Structural Optimization 7, 141–159 © Springer-Verlag 1994

Review Paper

Optimization methods for truss geometry and topology design

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Abstract Truss topology design for minimum external work (compliance) can be expressed in a number of equivalent potential or complementary energy problem formulations in terms of member forces, displacements and bar areas. Using duality principles and non-smooth analysis we show how displacements only as well as stresses only formulations can be obtained and discuss the implications these formulations have for the construction and implementation of efficient algorithms for large-scale truss topology design. The analysis covers min-max and weighted average multiple load designs with external as well as self-weight loads and extends to the topology design of reinforcement and the topology design of variable thickness sheets and sandwich plates. On the basis of topology design as an inner problem in a hierarchical procedure, the combined geometry and topology design of truss structures is also considered. Numerical results and illustrative examples are presented.

1 Introduction

The optimization of the geometry and topology of trusses can conveniently be formulated in terms of the well-known ground structure method. In this approach, the layout of a truss structure is found by allowing a certain set of connections between a fixed set of nodal points as potential structural or vanishing members. Topology design is inherently a discrete optimization problem, but for the truss problem the geometry allows for using the continuously varying cross-sectional bar areas as design variables, including the possibility of zero bar areas. This implies that the truss topology problem can be viewed as a standard sizing problem. Note that the basic combinatorial nature of topology design, namely finding the optimal set of vanishing truss members, remains in the sizing formulation. The sizing reformulation is possible for the simple reason that the truss, which is really a two- or three-dimensional continuum, is described geometrically as being one-dimensional.

The ground structure approach thus allows the truss topology design problem to be viewed as a sizing problem. However, the topology problem is unusual as a structural optimization problem as the number of design variables is typically several orders of magnitude bigger than the number of displacement variables. For most structural optimization problems described in the literature the opposite is the case. Also, for truss topology design the stiffness matrix of the full ground structure with certain members at zero gauge can be singular. This implies that most optimal designs have a singular stiffness matrix when described as part of the full ground structure, thus excluding the possibility of invoking standard structural optimization techniques.

It is broadly recognized that structural layout has an immense influence on structural performance, and recent years have seen a revived interest in this important area of structural optimization (Kirsch 1989; Rozvany 1992; Bendsøe and Mota Soares 1993). The study of fundamental properties of grid-like lay-outs was pioneered by Michell (1904); see also Hemp (1973), but this interesting field has only much later developed into what is now the well-established layout theory for frames and flexural systems (Rozvany 1976, 1989, 1992). The last couple of years witnessed the development of the so-called homogenization method for generating optimal topologies of structural elements (Bendsøe and Kikuchi 1988; Kikuchi and Suzuki 1991). The homogenization method predicts grid- and truss-like structures for structures with a low amount of available material and thus the homogenization method supplements analytical methods for the prediction of layout (Diaz and Belding 1992; Kikuchi and Suzuki 1991). The application of numerical methods to discrete truss topology problems and similar structural systems, which is the subject of this paper, has a shorter history with early contributions by, for example, Dorn *et al.* (1964), Fleron (1964), Pedersen (1970, 1972, 1973). The numerical methods developed for truss topology design over the period 1964 to 1990 are described in the review papers by Kirsch (1989) and Topping (1992) with the former also containing a survey of layout theory. This paper thus concentrates on more recent developments.

Truss topology design problems were in early work typically formulated in terms of member forces, ignoring kinematic compatibility to obtain a linear programming (LP) problem in member areas and forces. The resulting topology and force field are then often employed as a starting point for a more complicated design problem formulation, with heuristics, branch and bound techniques, etc., being used to link the two model problems (Kirsch 1989; Ringertz 1986). Alternatively, when displacement formulations are used, then (small) non-zero lower bounds on the cross-sectional areas have been imposed in order to have a positive definite stiffness matrix. This means that standard techniques for optimal structural design can be used, albeit imposing very tight restrictions on the size of problem that can be handled. Also, it allows for the use of standard optimality criteria methods for large scale design problems (Taylor and Rossow 1977; Zhou and Rozvany 1992/1993).

In the simultaneous analysis and design approach, the design variables and state variables are not distinguished and the full problem is solved by one unified numerical optimization procedure. However, unless specially developed numerical solution procedures are used, only very small problems can be treated (Saka 1980; Ringertz 1988; Bendsøe *et al.* 1991; Sankaranaryanan *et al.* 1993). A recent interesting development is the use of simulated annealing and genetic algorithm techniques for the topology problems in their original formulation as discrete selection problems, but also these fairly general approaches are with the present technology restricted to fairly small scale problems (Fleury 1993; Grierson and Pak 1993; Hajela *et al.* 1993).

In this paper, we will give a survey of classical truss topology methodology and investigate various new formulations of and numerical methods for truss topology design. We seek specifically to be able to handle problems with a large number of potential structural elements, using the ground structure approach. For this reason we consider the simplest possible optimal design problem, namely the minimization of compliance (maximization of stiffness) for a given weight of the structure. The analysis is general enough to encompass multiple load problems in the worst-case and weighted-average formulations, the case of external load plus selfweight and the problem of determining the optimal topology of the reinforcement of a structure as, for example, seen in fail-safe design. It turns out that for these problems a number of equivalent problem statements can be given, among them problems involving the nodal displacements only or in the member forces only. With these reformulations on hand, it is possible to devise very efficient algorithms that can handle large scale problems. The formulations are obtained through duality principles and the resulting formulations in displacements or stresses correspond to equilibrium problems for an optimal global strain energy and an optimal global complementary energy, respectively. Analogous formulations for continuum structures have been derived in analyses of the homogenization modelling for topology design (Allaire and Kohn 1993; Jog *et al.* 1994) as well as in analyses of the simultaneous design of material and structure (Bendsøe *et al.* 1992).

Since the positions of nodal points are not used as design variables in the ground structure approach, a high number of nodal points should be used in the ground structure in order to obtain efficient topologies. A drawback of the method is that the optimal topologies are very sensitive to the layout of nodal points, at least if the number of nodal points is relatively low. This makes it natural to consider an extension of the ground structure approach and to include the optimization of the nodal point location for a given number and connectivity of nodal points (see, e.g. Kirsch 1989). With very efficient tools on hand for the topology design with fixed nodal position it seems natural to treat the variation of nodal position as an outer optimization in a two-level hierarchical formulation. As the objective function (i.e. the compliance) for the optimal topology depends on the geometry variables in a nonsmooth way, this outer minimization requires nonsmooth optimization techniques. It is shown how very efficient designs can be achieved by these means.

2 Truss topology problem formulation

In the ground structure approach for truss topology design, a set of n chosen nodal points (N degrees of freedom) and m possible connections are given, and one seeks to find the optimal substructure of this structural universe. In some papers on the ground structure approach, the ground structure is always assumed to be the set of all possible connections between the chosen nodal points, but here we allow the ground structure to be any given set of connectivities (see Fig. 1). This approach may lead to designs that are not the best ones for the chosen set of nodal points, but the approach implicitly allows for restrictions on the possible spectrum of possible member lengths (see, e.g. Fig. 1B where only two bar lengths appear), as well as for the study of the optimal subset of members of a given truss-layout.

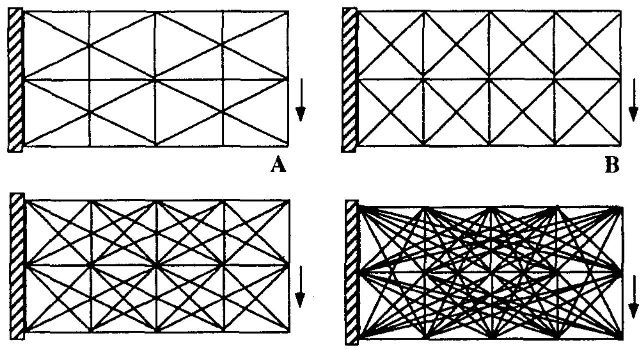

Fig. 1. Ground structures for transmitting a vertical force to a vertical line of supports. (A)-(C) Truss ground structures of variable complexity in a rectangular domain with a regular 5 by 3 nodal layout. (D) All connections between the nodal points are included

Let a_i and ℓ_i denote the cross-sectional area and length of bar number i, respectively, and we assume that all bars are made of linear elastic materials, with Young's moduli E_i. The volume of the truss is

$$V = \sum_{i=1}^{m} a_i \ell_i \,. \tag{1}$$

In order to simplify the notation at a later stage, we introduce the bar volumes $t_i = a_i \ell_i$, $i = 1, \ldots, m$, as the fundamental design variables. Static equilibrium is expressed as

$$\mathbf{Bq} = \mathbf{p} \,, \tag{2}$$

where $\mathbf{q}$, $\mathbf{p}$ are the member force and nodal force vectors, respectively, of the free degrees of freedom. The ground structure is chosen so that the matrix $\mathbf{B}$ has full rank and so that $m \geq N$, excluding mechanisms and rigid body motions. The stiffness matrix of the truss is written as

$$\mathbf{K}(\mathbf{t}) = \sum_{i=1}^{m} t_i \mathbf{K}_i \,, \tag{3}$$

where $t_i\mathbf{K}_i$ is the element stiffness matrix for bar number i, written in global coordinates. Note that $\mathbf{K}_i = \frac{E_i}{\ell_i^2}\mathbf{b}_i\mathbf{b}_i^T$ where $\mathbf{b}_i$ is the i-th column of $\mathbf{B}$.

