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Necessary conditions for optimal design of structures with a nonsmooth eigenvalue based criterion

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Abstract Necessary conditions of optimality for maximizing the buckling load for single or multimodal structures are derived using generalized gradients. The possible design dependence of the pre-buckling displacement is taken into account and implies the appearance of a number of adjoint problems not normally present in, for example, standard vibration problems. The relation to other equivalent forms of the necessary conditions are pointed out and the formulae are demonstrated on simple example problems.

1 Introduction

Optimal structural design with eigenvalue criteria (vibration and stability problems) is an important class of problems that has the complication of the non-differentiability of the eigenvalues at multimodal solutions [see, for example, Zyczkowski and Gajewski (1988), and Olhoff (1987) for surveys]. The occurrence of multimodal optimal designs is now well-established (cf. Olhoff and Rasmussen 1977), as is the derivation of directional derivatives for eigenvalues in vibration problems [see e.g. Haug *et al.* (1986)]. An overview of recent developments can be found in the report of Seyranian *et al.* (1994).

In this note we employ tools of nonsmooth analysis to derive necessary conditions of optimality for maximizing the buckling load for structures. With the use of mathematical programming methods for the nondifferentiable problems in mind, the conditions are obtained by use of the generalized gradient concept (Clarke 1983). The results obtained are generalizations of necessary conditions for vibration problems and have an additional complication in the appearance of a number of adjoint problems, which are associated with the design dependence of the pre-buckling displacement field. As for the simpler cases, we can also here show, by using some basic facts from linear algebra, the equivalence between a number of different formulations of the necessary conditions.

The approach we take is similar in concept to recent work by Overton (1993), and Overton and Womersley (1993), and as such is geared towards the application of algorithms for nonsmooth optimization (see e.g. Kiwiel 1985; Schramm and Zowe 1992). The application of such algorithms is not dealt with here, but the reader is referred to the literature.

We emphasize here that eigenvalue problems are just one class of nondifferentiable problems arising naturally in structural optimization, and for these problems the non-differentiability is directly associated with the optimization criterion employed. Another more subtle type of nondifferentiability can arise indirectly from an underlying analysis model which results in displacements that are non-differentiable as functions of design. Example problems are elasto-plastic problems and contact problems (see e.g. Sokolowski 1988).

2 The eigenvalue problem

Let us consider the generalized eigenvalue problem,

$$\mathbf{K}(\mathbf{b})\phi - P\mathbf{G}(\mathbf{b},\mathbf{u})\phi = 0\,, \tag{1}$$

where $\mathbf{b} = \{b_e\} \in \mathbf{R}^M$ denotes the vector of design variables and $\mathbf{u} = \{u_i\} \in \mathbf{R}^N$ is the displacement vector solution of a finite element approximation of a linear elasto-static problem,

$$\mathbf{K}(\mathbf{b})\mathbf{u} = \mathbf{f}\,. \tag{2}$$

This set of equations can be easily identified with, for example, a finite element model for the classical buckling problem of structures where $\mathbf{K}$ is the stiffness matrix, $\mathbf{G}$ the geometric matrix, the eigenvalues P are the buckling load factors, and $\mathbf{u}$ the pre-bifurcation nodal displacements vector, due to the applied load $\mathbf{f}$. It should be recognized that the geometric matrix $\mathbf{G}$ is given indirectly as a function of displacement through the pre-bifurcation stress field.

For this model we assume that for all the admissible designs $\mathbf{b}$ the matrices $\mathbf{K}$ and $\mathbf{G}$ are real, symmetric and smooth functions of the displacement $\mathbf{u}$ and design variables $\mathbf{b}$. Furthermore $\mathbf{K}$ is assumed (uniformly) positive definite for all allowed designs and the eigenvalues P are different from zero.

