N < \lambda_{N+1} \le \cdots \le \lambda_n$. The necessary optimality conditions of max-min are well-known (see Demyanov and Malozemow 1972)

$$\sum_{s=1}^{N} \gamma_{ss} \nabla K_{ss} - \gamma_0 \mathbf{f}_0 = \mathbf{0}, \tag{93}$$

$$\gamma_{ss} \ge 0, \quad s = 1, \ldots, N. \tag{94}$$

These conditions cannot be directly derived from the results presented in Section 4.4 since $\mathbf{f}_{sk} = 0$, $s, k = 1, \ldots, N$, $s \ne k$, whereby the vectors $\mathbf{f}_0$, $\mathbf{f}_{sk}$, $s, k = 1, \ldots, N$, $k \ge s$, will always be linearly dependent [in the linear combination in (73) we may take $\gamma_0 = 0$, $\gamma_{ii} = 0$, $i = 1, \ldots, N$, and take $\gamma_{ij}, i \ne j$, arbitrarily].

However, it is obvious that Lemma 2 and Theorem 2 in Section 4.4 remain valid if we leave out the vectors $\mathbf{f}_{sk}, s \ne k$, and only retain the vectors $\mathbf{f}_0$, $\mathbf{f}_{ss}$, $s = 1, \ldots, N$, in the formulations. With $\mathbf{f}_{ss} = \nabla K_{ss}$ we then obtain (93). Assuming that the rank of the matrix of the vectors $\mathbf{f}_0$ and $\mathbf{f}_{ss}$, $s = 1, \ldots, N$, is equal to N, we may construct an admissible vector of variation $\mathbf{e}$ from

$$\mathbf{f}_{ss}^T \mathbf{e} = \nu_s^0 > 0, \quad s = 1, \ldots, N, \quad \mathbf{f}_0^T \mathbf{e} = 0, \tag{95}$$

and show that all the coefficients γ_{ss}, $s = 1, \ldots, N$, are of the same sign, and this way establish (94).

5 Optimization of columns on an elastic foundation

In this section the optimization of single and bimodal eigenvalues will be exemplified by the maximization of the buckling load P of thin elastic columns of variable, but geometrically similar cross-sections with the relationship $I(x) = \alpha A^2(x)$ between the second area moment $I(x)$ and cross-sectional area $A(x)$ with the constant α given by the cross-sectional geometry. The columns are resting on an elastic foundation with the stiffness modulus C, are made of a material with Young's modulus E, and have a given total volume V and length L. Similar problems with a linear relationship between $I(x)$ and $A(x)$ admit analytical solutions and were considered by Plaut *et al.* (1986).

In the following, continuum and discrete problem formulations will be considered in Sections 5.1 and 5.2, respectively, a numerical solution procedure will be developed in Section 5.3, and numerical examples will be presented in Section 5.4.

5.1 Continuum problem

Consider first a continuum formulation of the problem in nondimensional form. If we nondimensionalize the coordinate x by means of L and define the dimensionless cross-sectional area function $a(x)$, buckling load λ and foundation modulus c by

$$a(x) = A(x)L/V, \quad \lambda = PL^4/(E\alpha V^2),$$
$$c = CL^6/(E\alpha V^2), \tag{96}$$

then the lateral deflection $y(x)$ at buckling is governed by the eigenvalue problem consisting of the differential equation

$$[a^2(x)y''(x)]'' = -\lambda y''(x) - cy(x), \tag{97}$$

and a set of boundary conditions. We shall assume that the columns have "classical" boundary conditions, i.e. clamped, simply supported or free ends. If e.g. the column is clamped and simply supported at the ends $x = 0$ and $x = 1$, respectively, the boundary conditions are

$$y(0) = y'(0) = 0, \quad y(1) = (a^2 y'')_{x=1} = 0. \tag{98}$$

The continuum problem possesses an infinite number of eigenvalues λ_j, $0 < \lambda_1 \le \lambda_2 \le \ldots$, and corresponding eigenfunctions $y_j(x)$. It will be assumed in the following that the latter are orthonormalized

$$\int_0^1 y_j'(x) y_k'(x)\, \mathrm{d}x = \delta_{jk}. \tag{99}$$

The nondimensional optimization problem consists in determining the cross-sectional area function $a(x)$, which for a given value of the foundation modulus c maximizes the smallest eigenvalue,

$$\max_{a(x)} \min_{j=1,\ldots,n} \lambda_j, \tag{100}$$

subject to the condition of given column volume

$$\int_0^1 a(x)\, \mathrm{d}x = 1. \tag{101}$$

If the maximized smallest eigenvalue λ_1 is simple, $\lambda_1 < \lambda_2 \le \lambda_3 \le \ldots$, then the necessary optimality condition takes the form of (56) with (see e.g. Tadjbakhsh and Keller 1962)

$$\nabla\lambda_1 = 2a(x)\left[y_1''(x)\right]^2, \quad f_0 = 1. \tag{102}$$

The positive real constant γ_0 in (56) is a Lagrangian multiplier to be determined by the dimensionless volume constraint (101).