The problem of finding the minimum compliance truss for a given volume of material (the stiffest truss) has the well-known formulation

$$\min_{\mathbf{u},\,\mathbf{t}} \mathbf{p}^T\mathbf{u}\,, \quad \text{subject to:} \quad \sum_{i=1}^{m} t_i\mathbf{K}_i\mathbf{u} = \mathbf{p}\,,$$

$$\sum_{i=1}^{m} t_i = V\,, \quad t_i \geq 0\,,\ i = 1, \ldots, m\,, \tag{4}$$

where the design variable t_i (control variables) and the displacements u_i (state variables) appear as independent variables.

The problem (4) is well studied in the case of an imposed non-negative lower bound on the volumes t_i (see, e.g. Svanberg 1984). In this case the stiffness matrix $\mathbf{K}(\mathbf{t})$ is positive definite for all $\mathbf{t} > \mathbf{0}$ and the displacements can be removed from the problem. The resulting problem in bar volumes turns out to be convex and existence of solutions is assured (Svanberg 1984). Allowing for zero lower bounds complicates the analysis, but it also provides valuable insight.

The zero lower bound on the variables t_i means that bars of the ground structure can be removed and the problem statement thus covers topology design. It should be emphasized that the basic combinatorial problem of topology design is still present in the problem formulation (4), even though it is not formulated as a 0-1 problem. In problem (4) the combinatorial problem is hidden in the non-negativity constraint on the bar volumes. A solution procedure for (4) will, through the identification of the active constraints, solve the combinatorial problem as to which bars should be present in the optimal topology.

The zero lower bound in problem (4) implies that the stiffness matrix is not necessarily positive definite and the state vector $\mathbf{u}$ cannot be removed by a standard adjoint method. Removing $\mathbf{u}$ from the formulation is not very important for the size of the problem, as typically, the number m of bars is much greater than the number of degrees of freedom. In the complete ground structure we connect all nodes, having $m = n(n-1)/2$, while the degree of freedom N is only of the order $2n$ or $3n$ (for planar and 3-D trusses); for this situation we have a fully populated stiffness matrix lacking any sparsity and bandedness.

Our aim in this paper is to study large scale truss topology problems and for this reason we employ the simplest possible design formulation as stated in problem (4). For more general problem statements involving, for example, stress and displacement constraints, a suitable formulation is to use a full parametrization of the state of the system in terms of independent fields of member forces, member strains and displacements [an extended simultaneous analysis and design formulation for this case is discussed by Sankaranaryanan *et al.* (1993)]. Such an approach allows the incorporation such constraints in a consistent way (Cheng and Jiang 1992). Local buckling of the individual bars of the ground structure can also be treated in this framework, but the formulation of a suitable problem statement that covers global buckling and the shift of member lengths in the buckling expressions, when inner nodes can be removed from the truss, is yet to be seen. The extended problem statements can be solved by a number of methods, all of which presently suffer from the inability to handle large scale problems. For a moderate number of active displacement constraints an optimality criteria approach seems to be viable (Zhou and Rozvany 1992/1993). Use of a conjugate gradient method for a penalized version of the general statement has been investigated (Sankaranaryanan *et al.* 1993), as has the use of interior penalty methods together with sparse matrix techniques (Ringertz 1988). For the case of a single loading, local stability constraints can be efficiently handled in a force formulation and solved by a modified SIMPLEX algorithm, as described by Pedersen (1993).

Problem (4) can result in an optimal topology that is a mechanism; this mechanism is in equilibrium under the given load, and infinitesimal bars can be added to obtain a stable structure (Dorn *et al.* 1964; Fleron 1964; Kirsch 1989; Ringertz 1985). Also, if the optimal topology has straight bars with inner nodal points (hinges), these nodal points should be ignored. The resulting truss maintains the stiffness and the equilibrium of the original optimal topology.

We can formulate the case of *multiple loads* by treating the problem of minimizing a weighted average of the compliances. Such a formulation has proven to be very efficient and useful in the homogenization method, (Diaz and Bendsøe 1992). For a set of M different load cases $\mathbf{p}^k$, $k = 1, \ldots, M$, and weighting factors W^k, $k = 1, \ldots, M$, we formulate the multiple load problem as

$$\min_{\mathbf{u},\,\mathbf{t}} \sum_{k=1}^{M} W^k {\mathbf{p}^k}^T \mathbf{u}^k\,, \quad \text{subject to:} \quad \sum_{i=1}^{m} t_i\mathbf{K}_i\mathbf{u}^k = \mathbf{p}^k\,,$$

$$k = 1, \ldots, M\,, \quad \sum_{i=1}^{m} t_i = V\,, \quad t_i \geq 0\,,\ i = 1, \ldots, m\,. \tag{5}$$

Let us introduce an extended displacement vector $\hat{\mathbf{u}} = (\mathbf{u}^1, \ldots, \mathbf{u}^M)$ of all the displacement vectors $\mathbf{u}^k$, $k = 1, \ldots, M$, an extended force vector $\hat{\mathbf{p}} = (W^1\mathbf{p}^1, \ldots, W^M\mathbf{p}^M)$ of the weighted force vectors $W^k\mathbf{p}^k$, $k = 1, \ldots, M$, and the extended element stiffness matrices as the block diagonal matrices

$$\hat{\mathbf{K}}_i = \begin{bmatrix} W^1\mathbf{K}_i & & & \\ & W^2\mathbf{K}_i & & \\ & & \ddots & \\ & & & W^M\mathbf{K}_i \end{bmatrix}. \tag{6}$$

Then problem (5) can be written as

$$\min_{\mathbf{u},\,\mathbf{t}} \hat{\mathbf{p}}^T\hat{\mathbf{u}}\,, \quad \text{subject to:} \quad \sum_{i=1}^{m} t_i\hat{\mathbf{K}}_i\hat{\mathbf{u}} = \hat{\mathbf{p}}\,,$$

$$\sum_{i=1}^{m} t_i = V\,, \quad t_i \geq 0\,,\ i = 1, \ldots, m\,. \tag{7}$$

which is precisely of the same form as problem (4). Note that in problem (7) it is possible to refer each loading case to a distinct ground sub-structure, and that it thus is possible to cover *fail-safe design* along the lines described by Taylor (1987).

We also note that the problem formulation (4) covers the finite element formulation of the minimum compliance design of continuum problems that exhibit a linear relation between rigidity and the relevant design variable, as exemplified by design of variable thickness sheets, the design of sandwich plates or the simultaneous design of structure and material (Rossow and Taylor 1973; Bendsøe *et al.* 1992). In these cases the matrices $\mathbf{K}_i$ should be interpreted as the specific element stiffness matrices, and the design variables t_i are the element thicknesses (volumes). For these cases and for the multiple load formulation (7), the extended element stiffness matrices no longer have the form of dyadic products. In order to cover all three cases by one formulation, we shall write in the following (4) in a generalized form

$$\min_{\mathbf{u},\mathbf{t}} \mathbf{p}^T\mathbf{u}\,, \quad \text{subject to}: \sum_{i=1}^{m} t_i \mathbf{A}_i \mathbf{u} = \mathbf{p}\,,$$

$$\sum_{i=1}^{m} t_i = V\,, \quad t_i \geq 0\,,\ i = 1, \ldots, m\,, \tag{8}$$

where $\mathbf{A}_i$ are positive semi-definite, symmetric matrices that satisfy that the matrix $\mathbf{A}(\mathbf{t}) = \sum_{i=1}^{m} t_i \mathbf{A}_i$ is positive definite if all t_i's are positive. For trusses this means that the number of bars in the ground structure exceeds the number of degrees of freedom and that the compatibility matrix has full rank.

Note that the formulations above lend themselves to natural extensions, such as to the problem of finding the optimal topology of the reinforcement of a given structure and the optimal topology problem with self-weight taken into consideration.