3 The optimization problem

The optimization problem is stated as the maximization of the buckling critical load of a structural component. To limit the amount of resource, a scalar global constraint on the design variables is introduced and we also assume that we impose simple bound constraints on the design variables

$$V(\mathbf{b}) \le 0, \quad \underline{b}_e \le b_e \le \overline{b}_e, \quad e = 1,\dots,M\,. \tag{3}$$

The optimal design problem is then to maximize the lowest *positive* eigenvalue of (1), (2) (the critical load factor P_{cr}), subject to the constraints (3). This is equivalent to minimizing the maximum of the *inverse* (of all) eigenvalues. With this in mind and using the Rayleigh variational principle the optimal design problem is thus stated as

$$\min_{\substack{\mathbf{b},\mathbf{u}\\ V(\mathbf{b})\le 0,\, \underline{b}\le b\le \bar{b}\\ \mathbf{K}(\mathbf{b})\mathbf{u}=\mathbf{f}}} \quad \max_{\substack{\phi\neq 0\\ \phi\in\mathbf{R}^N}} \frac{\phi^T\mathbf{G}(\mathbf{b},\mathbf{u})\phi}{\phi^T\mathbf{K}(\mathbf{b})\phi}, \tag{4}$$

where the minimization with respect to $\mathbf{b}$ and $\mathbf{u}$ is subject to the constraints (3) as well as the pre-bifurcation equilibrium constraint (2). Note that we in (4) also achieve that the normalization of the eigenvectors is performed with respect to the positive definite matrix $\mathbf{K}$. It is implicitly assumed that the design constraints (3) will ensure a bounded and positive definite stiffness matrix $\mathbf{K}$ and that the stiffness and geometric stiffness matrices are smooth matrix functions. The optimization problem defined above is a nonsmooth optimization problem in $\mathbf{b}$, $\mathbf{u}$, if P_{cr} is a multiple eigenvalue. To overcome this problem we will in the following use the generalized gradient concept (Clarke 1983) .

We emphasize here a basic difference between this model and other models for structural eigenvalue optimization (see e.g. Haug *et al.* 1986). The present model assumes that $\mathbf{G}$ also depends implicitly on design via the pre-bifurcation displacement $\mathbf{u}$, thus explaining the additional equilibrium constraint (2). This constraint is not present in optimization models for natural frequency optimization problems without membrane forces [*with* such forces, (4) covers this case also, with $\mathbf{G}$ being the stiffness matrix, $\mathbf{K}$ the mass matrix and the problem stated as a max-min problem].

4 Necessary conditions for the optimal solution

The necessary conditions for the optimization problem (4) are obtained via the associated Lagrangian for the outer minimization in the design and equilibrium variables $(\mathbf{b},\mathbf{u})$,

$$L = \left[\max_{\phi\neq 0} \frac{\phi^T\mathbf{G}(\mathbf{b},\mathbf{u})\phi}{\phi^T\mathbf{K}(\mathbf{b})\phi}\right] + \mathbf{v}^T[\mathbf{K}(\mathbf{b})\mathbf{u}-\mathbf{f}] +$$

$$\Lambda[V(\mathbf{b})] + \sum_{e=1}^{N}[\eta_1^e(b_e-\bar{b}_e) - \eta_2^e(b_e-\underline{b}_e)]\,. \tag{5}$$

In this augmented functional, $\mathbf{v}$ is the adjoint displacement vector [Lagrange multiplier of the equilibrium constraint (2)] and the Lagrange multipliers associated with the design constraints (3) satisfy the inequalities $\Lambda \ge 0$; $\eta_1^e \ge 0$, $\eta_2^e \ge 0$, $e = 1,2,\ldots,N$.

Based on this Lagrangian, the necessary conditions for a minimum can be identified with the condition of stationarity at the optimal solution $\hat{\mathbf{b}}$, $\hat{\mathbf{u}}$, stated in terms of generalized gradients for nonsmooth functions (Clarke 1983). The stationarity condition is thus expressed as

$$0 \in \partial_{\mathbf{b},\mathbf{u}}L\,, \tag{6}$$

where the generalized gradient $\partial_{\mathbf{b},\mathbf{u}}L$ is the convex set in $\mathbf{R}^M \times \mathbf{R}^N$ defined by