Consider now the bimodal case where the smallest eigenvalue $\lambda_1 = \lambda_2 < \lambda_3 \le \ldots$ is associated with two linearly independent eigenfunctions satisfying (97), the appropriate boundary conditions, and the orthonormality condition (99).

If we consider a varied cross-sectional area function in the form $a(x)+\varepsilon e(x)$, then the double eigenvalue $\lambda_1 = \lambda_2 = \tilde{\lambda}$ associated with $a(x)$ will generally split into two distinct ones, $\lambda_j = \tilde{\lambda} + \varepsilon\mu_j$, $j = 1,2$. By using a perturbation technique as in Section 2.3, we derive the following quadratic equation for determining the two directional derivatives (design sensitivities) $\mu = \mu_1 = \Delta\lambda_1/\Delta\varepsilon$ and $\mu = \mu_2 = \Delta\lambda_2/\Delta\varepsilon$ of the double eigenvalue,

$$\det\left[2\int_0^1 a(x)y_s''(x)y_k''(x)e(x)\,\mathrm{d}x - \mu\delta_{sk}\right] = 0\,,$$

$$s,k = 1,2\,. \tag{103}$$

This equation is analogous to (27) (with $N = 2$).

In problems where the boundary conditions and the design function $a(x)$ are symmetric, $a(x) = a(1-x)$, it is useful to apply symmetric and antisymmetric eigenfunctions $y_1(x) = y_1(1-x)$ and $y_2(x) = -y_2(1-x)$ in (103). Since the otherwise arbitrary "direction" function $e(x)$ must be symmetric along with $a(x)$, we see that the mixed term in (103) vanishes:

$$2\int_0^1 a(x)y_1''(x)y_2''(x)e(x)\,\mathrm{d}x = 0\,, \tag{104}$$

since its integrand will be antisymmetric for any arbitrary admissible function $e(x)$. Hereby (103) gives the following simple expressions for the directional derivatives of the double eigenvalue:

$$\mu_1 = 2\int_0^1 a(x)\left[y_1''(x)\right]^2 e(x)\,\mathrm{d}x\,,$$

$$\mu_2 = 2\int_0^1 a(x)\left[y_2''(x)\right]^2 e(x)\,\mathrm{d}x\,. \tag{105}$$

These expressions hold when the double eigenvalue can be treated as an intersection of two differentiable functionals, and are identical to those resulting from purely single modal formulations. However, in the general case of non-symmetric boundary conditions and designs, the mixed term in (104) does not vanish.

5.2 Columns with piecewise constant cross-sections

Let us now cast the optimization problem in discrete form by assuming the column to be composed of segments $x_{i-1} \le x \le x_i$ of prescribed lengths $\ell_i = x_i - x_{i-1}$, $i = 1,\dots,I$, (where $x_0 = 0$ and $x_I = 1$), and individual constant values a_i of the cross-sectional areas, i.e. $a(x) \equiv a_i$, $x_{i-1} < x < x_i$, $i = 1,\dots,I$. We treat the values of a_i as design variables, assemble them in the vector $\mathbf{a} = (a_1,\dots,a_I)$, and let $\mathbf{e} = (e_1,\dots,e_I)$ denote the corresponding unit direction vector of arbitrary variations.

Defining the vectors $\mathbf{f}_{sk}$, $s,k = 1,2$ as

$$\mathbf{f}_{sk} = \left[2a_1\int_0^{x_1} y_s''(x)y_k''(x)\,\mathrm{d}x,\dots,2a_i\int_{x_{i-1}}^{x_i} y_s''(x)y_k''(x)\,\mathrm{d}x,\dots,\right.$$

$$\left.2a_I\int_{x_{I-1}}^1 y_s''(x)y_k''(x)\,\mathrm{d}x\right]\,,\quad s,k = 1,2\,, \tag{106}$$

it is easily seen that the quadratic equation (103) for the two directional derivatives $\mu = \mu_1$ and $\mu = \mu_2$ of the bimodal eigenvalue precisely takes the form of (58) with solutions given by (59). Similarly, we easily identify the vector $\mathbf{f}_0$ in the volume constraint (101) as

$$\mathbf{f}_0 = (x_1,\dots,x_i - x_{i-1},\dots,1 - x_{I-1}) = (\ell_1,\dots,\ell_i,\dots,\ell_I)\,, \tag{107}$$

for the discrete problem. In the examples in Section 5.4, the integrals in (106) are computed by numerical integration with the functions $y_1''(x)$ and $y_2''(x)$ being represented by their discrete values at sets of densely spaced mesh points embedded in each of the column segments.