For the reinforcement problem (see, e.g. Olhoff and Taylor 1983), using the ground structure approach, we divide a given ground structure into the set S of bars of with fixed size and the set R of possible reinforcing bars. Typically S and R will be chosen as disjoint. We prefer here to choose R to contain S as a subset; in this way non-zero lower bounds on the design variables can easily be included in the general problem analysis. The bars (elements) of the given structure have given bar volumes s_i, $i \in S$, and the optimal reinforcement t_i, $i \in R$, is the solution of the minimum compliance problem

$$\min_{\mathbf{u},\mathbf{t}} \mathbf{p}^T\mathbf{u}\,, \quad \text{subject to}: \sum_{i\in R} t_i \mathbf{A}_i \mathbf{u} + \sum_{i\in S} s_i \mathbf{A}_i \mathbf{u} = \mathbf{p}\,,$$

$$\sum_{i\in R} t_i = V\,, \quad t_i \geq 0\,,\ i \in R\,. \tag{9}$$

This problem can be solved by analogous means and can be used for the other topology design problems formulated above. Note that a reinforcement formulation in connection with a multiple load formulation with distinct sub-ground structures of a common ground structure will allow for a very general fail-safe design formulation.

For the important case of optimization with loads due to the weight of the structure taken into account, we employ the standard assumption that the weight of a bar is carried equally by the joints at its ends, thus neglecting bending effects. With $\mathbf{g}_i$ denoting the specific nodal gravitational force vector due to the self-weight of bar number i, the problem of finding the optimal topology with self-weight loads and external loadings takes the form

$$\min_{\mathbf{u},\mathbf{t}} \left[\mathbf{p}^T\mathbf{u} + \left(\sum_{i=1}^{m} t_i \mathbf{g}_i \right)^T \mathbf{u} \right],$$

$$\text{subject to}: \sum_{i=1}^{m} t_i \mathbf{A}_i \mathbf{u} = \mathbf{p} + \sum_{i=1}^{m} t_i \mathbf{g}_i\,, \quad \sum_{i=1}^{m} t_i = V\,,$$

$$t_i \geq 0\,, \quad i = 1, \ldots, m\,. \tag{10}$$

Note that for the problem with self-weight, any feasible truss design (i.e. $\sum_{i=1}^{m} t_i = V$, $t_i \geq 0$, $i = 1, \ldots, m$), for which the self-weight load equilibrates the external load is an optimal design with compliance zero and zero displacement field (compliance is non-negative in all cases). Thus to avoid trivial situations, it is natural to assume that the set

$$\left(t_i \,\middle|\, \sum_{i=1}^{m} t_i = V,\ t_i \geq 0\,,\ i = 1, \ldots, m\,,\ \mathbf{p} + \sum_{i=1}^{m} t_i \mathbf{g}_i = 0 \right), \tag{11}$$

is empty.

We complete this exposition of problem statements by stating the reinforcement problem, with self-weight loads, and general stiffness matrices and loads, so that all cases above are covered as special cases,

$$\min_{\mathbf{u},\mathbf{t}} \left[\mathbf{p}^T\mathbf{u} + \left(\sum_{i\in R} t_i \mathbf{g}_i \right)^T \mathbf{u} + \left(\sum_{i\in S} s_i \mathbf{g}_i \right)^T \mathbf{u} \right],$$

$$\text{subject to}: \sum_{i\in R} t_i \mathbf{A}_i \mathbf{u} + \sum_{i\in S} s_i \mathbf{A}_i \mathbf{u} = \mathbf{p} + \sum_{i\in R} t_i \mathbf{g}_i + \sum_{i\in S} s_i \mathbf{g}_i\,,$$

$$\sum_{i\in R} t_i = V\,, \quad t_i \geq 0\,, \quad i \in R\,. \tag{12}$$

For the developments to follow it is convenient to rewrite the problem statements in terms of a minimum potential energy formulation of the equilibrium constraint. Making use of the fact that the potential energy at equilibrium equals the negative of one half the compliance, (12) can be rewritten as a max min problem in the form

$$\max_{\substack{\mathbf{t}\geq 0 \\ \sum_{i\in R} t_i = V}} \min_{\mathbf{u}} \left[\frac{1}{2}\mathbf{u}^T \left(\sum_{i\in R} t_i \mathbf{A}_i + \sum_{i\in S} s_i \mathbf{A}_i \right) \mathbf{u} - \left(\mathbf{p} + \sum_{i\in R} t_i \mathbf{g}_i + \sum_{i\in S} s_i \mathbf{g}_i \right)^T \mathbf{u} \right]. \tag{13}$$

This is a saddle point problem for a concave-convex problem, and we shall in the following use that the max and min operators in (13) can be interchanged.

3 Optimality criteria methods

For the sake of completeness of the presentation, we will in this section derive the optimality conditions for the general minimum compliance problem (10) with self-weight, and show how these conditions constitute the basis for the well-known optimality criteria method for the numerical solution of the general layout and topology design problem.

In order to obtain the necessary conditions for optimality for problem (10), we introduce Lagrange multipliers $\tilde{\mathbf{u}}$, Ξ, μ_i,

$i = 1, \ldots, m$, for the equilibrium constraint, the volume constraint and the zero lower bound constraint, respectively. The necessary conditions are thus found as the conditions of stationarity of the Lagrangian

$$L = \left(\mathbf{p} + \sum_{i=1}^{m} t_i \mathbf{g}_i\right)^T \mathbf{u} - \tilde{\mathbf{u}}^T \left(\sum_{i=1}^{m} t_i \mathbf{A}_i \mathbf{u} - \mathbf{p} - \sum_{i=1}^{m} t_i \mathbf{g}_i\right) + \Xi\left(\sum_{i=1}^{m} t_i - V\right) + \sum_{i=1}^{m} \mu_i(-t_i) . \tag{14}$$

By differentiation we obtain the necessary conditions

$$\sum_{i=1}^{m} t_i \mathbf{A}_i \tilde{\mathbf{u}} = \mathbf{p} + \sum_{i=1}^{m} t_i \mathbf{g}_i ; \quad \tilde{\mathbf{u}}^T(\mathbf{A}_i \mathbf{u} - 2\mathbf{g}_i) = \Xi - \mu_i ,$$

$$\mu_i \leq 0 , \quad \mu_i t_i = 0 , \quad i = 1, \ldots, m ; \quad \Xi \geq 0 . \tag{15}$$

If we impose a small non-negative lower bound on the areas, the stiffness matrix $\mathbf{A}$ is positive definite and thus $\tilde{\mathbf{u}}$ is the unique Lagrange multiplier for the equilibrium constraint, but the situation with a lower bound is not so straightforward.

Now let $\lambda(\mathbf{u})$ denote the maximal mutual energy with self-weight $\mathbf{u}^T(\mathbf{A}_i\mathbf{u} - 2\mathbf{g}_i)$ of the individual bars, i.e.

$$\lambda(\mathbf{u}) = \max\left\{\mathbf{u}^T(\mathbf{A}_i \mathbf{u} - 2\mathbf{g}_i) | i = 1, \ldots, m\right\} , \tag{16}$$

and let $J(\mathbf{u})$ denote the set of bars for which the mutual energy attains this maximum level

$$J(\mathbf{u}) = \left\{ i | \mathbf{u}^T(\mathbf{A}_i \mathbf{u} - 2\mathbf{g}_i) = \lambda(\mathbf{u}) \right\} , \tag{17}$$

We also define non-dimensional element volumes $\tilde{t}_i = t_i/V$. Then the necessary conditions are satisfied with

$$\tilde{\mathbf{u}} = \mathbf{u} ; \; t_i = \tilde{t}_i V , \; i \in J(\mathbf{u}) ; \; t_i = 0 , \; i \notin J(\mathbf{u}) ; \; \Xi = \lambda(\mathbf{u}) ;$$

$$\mu_i = 0 , \; i \in J(\mathbf{u}) ; \; \mu_i = \lambda(\mathbf{u}) - \mathbf{u}^T(\mathbf{A}_i \mathbf{u} - 2\mathbf{g}_i) , \; i \notin J(\mathbf{u}) , \tag{18}$$

provided that there exist a displacement field $\mathbf{u}$ with corresponding set $J(\mathbf{u})$ and non-dimensional element volumes $\tilde{t}_i$, $i \in J(\mathbf{u})$, such that

$$V \sum_{i \in J(\mathbf{u})} \tilde{t}_i \mathbf{A}_i \mathbf{u} = \mathbf{p} + V \sum_{i \in J(\mathbf{u})} \tilde{t}_i \mathbf{g}_i ; \quad \sum_{i \in J(\mathbf{u})} \tilde{t}_i = 1 . \tag{19}$$

The optimality conditions (19) state that a convex combination of the gradients of the quadratic functions $V(1/2\mathbf{u}^T\mathbf{A}_i\mathbf{u} - g_i^T\mathbf{u})$, $i \in J(\mathbf{u})$, equals the load vector $\mathbf{p}$.