$$\partial_{\mathbf{b},\mathbf{u}}L = \mathrm{co}\left\{ \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}, \begin{cases} x_e = \dfrac{\phi^T\left[\dfrac{\partial\mathbf{G}}{\partial b_e} - \dfrac{1}{P_{\mathrm{cr}}}\dfrac{\partial\mathbf{K}}{\partial b_e}\right]\phi}{\phi^T\mathbf{K}\phi} \\ y_i = \dfrac{\phi^T\dfrac{\partial\mathbf{G}}{\partial u_i}\phi}{\phi^T\mathbf{K}\phi} \end{cases} : \phi\neq 0, \phi\in\Gamma(\hat{\mathbf{b}},\hat{\mathbf{u}}) \right\} + \begin{pmatrix} \left\{\mathbf{v}^T\dfrac{\partial\mathbf{K}}{\partial b_e}\hat{\mathbf{u}}\right\} \\ \mathbf{v}^T\mathbf{K}\end{pmatrix} + \begin{pmatrix}\mathbf{z}\\ 0\end{pmatrix}, \tag{7}$$

with $\mathbf{z} = \{z_e\} = \left\{\Lambda\dfrac{\partial V(\hat{\mathbf{b}})}{\partial b_e} + \eta_1^e - \eta_2^e\right\}$ and with $\hat{\mathbf{b}}$, $\hat{\mathbf{u}}$ connected via the equilibrium equation (2) [see Clarke (1983) and Kiwiel (1985, Lemma 2.5) for the derivation of the generalized gradient of a maximum function]. In (7) and in the following the dependence of the $\mathbf{G}$ and $\mathbf{K}$ matrices on $\mathbf{u}$ and $\mathbf{b}$ is implicitly assumed.

In the previous expression "co" denotes the convex hull, i.e. the convex combinations of all elements in the set, the indices "e" and "i" range over all the design variables (M) and the displacement degrees of freedom (N), respectively, and $\Gamma(\hat{\mathbf{b}},\hat{\mathbf{u}})$ is the subspace of *all* the eigenvectors associated with the critical load factor. Note that this set includes all the eigenvectors for which the maximum of the Rayleigh quotient ($\lambda_1 = 1/P_{\mathrm{cr}}$) is achieved. To characterize the generalized gradient $\partial_{\mathbf{b},\mathbf{u}}L$ one needs only to consider the eigenvectors in $\Gamma(\hat{\mathbf{b}},\hat{\mathbf{u}})$ normalized with respect to the stiffness matrix $\mathbf{K}$, i.e. $\phi^T\mathbf{K}\phi = 1$. This follows from the fact that proportional eigenvectors will give rise to the same element in $\partial_{\mathbf{b},\mathbf{u}}L$.

Let us assume now that, at the optimal solution, the critical load factor has multiplicity "m" and let ϕ_p, $p = 1,\ldots,m$ be any set of m orthonormal (with respect to $\mathbf{K}$) eigenvectors corresponding to the critical load factor P_{cr}. Any normalized eigenvector $\phi \in \Gamma(\hat{\mathbf{b}},\hat{\mathbf{u}})$ can be represented as a linear combination of the eigenvectors ϕ_p, $p = 1,\ldots,m$ in the form (summation over repeated indices apply here and in the following) $\phi = \alpha_p\phi_p$, $\boldsymbol{\alpha} \in \mathbf{R}^m$ and $\|\boldsymbol{\alpha}\| = 1$. Substituting this in the Lagrangian generalized gradient (7) one obtains

$$\partial_{\mathbf{b},\mathbf{u}}L = \mathrm{co}\left\{ \begin{pmatrix} \left\{\alpha_p\alpha_q\phi_p^T\left[\dfrac{\partial\mathbf{G}}{\partial b_e} - \dfrac{1}{P_{\mathrm{cr}}}\dfrac{\partial\mathbf{K}}{\partial b_e}\right]\phi_q\right\} \\ \left\{\alpha_p\alpha_q\phi_p^T\dfrac{\partial\mathbf{G}}{\partial u_i}\phi_q\right\} \end{pmatrix} : \boldsymbol{\alpha}\in\mathbf{R}^m, \|\boldsymbol{\alpha}\| = 1 \right\} + \begin{pmatrix} \left\{\mathbf{v}^T\dfrac{\partial\mathbf{K}}{\partial b_e}\hat{\mathbf{u}}\right\} \\ \mathbf{v}^T\mathbf{K}\end{pmatrix} + \begin{pmatrix}\mathbf{z}\\ 0\end{pmatrix}. \tag{8}$$