5.3 Method of optimization

In order to exemplify the results in Section 4 and their applicability for design problems, we shall now develop an iterative numerical method for the solution of optimization problems for stepped columns on an elastic foundation. For these problems it is unknown *a priori* whether the optimum buckling load is a simple or a double eigenvalue. Extension of the method to any degree of multiplicity of the optimum eigenvalue is straightforward.

Assume first that at a given iteration stage the design is associated with a bimodal eigenvalue $\lambda_1 = \lambda_2 = \tilde{\lambda}$, or that the two lowest eigenvalues $\lambda_1 < \lambda_2$ are very close such that $\lambda_2 - \lambda_1 \le \delta$ where δ is a small tolerance which we assume to be specified.

Guided by the results stated in Lemma 1 and Theorem 1 in Section 4.3, we take the vector of increments $\mathbf{\Delta a}$ of the vector of design variables $\mathbf{a}$ as

$$\mathbf{\Delta a} = k(\gamma_{11}\mathbf{f}_{11} + 2\gamma_{12}\mathbf{f}_{12} + \gamma_{22}\mathbf{f}_{22} - \gamma_0\mathbf{f}_0)\,, \tag{108}$$

where k is a move-limit type of scaling factor. We may only need to apply k in iteration steps in the beginning of the computational procedure where the vector resulting from the expression in the parenthesis may become sufficiently large to hamper convergence. Application of k is not necessary later where the expression in the parenthesis, and hence $\mathbf{\Delta a}$, will tend towards the null vector, cf. (66), as we approach the optimum solution.

In (108), the vectors $\mathbf{f}_{11}$, $\mathbf{f}_{12}$, $\mathbf{f}_{22}$ are defined by (106), the vector $\mathbf{f}_0$ by (107), and γ_{11}, $2\gamma_{12}$, γ_{22} and γ_0 are to be determined. Both (66) and (108) permit us to normalize the latter quantities as

$$\gamma_{11} = \gamma_{11}^*/\,\|\gamma^*\|\,,\quad 2\gamma_{12} = 2\gamma_{12}^*/\,\|\gamma^*\|\,,$$

$$\gamma_{22} = \gamma_{22}^*/\|\gamma^*\|\,,\quad \gamma_0 = \gamma_0^*/\|\gamma^*\|\,, \tag{109}$$

where

$$\|\gamma^*\| = \sqrt{\gamma_{11}^{*\,2} + (2\gamma_{12}^*)^2 + \gamma_{22}^{*\,2} + \gamma_0^{*\,2}}\,. \tag{110}$$

Equations for obtaining γ_{11}^*, $2\gamma_{12}^*$, γ_{22}^* and γ_0^* [and hence $\|\gamma^*\|$ by (110)] will be established below.

Now, instead of using the unit vector $\mathbf{e}$ and the directional derivatives $\mu_1 = \Delta\lambda_1/\Delta\varepsilon$ and $\mu_2 = \Delta\lambda_2/\Delta\varepsilon$ as in (58) and (59), we wish to work with the increments $\Delta\lambda_1$ and $\Delta\lambda_2$ of the double eigenvalue that directly correspond to a given vector $\mathbf{\Delta a}$ of increments of the design variables. To this end, we multiply each component in (58) by ε, and obtain the following quadratic equation for the increments $\Delta\lambda = \Delta\lambda_1$ and $\Delta\lambda = \Delta\lambda_2$,

$$\det \begin{bmatrix} \mathbf{f}_{11}^T\mathbf{\Delta a} - \Delta\lambda & \mathbf{f}_{12}^T\mathbf{\Delta a} \\ \mathbf{f}_{12}^T\mathbf{\Delta a} & \mathbf{f}_{22}^T\mathbf{\Delta a} - \Delta\lambda \end{bmatrix} = 0\,, \tag{111}$$

just as we earlier transformed (27) into (28).