It can be shown (see below) that there does indeed exist a pair $(\mathbf{u}, \mathbf{t})$ which is a solution to the reduced optimality conditions (19). This implies that there exists an optimal truss that has bars with constant mutual energies and the set $J(\mathbf{u})$ is the set of these active bars. Note that a pair $(\mathbf{u}, \mathbf{t})$ satisfying the necessary conditions (19) for problem (10) is automatically a minimizer for the non-convex minimum compliance problem. This can be shown by copying the proof of Taylor (1969), who treated the case with a uniform, positive lower bound on the areas. For any design s_i, $i = 1, \ldots, m$, satisfying the volume constraint and with corresponding displacement field v we have that

$$\left(\mathbf{p} \sum_{i=1}^{m} t_i \mathbf{g}_i\right)^T \mathbf{u} = 2\left(\mathbf{p} + \sum_{i=1}^{m} t_i \mathbf{g}_i\right)^T \mathbf{u} - \sum_{i=1}^{m} t_i \mathbf{u}^T \mathbf{A}_i \mathbf{u} =$$

$$2\mathbf{p}^T\mathbf{u} - \sum_{i=1}^{m} t_i \mathbf{u}^T(\mathbf{A}_i \mathbf{u} - 2\mathbf{g}_i) = 2\mathbf{p}^T\mathbf{u} - \sum_{i=1}^{m} t_i \lambda(\mathbf{u}) =$$

$$2\mathbf{p}^T\mathbf{u} - V\lambda(\mathbf{u}) = 2\mathbf{p}^T\mathbf{u} - \sum_{i=1}^{m} s_i \lambda(\mathbf{u}) \leq$$

$$2\mathbf{p}^T\mathbf{u} - \sum_{i=1}^{m} s_i \mathbf{u}^T(\mathbf{A}_i \mathbf{u} - 2\mathbf{g}_i) \leq$$

$$2 \max_{\mathbf{w}} \left[\left(p + \sum_{i=1}^{m} s_i \mathbf{g}_i\right)^T \mathbf{w} - \frac{1}{2}\sum_{i=1}^{m} s_i \mathbf{w}^T \mathbf{A}_i \mathbf{w}\right] =$$

$$2\left(\mathbf{p} + \sum_{i=1}^{m} s_i \mathbf{g}_i\right)^T \mathbf{v} - \sum_{i=1}^{m} s_i \mathbf{v}^T \mathbf{A}_i \mathbf{v} = \left(\mathbf{p} + \sum_{i=1}^{m} s_i \mathbf{g}_i\right)^T \mathbf{v} , \tag{20}$$

where we have invoked the extremum principle for equilibrium. Finally note that the existence of solutions to the optimality condition (19) shows that there always exists an optimal solution with no more active bars than the degrees of freedom (dimension of u) plus 1; this follows from Caratheodory's theorem on convex combinations (see, e.g. Achtziger *et al.* 1992).

The optimality criterion (19) allows us to devise a very simple and effective method for solving the truss topology problem; the approach is well-known and well-documented in the literature under the label "optimality criterion method" (Olhoff and Taylor 1983; Rozvany 1989). The method is iterative and assigns material to members proportionally to the mutual energy of each member in order to reach the situation of constant mutual energy in the active bars

iteration step k:

t_i^{k-1} given,

compute displacement $\mathbf{u}_{k-1}$ from equilibrium eqs.

$$\zeta_i^k = \max\left(t_i^{k-1} \mathbf{u}_{k-1}^T \mathbf{A}_i \mathbf{u}_{k-1} , \, t_{\min}\right) ,$$

$$V^k = \sum_{i=1}^{m} \zeta_i^k , \quad t_i^k = \zeta_i^k (V/V^k) . \tag{21}$$

This method is an effective and general means for solving minimum compliance problems and has been used for topology as well as standard sizing problems (Bendsøe and Kikuchi 1988; Diaz and Bendsøe 1992; Olhoff and Taylor 1983; Rozvany 1989; Taylor and Rossow 1977; Zhou and Rozvany 1991). The optimality criteria algorithm can for the single loading case be viewed as a fully stressed design algorithm, and it can also be viewed as an implementation of a sequential quadratic programming technique for the topology design problem; this has been discussed in detail by Svanberg (1992a, b). The optimality criteria method involves assembly of the global stiffness matrix, as well as the solving the equilibrium problem at each iteration step, and this part of the algorithm is the most time-consuming. Note that the algorithm utilizes that the volume is linear in the design variables so that satisfying the volume constraint is just a rescaling of variables, but the algorithm does not take advantage of the fact that also the stiffness matrix is linear in the design variables. Also for the single load case truss topology problem (4) we have the coefficient matrices for the individual design variables in the form of dyadic products and this is also not taken into account.

It is the provision of a lower bound on the bar volumes that allows for the use of the very effective optimality criterion method. A similar efficiency can be obtained by considering the problem of taking the infimum of the compliances for all truss structures with positive bar volumes

$$\inf_{\mathbf{t}} \mathbf{p}^T \left[\sum_{i=1}^{m} t_i \mathbf{K}_i\right]^{-1} \mathbf{p}\,,$$

$$\text{subject to: } \sum_{i=1}^{m} t_i = V\,, \quad t_i > 0\,, \quad i = 1, \ldots, m\,. \tag{22}$$

As noted above this problem is convex; this follows from the fact that the compliance function, as a function of the design variables can be expressed as (see also Svanberg 1984)

$$\mathbf{p}^T \left[\sum_{i=1}^{m} t_i \mathbf{K}_i\right]^{-1} \mathbf{p} = \max_{\mathbf{v}} \left[2\mathbf{p}^T\mathbf{v} - \mathbf{v}^T \sum_{i=1}^{m} t_i \mathbf{K}_i \mathbf{v}\right]. \tag{23}$$

As the supremum (maximum) over a family of convex (linear in this case) functions is convex, the convexity of (22) follows. Also, it is readily seen that convexity extends to the problem with self-weight loads and to optimization problems where stiffness is a concave function of the design. The latter property is seldom seen in practice.

Problem (22) can be solved by, e.g. interior point barrier methods (Ben-Tal and Nemirovskii 1992, 1993; Ringertz 1989, 1992), so that the positivity constraint on the bar volumes is automatically satisfied. Problem (22) does not lend itself to the use of sparse techniques, as the inverse of the stiffness matrix and the Hessian matrix of the compliance are full matrices. Sparsity can, however, be utilized if the original problem

$$\inf_{\mathbf{u}, \mathbf{t}} \mathbf{p}^T\mathbf{u}\,, \quad \text{subject to: } \sum_{i=1}^{m} t_i \mathbf{K}_i \mathbf{u} = \mathbf{p}\,,$$

$$\sum_{i=1}^{m} t_i = V\,, \quad t_i > 0\,, \quad i = 1, \ldots, m\,, \tag{24}$$

in both the displacement and design variables is solved using interior point methods. Although the latter problem is not convex, finding a stationary solution provides also a stationary point for problem (22), and thus a minimizer for this convex problem (Ringertz 1992). This approach extends readily to all the problem types described above. The use of an interior barrier method for problems (24) or (22) involves the use of a suitable sequence of penalty parameters, which in effect corresponds to imposing a constraint of the type $t_i \geq t_{\min} > 0$, $i = 1, \ldots, m$ for a suitable small lower bound value $t_{\min}$.

4 Formulations in displacements only

We will now use the max-min formulation (13) of the truss topology design problem to derive a globally optimal strain energy functional that describes the energy of the optimal truss. This leads to an alternative, equivalent convex, but non-smooth formulation of the problem, for which a computationally effective steepest descent algorithm can be devised.