The optimality condition (8) can be further simplified by solving it, partially, for the adjoint displacement $\mathbf{v}$ (i.e. the Lagrange multiplier for the pre-bifurcation equilibrium constraint). With this in mind let us consider the solutions $\mathbf{v}^{pq} \in \mathbf{R}^N$, $p,q = 1,\ldots,m$ of

$$\mathbf{K}\mathbf{v}^{pq} = \left\{\phi_p^T\frac{\partial\mathbf{G}}{\partial u_i}\phi_q\right\}. \tag{9}$$

Because of symmetry, $\mathbf{v}^{pq} = \mathbf{v}^{qp}$, and we must solve (9) $\frac{1}{2}m(m+1)$ times to find all adjoint fields. However, this only involves different right-hand sides of the equation. With these adjoint fields, we see that we can write the necessary condition (6) as $\mathbf{0} \in \partial_b L$, where,

$$\partial_b L = \mathrm{co}\left\{ \mathbf{x}, x_e = \alpha_p \alpha_q \left[\phi_p^T \left(\frac{\partial \mathbf{G}}{\partial b_e} - \frac{1}{P_{\mathrm{cr}}} \frac{\partial \mathbf{K}}{\partial b_e} \right) \phi_q - \mathbf{v}^{pq} \frac{\partial \mathbf{K}}{\partial b_e} \hat{\mathbf{u}} \right] : \boldsymbol{\alpha} \in \mathbf{R}^m, \|\boldsymbol{\alpha}\| = 1 \right\} + \mathbf{z}, \quad (10)$$

where we should not forget that the adjoint displacements are solutions of the adjoint problems (9). Note that the first term in (10) is actually the generalized gradient of the inverse of the critical load as a function of the design variables only.

Finally, the condition (10) should be supplemented by the usual complementary slackness conditions, which we do not state here for the sake of brevity.

Before we describe some examples let us see how (10) particularises for two special cases. For the first case, we assume that we have a simple eigenvalue. Then the generalized gradient has only one element, the gradient of L and the necessary condition is $\nabla_b L = 0$, where

$$(\nabla_b L)_e = \boldsymbol{\phi}^T \left(\frac{\partial \mathbf{G}}{\partial b_e} - \frac{1}{P_{\mathrm{cr}}} \frac{\partial \mathbf{K}}{\partial b_e} \right) \boldsymbol{\phi} - \mathbf{v} \frac{\partial \mathbf{K}}{\partial b_e} \hat{\mathbf{u}} + \Lambda \frac{\partial V}{\partial b_e} + \eta_1^e - \eta_2^e . \quad (11)$$

In this expression the first two terms represent the e-th component of the simple eigenvalue gradient vector, the last three terms the components of the volume and bound constraint gradients, $\boldsymbol{\phi}$ is the normalized mode associated with the simple eigenvalue and the adjoint displacement $\mathbf{v}$ is the solution of the linear equation $\mathbf{K}\mathbf{v} = \left\{ \boldsymbol{\phi}^T \frac{\partial \mathbf{G}}{\partial u_i} \boldsymbol{\phi} \right\}$.

For the second case we consider a situation where the matrix $\mathbf{G}$ is only explicitly dependent on the design variables $\mathbf{b}$. Then the adjoint displacements $\mathbf{v}^{pq}$ are all equal to zero and (10) reduces to the gradient expression of a simple natural vibration frequency or of a buckling load factor for a structure with design independent pre-bifurcation stress field (see e.g. Haug *et al.* 1986).

Another aspect worth noting is that the necessary condition (10) can be given other equivalent forms, which generalizes the optimality condition presented by, for example, Seyranian *et al.* (1994) (for the case where $\mathbf{G}$ is independent of $\mathbf{u}$). First note that the set of convex combinations of symmetric matrices of the form $\boldsymbol{\alpha}\boldsymbol{\alpha}^T$, $\|\boldsymbol{\alpha}\| = 1$ is the same as the set of all positive semi-definite matrices with trace equal to 1. Thus $\mathbf{0} \in \partial_b L$ is equivalent to the existence of a positive semi-definite $m \times m$ matrix $\mathbf{H} = \{H_{pq}\}$ with $\mathrm{tr}\mathbf{H} = 1$, so that