Obviously, the increments $\mathbf{\Delta a}$ for the iterative computational procedure can be chosen in many ways. Equation (111) leads us to choose the following four simultaneous conditions as a basis for determining the four coefficients γ_{11}, $2\gamma_{12}$, γ_{22} and γ_0 in the expression for $\mathbf{\Delta a}$ in (108) (where we disregard the scaling factor k):

$$\Delta\lambda_1 = \mathbf{f}_{11}^T\mathbf{\Delta a} = 1/\,\|\gamma^*\|\,, \tag{112}$$

$$\Delta\lambda_2 = \mathbf{f}_{22}^T\mathbf{\Delta a} = (1+\lambda_1-\lambda_2)/\,\|\gamma^*\|\,, \tag{113}$$

$$\mathbf{f}_{12}^T\mathbf{\Delta a} = 0\,, \tag{114}$$

$$\mathbf{f}_{0}^T\mathbf{\Delta a} = 0\,. \tag{115}$$

Here, (115) expresses the volume constraint (which is linear in the design variables in the present case; the satisfaction of a nonlinear constraint requires a trivial modification of the approach). Equation (114) imposes a diagonalization of the matrix in (111), which reduces the solutions $\Delta\lambda = \Delta\lambda_1$ and $\Delta\lambda = \Delta\lambda_2$ of (111) to those expressed by the first equality signs in (112) and (113). These expressions imply that $\Delta\lambda_1$ and $\Delta\lambda_2$ are precisely the increments of λ_1 and λ_2, respectively, whose indices correspond to the directly identifiable modes $y_1(x)$ and $y_2(x)$ obtained from the solution of the eigenvalue problem for the current design.

It is seen from (112) and (113) that if $\lambda_1 = \lambda_2$ at a given iteration step, we specify $\Delta\lambda_1$ and $\Delta\lambda_2$ to be equal and positive with a view to obtaining an increase of the bimodal eigenvalue. If the two lowest eigenvalues are different (within the small difference δ defined earlier), we assign the index 1 to the smallest eigenvalue and the corresponding mode, prescribe the value $1/\,\|\gamma^*\|$ for its increment $\Delta\lambda_1$, see (112), and assign a slightly smaller value to the increment $\Delta\lambda_2$ of the next eigenvalue λ_2, see (113), in order to increase λ_1 while diminishing the difference between λ_1 and λ_2.

Now, by substituting (108) (where we disregard k) into (112)-(115) and making use of (109), we obtain the following system of equations for determining the unknown coefficients γ_{11}^*, $2\gamma_{12}^*$, γ_{22}^* and γ_0^*:

$$\begin{bmatrix} \mathbf{f}_{11}^T\mathbf{f}_{11} & \mathbf{f}_{11}^T\mathbf{f}_{22} & \mathbf{f}_{11}^T\mathbf{f}_{12} & \mathbf{f}_{11}^T\mathbf{f}_{0} \\ & \mathbf{f}_{22}^T\mathbf{f}_{22} & \mathbf{f}_{22}^T\mathbf{f}_{12} & \mathbf{f}_{22}^T\mathbf{f}_{0} \\ & & \mathbf{f}_{12}^T\mathbf{f}_{12} & \mathbf{f}_{12}^T\mathbf{f}_{0} \\ \text{symm} & & & \mathbf{f}_{0}^T\mathbf{f}_{0} \end{bmatrix} \begin{Bmatrix} \gamma_{11}^* \\ \gamma_{22}^* \\ 2\gamma_{12}^* \\ -\gamma_0^* \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1+\lambda_1-\lambda_2 \\ 0 \\ 0 \end{Bmatrix}\,. \tag{116}$$

Having solved (116), we compute $\|\gamma^*\|$ from (110), the normalized coefficients γ_{11}, $2\gamma_{12}$, γ_{22} and γ_0 from (109), and substitute these into (108) to obtain the new vector $\mathbf{\Delta a}$ of increments of design variables.

We note that the determinant of the coefficient matrix in (116) is nonnegative and that it only vanishes if the vectors $\mathbf{f}_{11}$, $\mathbf{f}_{12}$, $\mathbf{f}_{22}$ and $\mathbf{f}_0$ become linearly dependent, which is the necessary condition for an optimum bimodal solution, cf. Theorem 1 and (66) in Section 4.3. The latter implies, see (66), that the vector $\mathbf{\Delta a}$ in (108) vanishes at the bimodal optimum. In the above iterative computational procedure, the numerical values of γ_{11}^*, $2\gamma_{12}^*$, γ_{22}^* and γ_0^* obtained from (116), and hence the value of $\|\gamma^*\|$ from (110), rapidly increases as we approach the bimodal optimum point with linear dependence of the vectors $\mathbf{f}_{11}$, $\mathbf{f}_{12}$, $\mathbf{f}_{22}$ and $\mathbf{f}_0$. This implies that in addition to $\mathbf{\Delta a} \to \mathbf{0}$ in (108), due to $\|\gamma^*\| \to \infty$, in (112) and (113) we have $\Delta\lambda_1 \to 0$ and $\Delta\lambda_2 \to 0$ as we approach the bimodal optimum solution.