We recall that problem (13) has the form

$$\max_{\substack{\mathbf{t} \geq \mathbf{0} \\ \sum_{i \in R} t_i = V}} \min_{\mathbf{u}} \left[\frac{1}{2}\mathbf{u}^T \left(\sum_{i \in R} t_i \mathbf{A}_i + \sum_{i \in S} s_i \mathbf{A}_i\right) \mathbf{u} - \left(\mathbf{p} + \sum_{i \in R} t_i \mathbf{g}_i + \sum_{i \in S} s_i \mathbf{g}_i\right)^T \mathbf{u}\right], \tag{13}$$

and this problem is linear in the design variable and convex in the displacement variable. Thus the problem is concave-convex (with a compact $\mathbf{t}$-constraint set) and we can interchange the max and min operators, to obtain

$$\min_{\mathbf{u}} \max_{\substack{\mathbf{t} \geq \mathbf{0} \\ \sum_{i \in R} t_i = V}} \left[\frac{1}{2}\mathbf{u}^T \left(\sum_{i \in R} t_i \mathbf{A}_i + \sum_{i \in S} s_i \mathbf{A}_i\right) \mathbf{u} - \left(\mathbf{p} + \sum_{i \in R} t_i \mathbf{g}_i + \sum_{i \in S} s_i \mathbf{g}_i\right)^T \mathbf{u}\right]. \tag{25}$$

The inner problem is now a linear programming problem in the $\mathbf{t}$ variable. As

$$\sum_{i \in R} t_i \left(\mathbf{u}^T\mathbf{A}_i\mathbf{u} - 2\mathbf{g}_i^T\mathbf{u}\right) \leq \sum_{i \in R} t_i \max_{i \in R} \left(\mathbf{u}^T\mathbf{A}_i\mathbf{u} - 2\mathbf{g}_i^T\mathbf{u}\right) = V \max_{i \in R} \left(\mathbf{u}^T\mathbf{A}_i\mathbf{u} - 2\mathbf{g}_i^T\mathbf{u}\right), \tag{26}$$

when $\mathbf{t} \geq \mathbf{0}$, $\sum_{i \in R} t_i = V$, and as the equality holds for if all material is assigned to a bar with maximum specific energy $\mathbf{u}^T\mathbf{A}_i\mathbf{u} - 2\mathbf{g}_i^T\mathbf{u}$, we see that the problem (13) can be reduced to (Ben-Tal and Bendsøe 1992)

$$\min_{\mathbf{u}} \max_{i \in R} \left[\frac{V}{2}\left(\mathbf{u}^T\mathbf{A}_i\mathbf{u} - 2\mathbf{g}_i^T\mathbf{u}\right) - \left(\mathbf{p} + \sum_{i \in S} s_i \mathbf{g}_i - \sum_{i \in S} \frac{1}{2} s_i \mathbf{A}_i \mathbf{u}\right)^T \mathbf{u}\right]. \tag{27}$$

This is an unconstrained, *convex and nonsmooth* problem in the displacement variable $\mathbf{u}$ only, with optimal value minus one half of the optimal value for the problem (12). For completeness let us state the equivalent problems for the specific cases discussed in Section 2. For the single load case we have

$$\min_{\mathbf{u}} \left[\max_{i=1,\ldots,m} \left(\frac{V}{2}\mathbf{u}^T\mathbf{K}_i\mathbf{u} - \mathbf{p}^T\mathbf{u}\right)\right], \tag{28}$$

for the truss case, and for the general problem (8) becomes

$$\min_{\mathbf{u}} \left[\max_{i=1,\ldots,m} \left(\frac{V}{2}\mathbf{u}^T\mathbf{A}_i\mathbf{u} - \mathbf{p}^T\mathbf{u}\right)\right]. \tag{29}$$

For the multiple load case this can also be written as

$$\min_{\mathbf{u}^k} \left\{\max_{i=1,\ldots,m} \left[\sum_{k=1}^{M} W^k \left(\frac{V}{2}{\mathbf{u}^k}^T\mathbf{K}_i\mathbf{u}^k - {\mathbf{p}^k}^T\mathbf{u}^k\right)\right]\right\}. \tag{30}$$

For the reinforcement problem, without self-weight the equivalent statement is

$$\min_{\mathbf{u}}\left\{\max_{i\in R}\left[\frac{V}{2}\mathbf{u}^T\mathbf{A}_i\mathbf{u}+\left(\sum_{i\in S}\frac{1}{2}s_i\mathbf{u}^T\mathbf{A}_i-\mathbf{p}^T\right)\mathbf{u}\right]\right\}, \quad (31)$$

and for the full topology problem with self-weight the statement becomes

$$\min_{\mathbf{u}}\left\{\max_{i=1,\dots,m}\left[\frac{V}{2}\left(\mathbf{u}^T\mathbf{A}_i\mathbf{u}-2\mathbf{g}_i^T\mathbf{u}\right)-\mathbf{p}^T\mathbf{u}\right]\right\}. \quad (32)$$

One can think of the resulting displacements only problems (27)–(32) shown above as equilibrium problems for a structure with a non-smooth, convex strain energy. This strain energy is the strain energy for a self-optimized structure which automatically adjusts its topology and sizing so as to minimize compliance for the applied load(s). This feature is also prominent in studies of the simultaneous design of material and structure (Bendsøe *et al.* 1992), as well as in the homogenization approach to optimum topology design of continuum structures, in which case the optimal structure will be a continuum with a fine microstructure, see, for example, Allaire and Kohn (1993).

It is possible to show existence of solutions to the problems (27)–(32) and to prove the equivalence between problem statements of the form (4), (7)–(10), (12), (13) and (27)–(32) (Ben-Tal and Bendsøe 1992). There is no uniqueness in the solutions and it is quite well-known that there are normally "many" solutions (subspaces). The equivalence of the problems is understood in the sense that for a solution $\mathbf{u}$ to, for example, problem (29) and the corresponding set $J(\mathbf{u})$ of active bars, there exists a corresponding set of bar volumes $\mathbf{t}$ satisfying the optimality condition

$$\sum_{i\in J(\mathbf{u})} t_i\mathbf{A}_i\mathbf{u}=\mathbf{p}; \qquad \sum_{i\in J(\mathbf{u})} t_i=V,$$
$$t_i=0,\ i\notin J(\mathbf{u}); \quad t_i\ge 0,\ i=1,\dots,m, \quad (33)$$

and these optimality conditions are precisely the optimality conditions for the min-max problem (29).

It turns out that it is advantageous to consider algorithms for the equivalent problems (27)–(32) and we will here describe an "ε-steepest descent" method for the non-smooth problems (Demyanov and Malozemov 1974; Ben-Tal and Bendsøe 1992). The algorithm is not the most effective that can be devised (see later sections), but it is physically intuitive, and it is closely related to optimality criteria type algorithms found in the literature. Also, it is an algorithm in the displacement variables only, as the design variables have been removed through a duality argument. Note that the standard procedure in design problems is to solve for the design variables, with the displacements removed via the state equation and adjoint equation. The algorithm generates the solution $\mathbf{u}$ as well as the bar volumes $\mathbf{t}$. For generality we describe the algorithm for the general reinforcement problem with external loads as well as loads due to self-weight. The algorithm consists of the following very intuitive steps.

Algorithm (35) for the problem (27):

$$\min_{\mathbf{u}} F(\mathbf{u}), \quad \text{with } F(\mathbf{u})=\frac{V}{2}\max_{i\in R}\left(\mathbf{u}^T\mathbf{A}_i\mathbf{u}-2\mathbf{g}_i^T\mathbf{u}\right)+$$
$$\frac{1}{2}\sum_{i\in S}s_i\left(\mathbf{u}^T\mathbf{A}_i\mathbf{u}-2\mathbf{g}_i^T\mathbf{u}\right)-\mathbf{p}^T\mathbf{u}. \quad (34)$$

0. Compute an initial guess of displacement field $\mathbf{u}$, for example by solving the equilibrium equations for a feasible set of bar volumes $\mathbf{t}$.

1. For present $\mathbf{u}$, compute $\lambda(\mathbf{u})=\max_{i\in R}\left(\mathbf{u}^T\mathbf{A}_i\mathbf{u}-2\mathbf{g}_i^T\mathbf{u}\right)$, indices

$$J(\mathbf{u})=\left(i\in R\,\middle|\,\mathbf{u}^T\mathbf{A}_i\mathbf{u}-2\mathbf{g}_i^T\mathbf{u}\ge\lambda(\mathbf{u})-\varepsilon\right),$$
$$J_U(\mathbf{u})=\left(i\in R\,\middle|\,\mathbf{u}^T\mathbf{A}_i\mathbf{u}-2\mathbf{g}_i^T\mathbf{u}\ge\lambda(\mathbf{u})-C\varepsilon\right),$$
$$C\approx 10, \quad (35)$$

and the displacement dependent load $\mathbf{f}=\mathbf{p}+\sum_{i\in S}s_i\mathbf{g}_i-\sum_{i\in S}s_i\mathbf{A}_i\mathbf{u}$.