$$H_{pq} \left[\phi_p^T \left(\frac{\partial \mathbf{G}}{\partial b_e} - \frac{1}{P_{\mathrm{cr}}} \frac{\partial \mathbf{K}}{\partial b_e} \right) \phi_q - \mathbf{v}^{pq} \frac{\partial \mathbf{K}}{\partial b_e} \hat{\mathbf{u}} \right] + z_e = 0 ,$$

$$e = 1, \ldots, M . \quad (12)$$

Moreover, (12) can be "diagonalized" in the following sense. Let $\mathbf{Q} = \{Q_{rs}\}$ be an orthonormal matrix, so that $\mathbf{Q}^T\mathbf{H}\mathbf{Q}$ is a diagonal matrix of the eigenvalues μ_i of $\mathbf{H}$, with $\mu_i \geq 0$, $\sum_{i=1}^{m} \mu_i = 1$. By considering the (orthonormal) eigenvectors $\tilde{\phi}_r = Q_{sr}\phi_s$, we can write (12) as

$$\sum_{i=1}^{m} \mu_i \left[\tilde{\phi}_i^T \left(\frac{\partial \mathbf{G}}{\partial b_e} - \frac{1}{P_{\mathrm{cr}}} \frac{\partial \mathbf{K}}{\partial b_e} \right) \tilde{\phi}_i - \tilde{\mathbf{v}}^{ii} \frac{\partial \mathbf{K}}{\partial b_e} \hat{\mathbf{u}} \right] + z_e = 0 ,$$

$$e = 1, \ldots, M . \quad (13)$$

where summation over repeated indices does not apply.

Observe that $\tilde{\mathbf{v}}^{ii}$ is now the adjoint displacements given by (9), but from the eigenvectors $\tilde{\phi}_r$. Also note that the necessary condition (12) is a condition for any initial choice of orthonormal eigenvectors, while (13) is a condition for a *specific* choice of orthonormal eigenvectors $\tilde{\phi}_r$. In this sense (12) and (13) are equivalent. The condition (13) is a generalization of the optimality conditions first stated by Olhoff and Rasmussen (1977), for bimodal isostatic buckling [the derivation for the general (isostatic) problem is carried out by Bendsøe *et al.* (1983), using a bound formulation]. The interested reader is also referred to Overton (1993), and to Overton and Womersley (1993), who discuss the various forms of the necessary conditions.

5 Examples

5.1 Example 1

Let us consider the problem (5) with data, $M = 1$, $N = 2$, $\underline{b} = -0.2$, $\overline{b} = 0.4$, no volume constraint (3) and

$$\mathbf{K} = \begin{bmatrix} 1+b & 0 \\ 0 & 1+b \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 2-u_1 & 0 \\ 0 & 1+u_2 \end{bmatrix}, \quad \mathbf{f} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}.$$

The generalized eigenvalue problem has eigenvalues $\lambda_1 = \frac{1}{P_1} = \frac{2b+1}{(1+b)^2}$ and $\lambda_2 = \frac{1}{P_2} = \frac{1}{1+b}$, and the eigenvalues plotted as functions of the design variable b are shown in the graph in Fig. 1.

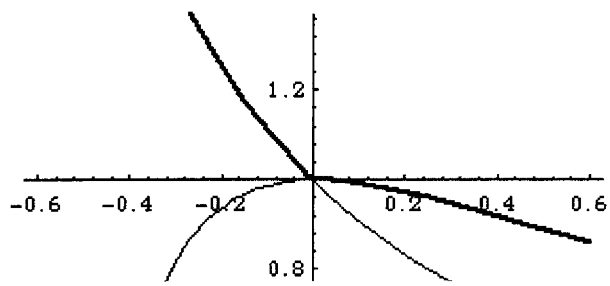

Fig. 1. Graphical representation of the eigenvalues λ_i ($i = 1, 2$) in dependence of the design variable b

From Fig. 1 it is clear that the point $\hat{b} = 0$ is a *multimodal* stationary point, which is not a local minimum. The global minimum is attained at $\hat{b} = 0.4$ with the upper bound constraint active, and this is a unimodal design. Let us now verify the necessary condition at these two candidate points.