Let us finally consider the case where the column design obtained at a given iteration stage is associated with a simple buckling eigenvalue λ_1 with $\lambda_2 - \lambda_1 > \delta$. In this case we wish to increase λ_1 by single modal steps of redesign. The eigenvalue increment and the volume constraint are then expressed by

$$\Delta\lambda_1 = \mathbf{f}_{11}^T\mathbf{\Delta a}\,, \tag{117}$$

$$\mathbf{f}_{0}^T\mathbf{\Delta a} = 0\,, \tag{118}$$

where $\mathbf{f}_{11}$ and $\mathbf{f}_0$ are defined by (106) and (107), and the vector of increments of the design variables is taken in the single modal form

$$\mathbf{\Delta a} = k(\mathbf{f}_{11} - \gamma_0\mathbf{f}_0)\,, \tag{119}$$

where k is again a positive scaling factor. The *a priori* unknown constant γ_0 is determined by substituting (119) into (118), which gives

$$\gamma_0 = \frac{\mathbf{f}_0^T\mathbf{f}_{11}}{\mathbf{f}_0^T\mathbf{f}_0}\,, \tag{120}$$

and substitution of (119) and (120) into (117) yields the Cauchy-Bunyakowski inequality

$$\Delta\lambda_1 = k\left(\mathbf{f}_{11}^T\mathbf{f}_{11} - \frac{(\mathbf{f}_0^T\mathbf{f}_{11})(\mathbf{f}_0^T\mathbf{f}_{11})}{\mathbf{f}_0^T\mathbf{f}_0}\right) \geq 0\,, \tag{121}$$

for the increment $\Delta\lambda_1$ of the eigenvalue. Thus, each step of redesign increases the eigenvalue λ_1 while satisfying the volume constraint, (118), and this continues until $\mathbf{f}_{11}$ and $\mathbf{f}_0$ become linearly dependent, i.e.

$$\mathbf{f}_{11} - \gamma_0\mathbf{f}_0 = \mathbf{0}\,, \tag{122}$$

which is the necessary condition for single modal optimality of the fundamental eigenvalue λ_1.

If, during this single modal iterative procedure, the distance between λ_2 and λ_1 decreases and we obtain $\lambda_2 - \lambda_1 \leq \delta$ at a certain stage, then we perform subsequent iterations using the bimodal eigenvalue optimization procedure described earlier. This may give rise to a decrease of the distance between the eigenvalues λ_3 and the bimodal eigenvalue $\lambda_1 = \lambda_2$, and if coalescence takes place, it is necessary to adopt a trimodal optimization scheme for subsequent iterations, and so on. Note that in addition to the (single or multimodal) fundamental eigenvalue subject to treatment at a given stage of the iterative procedure, it is also necessary to know the value of the next (higher order) eigenvalue in order to be able to capture its possible coalescence with the fundamental eigenvalue and update the subsequent computations. As mentioned earlier, it is straightforward to construct a multimodal scheme for optimization of eigenvalues of any multiplicity N by generalization of the method described above.

5.4 Numerical examples

Subject to some selected values of the elastic foundation modulus c, see (96), we now solve the dimensionless buckling load optimization problem for a stepped column with $I = 50$ segments of equal lengths $\ell_i = 1/50$ but individual cross-sectional areas a_i, $i = 1, \dots, I$, which play the role as design variables of the discretized problem. No minimum or maximum constraints are specified for the design variables. The optimization problems are solved by the method developed in Section 5.3, and we have chosen to determine the eigenvalues and eigenmodes in the steps of redesign by a finite difference procedure based on successive iterations (which involves further discretization of each of the segments of the column).

Examples will be presented for (symmetric) clamped-clamped boundary conditions and for (non-symmetric) clamped-simply supported conditions. In each of the examples, the initial column design (iteration 0) has been chosen to be uniform with $a_i = 1$, $i = 1, \dots, I = 50$, which meets the nondimensional volume constraint $\sum a_i \ell_i = 1$. For each value of c and set of boundary conditions, the uniform design will also be used as a reference for the gain of the optimization, and we shall denote the fundamental buckling eigenvalue of the uniform column by λ^u.

Figure 29 illustrates optimum designs of columns with both ends champed. The designs are shown to suitable scale, and the linear dimensions perpendicular to the column axes are proportional to the square root of the cross-sectional areas.