2. Compute descent direction $\mathbf{d}$ as

$$\mathbf{d}=-\left[\sum_{i\in J}t_i\left(\mathbf{A}_i\mathbf{u}-\mathbf{g}_i\right)-\mathbf{f}\right], \quad (36)$$

where t_i, $i\in J$ are found from

$$\min_{t\in J}\left[\left\|\sum_{i\in J}t_i\left(\mathbf{A}_i\mathbf{u}-\mathbf{g}_i\right)-\mathbf{f}\right\|^2-\sum_{i\in J}t_i\mathbf{u}^T\left(\mathbf{A}_i\mathbf{u}-2\mathbf{g}_i\right)\right],$$

$$\text{subject to: } \sum_{i\in J}t_i=V; \quad t_i\ge 0, \quad i\in J. \quad (37)$$

3. If $|\mathbf{d}|\le\delta$, stop. Else go to 4.

4. Compute a step size α^* for update, $\mathbf{u}:=\mathbf{u}+\alpha\mathbf{d}$, by a line search with the function

$$\Psi(\alpha)=F(\mathbf{u}+\alpha\mathbf{d})=\max_{i\in R}\Psi_i(\alpha), \quad \Psi_i(\alpha)=a_i\alpha^2+b_i\alpha+c_i,$$

$$a_i=\frac{V}{2}\mathbf{d}^T\mathbf{A}_i\mathbf{d}+\frac{1}{2}\sum_{i\in S}s_i\mathbf{d}^T\mathbf{A}_i\mathbf{d},\ b_i=\left[V\left(\mathbf{A}_i\mathbf{u}-\mathbf{g}_i\right)-\mathbf{f}\right]^T\mathbf{d},$$

$$c_i=\left[\frac{V}{2}\left(\mathbf{A}_i\mathbf{u}-2\mathbf{g}_i\right)-\left(\mathbf{f}+\frac{1}{2}\sum_{i\in S}s_i\mathbf{A}_i\mathbf{u}\right)\right]^T\mathbf{u}. \quad (38)$$

5. Update, $\mathbf{u}:=\mathbf{u}+\alpha^*\mathbf{d}$, and go to 1.

Here, ε is a relaxation on the activity set J which is crucial to guarantee the convergence of the algorithm, and δ determines the accuracy of the solution. Each iteration loop of the algorithm consists of first finding the set of almost active bars (Step 1). The descent direction (Step 2) is then found by first finding the bar volumes of these bars, which minimizes the error in equilibrium for the given estimate of displacement. This is a quadratic programming problem. The error is measured in a least squares sense and the descent direction is given as the vectorial error with this best fit of bar volumes. For ε small enough, the set of almost active bars equals the set of actually active bars, so it is natural to work with a decreasing sequence of the relaxation parameter ε, as well as with a decreasing sequence of equilibrium errors δ. The line search for the non-smooth function $\Psi(\alpha)$ (Step 4) is most conveniently carried out using a Golden Section method, using the set J_U of almost active bars as the basis for the search. The full search is only invoked if the update with this reduced set of bars does not result in an improvement of the functional.

Note that the algorithm above lends itself to an implementation that takes the fullest advantage of sparsity both in storage and computations. An efficient storage strategy is to store the bar-connectivity matrix (m by 2 matrix of integers) and the bar cosines (m by 2 or 3 matrix of reals) for all information on the matrices $\mathbf{A}_i$, $i=1,\dots,m$, and to base computations of the data such as $\mathbf{A}_i\mathbf{u}$, $\mathbf{u}^T\mathbf{A}_i\mathbf{u}$ on this information. Finally note that the least squares sub-problem of Step 2 are sparse problems in the t_i variables. For a proof of the convergence of the algorithm, we refer to Ben-Tal and Bendsøe (1993), where also the case of constrained bar areas is treated in full detail.

The algorithm above is conceptually similar to the algorithm given by Taylor and Rossow (1977) for the single load case, the difference being in the update scheme, which here is based on the formal identification of the equivalence between problems (8) and (29).

5 The truss topology problem as a linear programming problem

In the preceding section we saw how the minimum compliance truss topology problem can be reformulated as a nonsmooth, convex problem in the displacements only, and we will here use this equivalent problem formulation as the basis for generating other equivalent problem statements.

The truss problem (27) is, by introducing a bound formulation, equivalent to the convex problem (Achtziger *et al.* 1992),

$$\min_{\mathbf{u},\mu}\left(\sum_{i\in S}\frac{1}{2}s_i\mathbf{u}^T\mathbf{A}_i\mathbf{u}-\mathbf{p}^T\mathbf{u}-\sum_{i\in S}s_i\mathbf{g}_i^T\mathbf{u}+\mu^2\right),$$

$$\text{subject to}: \ \frac{V}{2}\left(\mathbf{u}^T\mathbf{A}_i\mathbf{u}-2\mathbf{g}_i^T\mathbf{u}\right)\le\mu^2\,,\quad i\in R\,, \tag{39}$$

which is a smooth, quadratic, positive semi-definite optimization problem with a large number of constraints. This problem lends itself to numerical treatment by invoking a *sparse SQP* method specially suited for problems with many constraints.

For the simpler case of the pure topology problem (no structure to reinforce), the problem becomes, up to a rescaling,

$$\min_{\mathbf{u}}\left(-\mathbf{p}^T\mathbf{u}\right),$$

$$\text{subject to}: \ \frac{V}{2}\left(\mathbf{u}^T\mathbf{A}_i\mathbf{u}-2\mathbf{g}_i^T\mathbf{u}\right)\le 1\,,\quad i=1,\dots,m\,. \tag{40}$$

Finally, if also selfweight is absent, the problem statement reduces further to

$$\min_{\mathbf{u}}\left(-\mathbf{p}^T\mathbf{u}\right),$$

$$\text{subject to}: \ \frac{V}{2}\mathbf{u}^T\mathbf{A}_i\mathbf{u}\le 1\,,\quad i=1,\dots,m\,, \tag{41}$$

i.e. a maximization of compliance, with constraints on the specific strain energies (mutual energies).

For the single load truss problem the element stiffness matrices are dyadic products and we obtain for the mutual energies

$$\mathbf{u}^T\mathbf{K}_i\mathbf{u}=\left(\frac{\sqrt{E_i}}{\ell_i}\mathbf{b}_i^T\mathbf{u}\right)^2. \tag{42}$$

This special form of the element mutual energies implies that (41) can be written in LP-form as (Achtziger *et al.* 1992)

$$\min_{\mathbf{u}}\left(-\mathbf{p}^T\mathbf{u}\right),$$

$$\text{subject to}: \ -1\le\sqrt{\frac{VE_i}{2}}\frac{\mathbf{b}_i^T\mathbf{u}}{\ell_i}\le 1\,,\quad i=1,\dots,m\,. \tag{43}$$

For suitable stress constraint values σ_i, problem (43) is the dual of the traditional force formulation for single load, plastic design (cf. Dorn *et al.* 1964; Fleron 1964; Hemp 1973; Kirsch 1989, 1992; Pedersen 1970, 1972, 1973; Topping 1992)

$$\min_{\mathbf{q}^+,\mathbf{q}^-}\sum_{i=1}^{m}\frac{\ell_i}{\sigma_i}\left(q_i^++q_i^-\right),$$

$$\text{subject to}: \ \mathbf{B}^T\left(\mathbf{q}^+-\mathbf{q}^-\right)=\mathbf{p}\,,\quad q_i^+\ge 0\,,\quad q_i^-\ge 0\,,$$

$$i=1,\dots,m\,. \tag{44}$$

Here q_i^+, q_i^- are the multipliers for the inequality constraints of (43). With a change of variables

$$t_i=\frac{\ell_i}{\sigma_i}\left(q_i^++q_i^-\right),\quad \mathbf{q}=\left(\mathbf{q}^+-\mathbf{q}^-\right), \tag{45}$$

we obtain the minimum weight plastic design formulation

$$\min_{\mathbf{q},\mathbf{t}}\sum_{i=1}^{m}t_i\,,$$

$$\text{subject to}: \ \mathbf{B}^T\mathbf{q}=\mathbf{p}\,,\quad -\sigma_it_i\le\ell_iq_i\le\sigma_it_i\,,$$

$$i=1,\dots,m\,,\quad t_i\ge 0\,,\quad i=1,\dots,m\,. \tag{46}$$

The member force formulations (44) and (45) are the traditional formulations for single load truss topology optimization. These are, of course, very efficient formulations and could be solved using *sparse*, primal-dual LP-methods. The force methods are at first glance problems in plastic design, because kinematic compatibility is ignored, and their use in elastic design is commonly justified by the possibility of finding statically determinate solutions. Note that the developments described above show that the minimum compliance design problem for a single load case is equivalent to a minimum weight plastic design formulation, in the sense that for a solution $\mathbf{t}$, $\mathbf{q}$ to the minimum weight plastic design problem with data V, σ_i there corresponds a solution $\mathbf{t}_C$, $\mathbf{x}_C$ to the minimum compliance problem with data V_C, E_i. The precise relations are (cf. Achtziger *et al.* 1992)

$$\sigma_i=\sqrt{E_i}\,,\quad \mathbf{t}_C=\frac{V_C}{V}\mathbf{t}\,,\quad \mathbf{x}_C=\frac{V_C}{V}\mathbf{x}\,, \tag{47}$$

where $\mathbf{x}$ is the dual variable of the minimum weight plastic design problem corresponding to the static equilibrium constraint $\mathbf{B}^T\mathbf{q}=\mathbf{p}$.