At $\hat{b} = 0$ the eigenvalue has multiplicity $m = 2$, $\lambda_1 = \lambda_2 = 1$, the displacement vector is $\hat{\mathbf{u}} = \{1\ 0\}^T$, and two orthonormal eigenvectors are $\phi_1 = (1\ 0)^T$, $\phi_2 = (0\ 1)^T$. The adjoint displacement vectors $\mathbf{v}^{11}$, $\mathbf{v}^{12} = \mathbf{v}^{21}$, $\mathbf{v}^{22}$ are solutions of

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\mathbf{v}^{ij} = \left\{ \begin{array}{c} \phi_i^T \begin{bmatrix}-1 & 0\\ 0 & 0\end{bmatrix} \phi_j \\ \phi_i^T \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \phi_j \end{array} \right\} \Rightarrow \mathbf{v}^{11} = \begin{Bmatrix}-1\\ 0\end{Bmatrix},$$

$$\mathbf{v}^{12} = \begin{Bmatrix}0\\ 0\end{Bmatrix}, \quad \mathbf{v}^{22} = \begin{Bmatrix}0\\ 1\end{Bmatrix},$$

so for the generalized gradient one obtains

$$\partial L_{b=0} = \mathrm{co}\left\{ -\begin{Bmatrix}\alpha_1\\ \alpha_2\end{Bmatrix}^T \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \begin{Bmatrix}\alpha_1\\ \alpha_2\end{Bmatrix} - \right.$$

$$\left. \alpha_i \alpha_j \mathbf{v}^{ij^T} \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \begin{Bmatrix}1\\ 0\end{Bmatrix} : \boldsymbol{\alpha} \in \mathbf{R}^2, \|\boldsymbol{\alpha}\| = 1 \right\} =$$

$$\mathrm{co}\{-\alpha_2^2 : |\alpha_2| \leq 1\},$$

and the necessary condition is satisfied for $\alpha_2 = 0$.

On the other hand, at upper bound constraint $\hat{b} = 0.4$ the eigenvalue is simple, $m = 1$, $\lambda_1 = \frac{45}{49}$, the displacement vector is $\hat{\mathbf{u}} = \left\{\frac{5}{7}\ 0\right\}^T$ and the objective function is differentiable since the eigenvalue is simple. The necessary condition is

$$\nabla L_{b=0.4} = \left\{ -\frac{45}{49} \begin{Bmatrix}\sqrt{5/7}\\ 0\end{Bmatrix}^T \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \begin{Bmatrix}\sqrt{5/7}\\ 0\end{Bmatrix} - \right.$$

$$\left. \mathbf{v}^T \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \begin{Bmatrix}5/7\\ 0\end{Bmatrix} + \eta_1 \right\} = 0, \quad \text{for } \eta_1 \geq 0,$$

with the adjoint displacement vector $\mathbf{v}$ given as the solution to

$$\begin{bmatrix}7/5 & 0\\ 0 & 7/5\end{bmatrix} \begin{Bmatrix}v_1\\ v_2\end{Bmatrix} =$$

$$\left\{ \begin{array}{c} \begin{Bmatrix}\sqrt{5/7}\\ 0\end{Bmatrix}^T \begin{bmatrix}-1 & 0\\ 0 & 0\end{bmatrix} \begin{Bmatrix}\sqrt{5/7}\\ 0\end{Bmatrix} \\ \begin{Bmatrix}\sqrt{5/7}\\ 0\end{Bmatrix}^T \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \begin{Bmatrix}\sqrt{5/7}\\ 0\end{Bmatrix} \end{array} \right\} \Rightarrow \begin{Bmatrix}v_1\\ v_2\end{Bmatrix} = \begin{Bmatrix}-25/49\\ 0\end{Bmatrix}.$$

Thus substituting in the necessary condition we obtain $\nabla L_{b=0.4} = 0$ for $\eta_1 = \frac{100}{343}$.

In this example we have shown that a multimodal point can be stationary without being a local minimum. One can also generate examples where such multimodal points are not even stationary points.