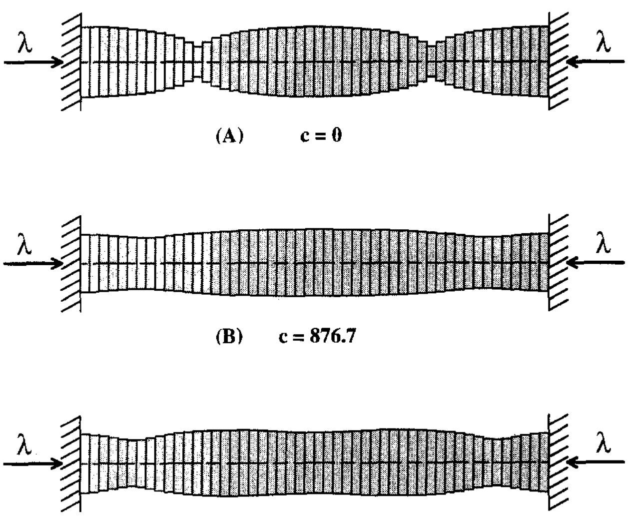

Fig. 29. Clamped-clamped optimum columns subject to different values c of modulus of elastic foundation. Columns have 50 segments of equal lengths. (a) $c = 0$, optimum $\lambda = 52.105$ is bimodal, $\lambda/\lambda^u = 1.320$. (b) $c = 876.7$, optimum $\lambda = 102.55$ is bimodal, $\lambda/\lambda^u = 1.039$. (c) $c = 1500$, optimum $\lambda = 123.91$ is bimodal, $\lambda/\lambda^u = 1.114$

All the designs in Fig. 29 are found to be symmetric and associated with a bimodal optimum buckling load. The buckling load of the optimum column without elastic foundation ($c = 0$) in Fig. 29a is $\lambda = \lambda_1 = \lambda_2 = 52.105$ and we have $\lambda/\lambda^u = 1.320$, i.e. the optimum (bimodal) buckling load is 32.0% higher than the (simple) buckling load of a corresponding uniform column with both ends clamped and the same volume and length.

It is interesting to note that the value $\lambda = 52.105$ for the stepped optimum column design in Fig. 29a with 50 design variables, is only marginally less than the bimodal buckling load $\lambda = 52.356$ for the classical clamped-clamped *continuum* column (with, in principle, infinitely many design variables) determined by Olhoff and Rasmussen (1977).

Figure 30 displays the iteration history behind the optimum solution in Fig. 29a, when using the method of Section 5.3 and starting out from uniform design. We see that the initially distinct eigenvalues λ_1 and λ_2 coalesce after around 10 iterations and that subsequent iterations are carried out by the bimodal eigenvalue optimization procedure. Figure 30 also shows our monitoring of λ_3 which turns out to remain distinct from λ_1 and λ_2. (This is the case for all the examples considered in this section.)

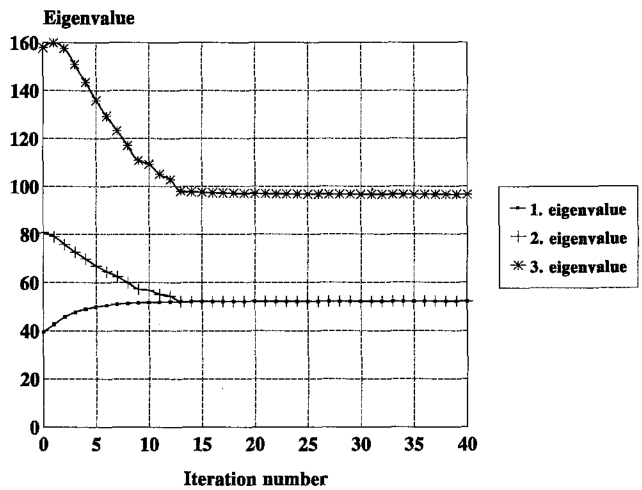

Fig. 30. Iteration history, optimization of clamped-clamped column with $c = 0$. The optimum design is shown in Fig. 29a

For the solution in Fig. 29a, the vectors $\mathbf{f}_{11}$, $\mathbf{f}_{12}$, $\mathbf{f}_{22}$ and $\mathbf{f}_0$ are linearly dependent, i.e. satisfy (66). The (normalized) values of the coefficients γ_{11}, $2\gamma_{12}$ and γ_{22} are determined to be $\gamma_{11} = 3.459 \cdot 10^{-4}$, $2\gamma_{12} = 0$ and $\gamma_{22} = 9.250 \cdot 10^{-3}$, which implies that the inequality for positive semidefiniteness in (67) is also satisfied. Thus, both necessary conditions (66) and (67) for a bimodal optimum are satisfied by the solution in Fig. 29a. The same holds true for the solutions in Figs. 29b and 29c. Here, we again find $2\gamma_{12} = 0$, which is not surprising since the optimum designs obtained and shown in Fig. 29 are symmetric.