Note that in problem (46) the stress constraints are written in terms of *member forces*, in order to give a consistent formulation. For some truss problems, the stress in a number of members will converge to a finite non-zero level as the member areas converge to zero, but the member forces will converge to zero (Cheng and Jiang 1992; Kirsch 1992). This fact should be observed for any truss design problem involving stress constraints.

The equivalence between the force methods and the minimum compliance problem for the *single load case* shows that *any* solution to the force LP-formulation leads to a minimum compliance topology design, within the frame-work of elastic designs. Such designs are uniformly stressed designs, as well as having a constant specific energy in all active bars. The existence of basic solutions to the linear programming problem (44) implies that there exist minimum weight truss topologies with a number of bars not exceeding the degrees of freedom. If there exists such a basic solution with only non-zero forces (areas), this is a statically determinate truss. Otherwise, the truss will have a unique force field for the given load but will be kinematically indeterminate; this may be the case even after nodes with no connected bars are removed [see also Kirsch (1992), for a discussion on this].

The equivalence between problems (44), (46) and (43) can also be found in the paper by Dorn *et al.* (1964), and the equivalence between problems (4) and (44) and (46) was indicated by Hemp (1973) and others. In the paper by Dorn *et al.* (1964), one can also find a lengthy discussion on how the force formulations are convenient for studying an eventual static determinacy of the solutions.

The linear programming formulations above hold *only* for the case of pure topology truss design with unconstrained design variables, a single loading case and excluding selfweight. Thus, the natural extension of the plastic design situation to an LP problem which caters for multiple loads and selfweight loads does not seem to have a natural equivalent statement in terms of displacements and compliances. Also, it is well-known that in this case it is most common that statically indeterminate solutions result, thus imposing a requirement for further redesign if kinematic compatibility is required, as for elastic design (Kirsch 1989, 1992; Topping 1992).

For the sake of completeness of presentation, note that in the reinforcement case without selfweight, the single load case problem can be reduced to a quadratic optimization problem with linear constraints

$$\min_{\mathbf{u},\mu} \left[\frac{1}{2}\mathbf{u}^T \left(\sum_{i\in S} s_i \mathbf{K}_i \right) \mathbf{u} - \mathbf{p}^T\mathbf{u} + \mu^2 \right] ,$$

$$\text{subject to: } \mu \le \sqrt{\frac{VE_i}{2\ell_i}} \mathbf{b}_i^T \mathbf{u} \le \mu , \quad i \in R . \tag{48}$$

Notice here that the matrix $\sum_{i\in S} s_i \mathbf{K}_i$ is positive semi-definite, but usually not positive definite. The problem statement (48) also represents a simplification of the minimum compliance problem for a single load case (no selfweight) and with lower bounds on the variables; the vector $\mathbf{s}$ represents the vector of lower bounds on the design variables.

6 The min-max multiple load truss topology problem

In this section we shall present a number of displacement based equivalent formulations for the worst case multiple load topology problem. This implies a min-max formulation of the minimum compliance problem. We shall use an infimum type formulation, using that the strain energy as a function of design is convex, thus suggesting the use of interior point algorithms, as indicated in the introductory section on truss design formulations. For trusses, the dyadic structure of the stiffness matrix again implies that these problem statements in the displacements can be given a dual representation in terms of member forces, which, however, for more than one load case is not the simple linear programming formulation for a multiple load plastic design.

In the following we will refrain from covering the problem of reinforcement, mainly to simplify notation. Also, the selfweight problem will play a minor role in the following. However, we begin with a general treatment that covers truss, variable thickness sheet and sandwich plate design.

The problem of worst case minimum compliance design for multiple loadings $\mathbf{p}^k$, $k = 1, \ldots, M$, reads

$$\min_{\mathbf{u}^k, \mathbf{t}} \max_{k=1,\ldots,M} \mathbf{p}^{k^T} \mathbf{u}^k ,$$

$$\text{subject to: } \sum_{i=1}^{m} t_i \mathbf{A}_i \mathbf{u}^k = \mathbf{p}^k , \quad k = 1, \ldots, M ,$$

$$\sum_{i=1}^{m} t_i = V , \quad t_i \ge 0 , \quad i = 1, \ldots, m , \tag{49}$$

where $\mathbf{u}^k$, $k = 1, \ldots, M$, are the displacements corresponding to the different loading cases. The discrete optimization over the compliance values can be converted into a smooth maximization by introducing a convex combination of weighting parameters λ^k, $k = 1, \ldots, M$, so that the problem becomes

$$\min_{\mathbf{u}^k, \mathbf{t}} \max_{\lambda^k} \sum_{k=1}^{M} \lambda^k \mathbf{p}^{k^T} \mathbf{u}^k ,$$

$$\text{subject to: } \sum_{i=1}^{m} t_i \mathbf{A}_i \mathbf{u}^k = \mathbf{p}^k , \quad k = 1, \ldots, M ,$$

$$\sum_{i=1}^{m} \lambda^k = 1 ; \quad \lambda^k \ge 0 , \quad k = 1, \ldots, M ,$$

$$\sum_{i=1}^{m} t_i = V , \quad t_i \ge 0 , \quad i = 1, \ldots, m , \tag{50}$$

which is very similar in form to the weighted average formulation. As for the standard single load problem and by similar means (Achtziger 1992, 1993), it is possible to generate a displacements only formulation in the form

$$\min_{\mathbf{u}^k, \lambda^k} \left\{ \max_{i=1,\ldots,m} \left[\sum_{k=1}^{M} \lambda^k \left(\frac{V}{2} \mathbf{u}^{k^T} \mathbf{A}_i \mathbf{u}^k - \mathbf{p}^{k^T} \mathbf{u}^k \right) \right] \right\} ,$$

$$\text{subject to: } \sum_{k=1}^{M} \lambda^k = 1 ; \quad \lambda^k \ge 0 , \quad k = 1, \ldots, M . \tag{51}$$

Solutions to this problem can be proved to exist by considering an equivalent smooth formulation in the form (Achtziger 1992, 1993)

$$\max_{\mathbf{u}^k} \sum_{k=1}^{M} \left(\mathbf{p}^{k^T} \mathbf{u}^k \right)^2 ,$$

$$\text{subject to: } \sum_{k=1}^{M} \mathbf{u}^{k^T} \mathbf{A}_i \mathbf{u}^k \le 1 , \quad i = 1, \ldots, m . \tag{52}$$

The optimal value of problem (51) equals minus one half the extremal value of problem (49), but the direct equivalence between the two problems (in the sense discussed earlier for the single load problem) may fail if a multiplier λ^k equals zero in the optimal solution to problem (51). If this is the case, we cannot guarantee equilibrium for this loading condition. However, a set of bar areas can be identified by considering the loadings with non-zero multipliers, and a minimum compliance truss will be generated for these loadings. This makes it natural to consider a slightly perturbed version of (49) and (51), where the multipliers are constrained as $\lambda^k \geq \varepsilon > 0$, $k = 1, \ldots, M$. For the resulting perturbed version of problem (51) we can write

$$\min_{\substack{\lambda^k \geq \varepsilon \\ \sum_{k=1}^M \lambda^k = 1}} \left\{ \min_{\mathbf{u}^k} \left\{ \max_{i=1,\ldots,m} \left[\sum_{k=1}^M \lambda^k \left(\frac{V}{2} \mathbf{u}^{k^T} \mathbf{A}_i \mathbf{u}^k - \mathbf{p}^{k^T} \mathbf{u}^k \right) \right] \right\} \right\}, \tag{53}$$

indicating that the inner problem in the displacements could be solved using the methods described in the preceding sections, with the outer problem solved using algorithms for convex non-differentiable optimization problems (Achtziger 1992, 1993).

Let us now in light of the perturbation introduced above consider the topology optimization problem as a limes inferior problem for a series of optimal design problems with decreasing positive lower bounds on the design variables. Rewriting (49) and imposing positive element volumes, we can remove the displacement variables by solving for the now unique displacements and write

$$\inf_{\mathbf{t}} \max_{k=1,\ldots,M} \mathbf{p}^{k^T} \left(\sum_{i=1}^m t_i \mathbf{A}_i \right)^{-1} \mathbf{p}^k ,$$

$$\text{subject to}: \sum_{i=1}^m t_i = V; \quad t_i > 0, \quad i = 1, \ldots, m. \tag{54}$$

Note that we have exchanged the min-operator with the inf.-operator as well as changing the constraint $t_i \geq 0$ to $t_i > 0$. This problem is a convex problem as was also seen for the single load problem, cf. Section. 3.