5.2 Example 2

Let us now consider an example with $M = 2$, $N = 2$ and

$$\mathbf{K} = \begin{bmatrix}(1+b_1)^2 & 0\\ 0 & (1+b_1)^2\end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} \dfrac{2(1+b_1)}{u_2} & \dfrac{b_2}{u_1} \\ \dfrac{b_2}{u_1} & \dfrac{2(1-b_1)}{u_2} \end{bmatrix}, \quad \mathbf{f} = \begin{Bmatrix}1\\ 2\end{Bmatrix}.$$

The generalized eigenvalue problem has eigenvalues $\lambda_1 = \frac{1}{P_1} = 1 - \sqrt{(b_1)^2 + (b_2)^2}$ and $\lambda_2 = \frac{1}{P_2} = 1 + \sqrt{(b_1)^2 + (b_2)^2}$, each of which are non-smooth, and for which it is clear that $\hat{\mathbf{b}} = \{0\ 0\}^T$ is a double eigenvalue stationary point (see Fig. 2). Let us check the necessary condition at this point.

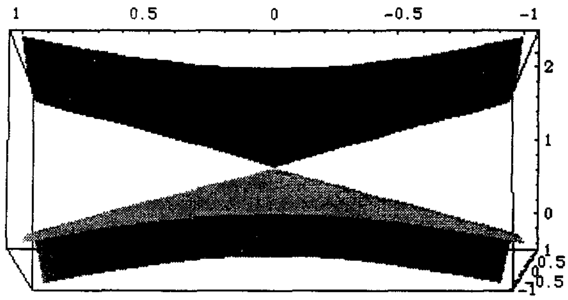

Fig. 2. Graphical representation of the eigenvalues λ_i $(i = 1, 2)$ in dependence of b_1 and b_2

For $\hat{\mathbf{b}} = \{0\ 0\}^T$ we have $\hat{\mathbf{u}} = \{1\ 2\}^T$, $\lambda_1 = \lambda_2 = 1$ and orthonormal eigenvectors $\boldsymbol{\phi}_1 = (1\ 0)^T$, $\boldsymbol{\phi}_2 = (0\ 1)^T$. So the generalized gradient becomes for this case

$$\partial L_{b=0} = \mathrm{co}\left\{ \left(\begin{array}{c} \boldsymbol{\alpha}^T \left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} - \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} \right) \boldsymbol{\alpha} \\ \boldsymbol{\alpha}^T \left(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \right) \boldsymbol{\alpha} \end{array} \right) - \right.$$

$$\left. \left\{ \begin{array}{c} \alpha_i \alpha_j \mathbf{v}^{ij^T} \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} \begin{Bmatrix}1\\ 2\end{Bmatrix} \\ \alpha_i \alpha_j \mathbf{v}^{ij^T} \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \begin{Bmatrix}1\\ 2\end{Bmatrix} \end{array} \right\} : \boldsymbol{\alpha} \in \mathbf{R}^2, \|\boldsymbol{\alpha}\| = 1 \right\},$$

with the adjoint displacement vectors $\mathbf{v}^{11}$, $\mathbf{v}^{12} = \mathbf{v}^{21}$, $\mathbf{v}^{22}$ given as solutions of

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\mathbf{v}^{ij} = \left\{ \begin{array}{c} \phi_i^T \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \phi_j \\ \phi_i^T \begin{bmatrix}-1/2 & 0\\ 0 & -1/2\end{bmatrix} \phi_j \end{array} \right\} \Rightarrow \mathbf{v}^{11} =$$

$$\begin{Bmatrix}0\\ -1/2\end{Bmatrix}, \quad \mathbf{v}^{12} = \begin{Bmatrix}0\\ 0\end{Bmatrix}, \quad \mathbf{v}^{22} = \begin{Bmatrix}0\\ -1/2\end{Bmatrix}.$$

Substituting this in the generalized gradient we obtain

$$\partial L_{b=0} = \mathrm{co}\left\{ \begin{Bmatrix}(\alpha_1)^2 - (\alpha_2)^2\\ 2\alpha_1\alpha_2\end{Bmatrix} : \boldsymbol{\alpha} \in \mathbf{R}^2, \alpha_1^2 + \alpha_2^2 = 1 \right\},$$

and the necessary condition is satisfied if we in the generalized gradient take the convex combination (combination factor of 0.5) for the choices $\boldsymbol{\alpha}^1 = \{1\ 0\}^T$, $\boldsymbol{\alpha}^2 = \{0\ 1\}^T$.

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