Figure 29b shows the optimum design of the stepped column with two clamped ends when the modulus of the elastic foundation is taken to be $c = 876.7$. The design is associated with a bimodal optimum buckling load $\lambda = \lambda_1 = \lambda_2 = 102.55$ which is only 3.9% larger than the buckling load of the corresponding uniform column with the same volume, length, foundation modulus, and boundary conditions. The interesting point associated with $c = 876.7$ is that at this value of the foundation modulus, the buckling load of the *uniform* comparison column is bimodal, $\lambda^u = \lambda_1^u = \lambda_2^u = 98.67$. This implies, as can be seen in Fig. 31, that when starting the optimization procedure from uniform design, the final optimum bimodal design in Fig. 29b will be obtained through an iteration history that exclusively encompasses steps of redesign involving bimodal buckling loads.

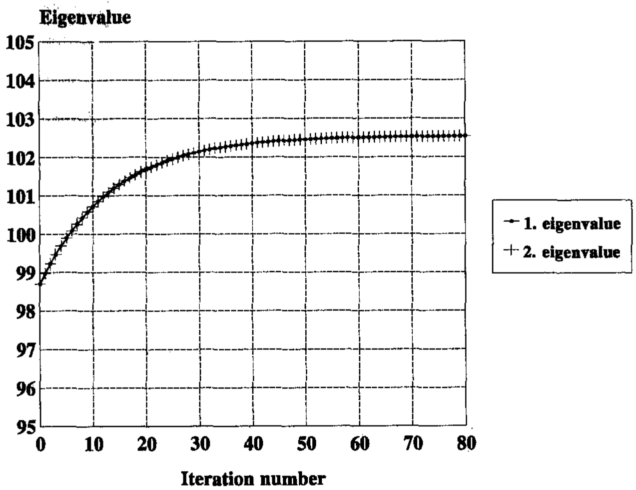

Fig. 31. Iteration history, optimization of clamped-clamped column with $c = 876.7$. The optimum design is shown is Fig. 29b

Figure 29c illustrates the stepped optimum column design for $c = 1500$. For this solution, the optimum bimodal buckling load is $\lambda = \lambda_1 = \lambda_2 = 123.91$, which is 11.4% larger than the (simple) buckling load of the corresponding uniform column on the same foundation. The iteration history is qualitatively similar to the one shown in Fig. 30. The solution in Fig. 29c illustrates the fact that the waviness of the optimum designs increases with increasing values of the modulus c of the elastic foundation.

Next, we consider optimum designs of stepped columns with one end clamped and the other simply supported, see Fig. 32. It is characteristic that both for these boundary conditions and for doubly clamped boundary conditions, corresponding optimum solutions for *continuum* columns are associated with bimodal buckling loads for any value $c \geq 0$ of the modulus of the elastic foundation. However, for the *stepped* clamped-simply supported columns under consideration, with the finite dimension $I = 50$ of the design space, the optimum designs are associated with simple buckling loads for values of c up to a threshhold value of approximately 250, and only associated with bimodal optimum buckling loads for larger values of c.

Figure 32a shows the stepped optimum column design for $c = 50$. Its buckling load $\lambda = \lambda_1 = 31.042$ is simple, and 27.8% larger than that of a corresponding uniform column on the same foundation. Figure 33 displays the iteration history, which entirely consists of single modal steps of redesign, and we have checked that the final solution satisfies the necessary condition in (56) for a single modal optimum solution.

For $c = 500$, the stepped optimum column, see Fig. 32b, has a bimodal buckling load $\lambda = \lambda_1 = \lambda_2 = 66.296$ which

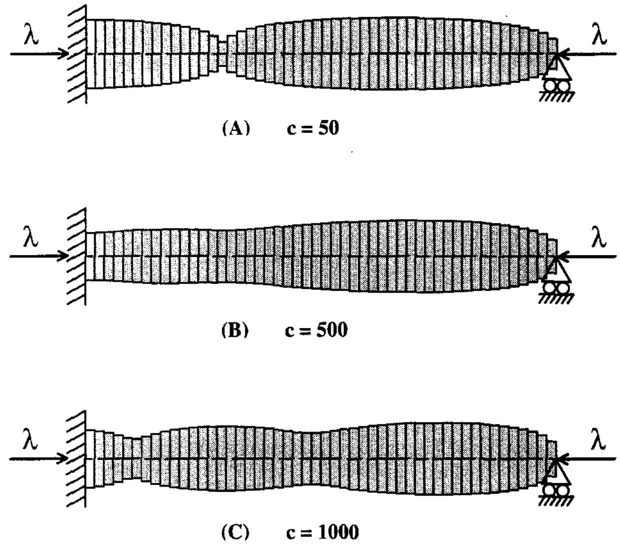

Fig. 32. Clamped-simply supported optimum columns subject to different values c of modulus of elastic foundation. Columns have 50 segments of equal lengths. (a) $c = 50$, optimum $\lambda = 31.042$ is simple, $\lambda/\lambda^u = 1.278$. (b) $c = 500$, optimum $\lambda = 66.296$ is bimodal, $\lambda/\lambda^u = 1.188$. (c) $c = 1000$, optimum $\lambda = 90.420$ is bimodal, $\lambda/\lambda^u = 1.214$

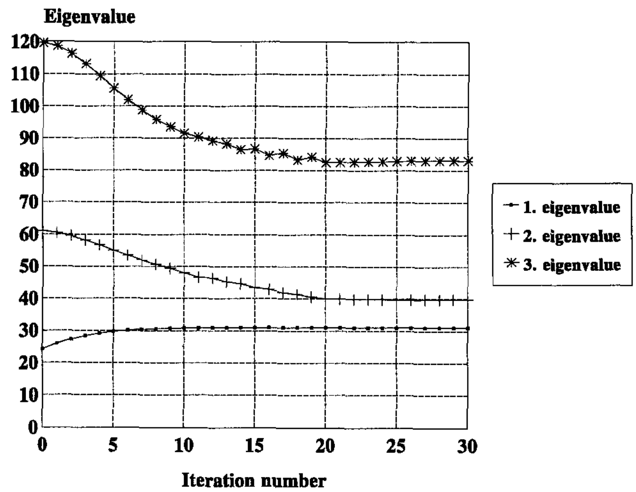

Fig. 33. Iteration history, optimization of clamped-simply supported column with $c = 50$. The optimum design is shown in Fig. 32a

is 18.8% larger than the (simple) buckling load of the corresponding uniform reference column. The iteration history is qualitatively similar to that shown in Fig. 30, and the final solution satisfies the necessary conditions in (66) for a bimodal optimum solution. The (normalized) values of γ_{11}, $2\gamma_{12}$ and γ_{22} associated with the solution are found to be $\gamma_{11} = 2.488{\cdot}10^{-3}, 2\gamma_{12} = 1.882{\cdot}10^{-3}$ and $\gamma_{22} = 1.319{\cdot}10^{-2}$, so the necessary condition in (67) is also satisfied. We note that the value of $2\gamma_{12}$ is now non-zero as the optimum design in Fig. 32b is nonsymmetric.

For $c = 1000$, we obtain the optimum design in Fig. 32c with the bimodal buckling load $\lambda = \lambda_1 = \lambda_2 = 90.420$, which is 21.4% higher than the (simple) buckling load of the corresponding uniform comparison column. The iteration history behind this optimum solution is also qualitatively similar to that in Fig. 30, and we again find that the final solution satisfies the necessary conditions (66) and (67) for bimodal optimality, and that $2\gamma_{12} \neq 0$. Figure 32b illustrates that also the clamped-simply supported optimum columns become increasingly wavy with increasing values of c.

6 Conclusions

Multiple eigenvalues often occur in symmetric structures and in complex structures that depend on many design parameters. Thus, multiparameter optimization problems should always be considered in this perspective. Simple examples show that multiple eigenvalues are not differentiable in the common sense. This creates serious problems for derivation of optimality conditions and numerical analysis in solving optimization problems.

The design sensitivity analysis based on a perturbation technique presented in this paper shows that directional derivatives of multiple eigenvalues are eigenvalues of a specially constructed symmetric matrix with the components $\mathbf{f}_{sk}^T\mathbf{e}$. The efficiency and accuracy of sensitivity analysis of simple and multiple eigenvalues with respect to six design parameters has been demonstrated by a thorough numerical study of a multiparameter vibration problem of an elastic square plate reinforced by stiffeners.

We show that linear dependence of the vectors $\mathbf{f}_{sk}$ and the gradient vector $\mathbf{f}_0$ of the constraint with the coefficients γ_{ij} and γ_0 respectively, constitutes the first necessary optimality condition, and that positive semidefiniteness of the matrix of the coefficients γ_{ij} is the second necessary optimality condition for an optimum multiple eigenvalue. It is demonstrated and illustrated by examples of optimization of columns on elastic foundations that these optimality conditions can be directly used in the development of a very efficient method for optimum design.

It is shown in the paper that there always exists a basis of eigenvectors corresponding to the multiple eigenvalue in which the matrix of the coefficients γ_{ij} is diagonal. For this case, the necessary optimality conditions attain the same form as those obtained by a Lagrange multiplier method applied to different and differentiable functionals. This circumstance, fortunately, confirms many papers on multimodal optimization which used a formal variational approach or a formal approach based on Pontryagin's maximum principle without taking care of the nondifferentiability of multiple eigenvalues.

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