Problem (54) lends itself to the application of interior point algorithms which automatically will enforce the constraints $t_i > 0$, as described by Ben-Tal and Nemirovskii (1992, 1993). With a bound formulation of (54) with bounding variable α, a possible logarithmic barrier function for the problem is of the form

$$\min_{\mathbf{t},\alpha} \left\{ - \sum_{k=1}^M \ln \left[\alpha - \mathbf{p}^{k^T} \left(\sum_{i=1}^m t_i \mathbf{A}_i \right)^{-1} \mathbf{p}^k \right] - \sum_{i=1}^m \ln(t_i) - \ln(\alpha_{\max} - \alpha) \right\}, \tag{55}$$

where $\alpha_{\max}$ is a suitable guaranteed upper bound on the optimal value of problem (54). Further details on the use of such interior point methods can be found in the papers by Ben-Tal and Nemirovskii (1992, 1993). Note that for efficiency, sparse matrix techniques should be employed. It is straightforward to use sparse matrix techniques if the equilibrium conditions are maintained as equality constraints, so that the Hessian for the problem remains sparse (see also Ringertz 1988).

Now returning to problem (53), we note that by a change of variables of $\mathbf{u}^k$ to $\frac{1}{\lambda^k}\mathbf{u}^k$, this problem can be stated as

$$\inf_{\substack{\lambda^k > 0 \\ \sum_{k=1}^M \lambda^k = 1}} \left\{ \min_{\mathbf{u}^k} \cdot \left\{ \max_{i=1,\ldots,m} \left[\sum_{k=1}^M \left(\frac{V}{2} \frac{1}{\lambda^k} \mathbf{u}^{k^T} \mathbf{A}_i \mathbf{u}^k - \mathbf{p}^{k^T} \mathbf{u}^k \right) \right] \right\} \right\}, \tag{56}$$

which is now jointly convex on the feasible set in both the multipliers and the displacements. Here we have again used the inf-operator to indicate the use of a decreasing sequence of lower bounds on the multipliers λ^k, as well as to indicate the natural choice of interior penalty methods for the numerical solving of this problem. It has turned out that the use of a so-called "Penalty/Barrier/Multiplier Method" is a very effective means for the solving of the min-max topology problem with linear cost and stiffness (Ben-Tal and Nemirovskii 1992, 1993).

We describe briefly the "Penalty/Barrier/Multiplier (PBM) Method" for a general non-linear program (see Zibulevsky and Ben-Tal 1993),

$$\min_{\mathbf{x}} \left[f_0(\mathbf{x}) \,|\, f_i(\mathbf{x}) \leq 0, \quad i \in I \right]. \tag{57}$$

Consider now the strictly increasing and strictly convex, smooth function

$$\varphi_\rho(t) = \begin{cases} \frac{1}{8\rho} t^2 + t & \text{if } t \geq -2\rho \\ -\rho \left[\log\left(\frac{t}{-2\rho} \right) + \frac{3}{2} \right] & \text{if } t < -2\rho \end{cases}, \tag{58}$$

composed of a logarithmic branch and a quadratic branch. Since $\varphi_\rho(t) \leq 0$ if and only if $t \leq 0$, it follows that the problem (57) is equivalent to the problem,

$$\min_{\mathbf{x}} \left\{ f_0(\mathbf{x}) \,|\, \varphi_\rho \left[f_i(\mathbf{x}) \right] \leq 0, \quad i \in I \right\}. \tag{59}$$

The Lagrangian corresponding to problem (59) is

$$F_\rho(\mathbf{x}, \boldsymbol{\mu}) = f_0(\mathbf{x}) + \sum_{i \in I} \mu_i \varphi_\rho \left[f_i(\mathbf{x}) \right], \tag{60}$$

and the PBM method consists in minimizing this combined penalty, barrier and multiplier function. At the j-th iteration step of the PBM method the penalty parameter $\rho_j > 0$ and the current estimate of the Lagrange multipliers $\left(\mu_i^j : i \in I \right)$ are given. The update of the variables $\mathbf{x}^j$ are computed by a Newton method for the minimization of (60), i.e.

$$x^{j+1} = \arg\min_{\mathbf{x}} f_{\rho^j}(\mathbf{x}, \boldsymbol{\mu}^j). \tag{61}$$

The multipliers are then updated by the rule $\boldsymbol{\mu}^{j+1} = \boldsymbol{\mu}^j \frac{d\varphi_{\rho^j}}{dt} \left[f_i(\mathbf{x}^{j+1}) \right]$, $i \in I$ and the penalty parameter by the update formula $\rho_{j+1} = \alpha \rho_j$, with a parameter α, $0 < \alpha < 1$. For details on motivation, convergence properties and implementation of the PBM method we refer to Ben-Tal *et al.* (1992), and Zibulevsky and Ben-Tal (1993).

In order to apply the PBM method to the min-max multiple load truss topology design problem, (56) is used in a form where the discrete maximization over bar numbers is removed by a bound formulation

$$\inf_{\substack{\lambda^k > 0, \mathbf{u}^k, \tau \\ \sum_{k=1}^M \lambda^k = 1}} \left(V\tau - \sum_{k=1}^{M} \mathbf{p}^{k^T} \mathbf{u}^k \right) ,$$

$$\text{subject to:} \ \sum_{k=1}^{M} \frac{1}{2\lambda^k} \mathbf{u}^{k^T} \mathbf{A}_i \mathbf{u}^k - \tau \le 0 \, , \ i = 1, \dots, m \, . \quad (62)$$

Note that (62) is a smooth convex optimization problem. It can be shown from the Karush-Kuhn-Tucker conditions of problem (56), that the Lagrange multipliers for the constraints on the mutual energies are precisely the optimal volumes of the bars in the optimal topology. Hence the optimal bar volumes are approximated at each iteration step of the PBM method by the multipliers $\left(\mu_i^j : i = 1, \dots, m\right)$.

We shall now show that by deriving the dual formulations of (56) one can for the truss case generate what amounts to stress based min-max minimum compliance formulations. The basis for this derivation is again, as in the earlier development, the dyadic structure of the individual member stiffness matrices. Expressing the maximization over the bar numbers (the inner problem) with a bounding variable and using auxiliary variables $c_i^k = \mathbf{b}_i^T \mathbf{u}^k$ (member elongations), where $\mathbf{b}_i^T$ is the i-th row of the matrix $\mathbf{B}$, the equivalent convex dual problem can be derived to have the form

$$\inf_{\mathbf{t}} \min_{\mathbf{q}^k} \left[\max_{k=1,\dots,M} \left(\frac{1}{2} \sum_{i=1}^{m} \frac{\ell_i^2}{E_i} \frac{(q_i^k)^2}{t_i} \right) \right] ,$$

$$\text{subject to:} \ \mathbf{B}^T \mathbf{q}^T = \mathbf{p}^k \, , \quad k = 1, \dots, M \, ;$$

$$\sum_{i=1}^{m} t_i = V \, , \quad t_i > 0 \, , \quad i = 1, \dots, m \, . \quad (63)$$

With $\overline{\lambda}^k$, $\overline{\mathbf{u}}^k$ denoting the Lagrange multipliers for a bound constraint formulation of the maximization over k and the equilibrium constraint, respectively, we can for an optimum q^k, t of (63) with $\overline{\lambda}^k > 0$, $k = 1, \dots, M$, identify $\mathbf{u}_k = \overline{\mathbf{u}}^k / \overline{\lambda}^k$, t as a solution to our original problem statement (49) in displacements and bar areas. Also, we can show, from the Karush-Kuhn-Tucker optimality conditions that $q_i^k = \frac{E_i}{\ell_i^2} t_i \mathbf{b}_i^T \mathbf{u}^k$, i.e. compatibility of stresses and displacements is automatically assured.

The problem (63) is the minimum compliance problem formulated in terms of the complementary energy, written for the worst-case multiple load situation. This type of formulation has been used to great advantage for studies of minimum compliance problems of continuum structures using a homogenization modelling (Allaire and Kohn 1993) but is usually not seen employed for truss design. For the single loading case we have the convex problem formulation

$$\inf_{\mathbf{t}} \min_{\mathbf{q}} \frac{1}{2} \sum_{i=1}^{m} \frac{\ell_i^2}{E_i} \frac{(q_i)^2}{t_i} ,$$

$$\text{subject to:} \ \mathbf{B}^T \mathbf{q} = \mathbf{p} \, ; \quad \sum_{i=1}^{m} t_i = V \, , \quad t_i > 0 \, ,$$

$$i = 1, \dots, m \, . \quad (64)$$

Finally, we will consider the elimination of the bar volumes from the problem (63), by directly solving for these variables. This corresponds to the elimination of bar volumes in the displacements (strain) based formulation as carried out in Section 4. Expressing the maximization over loading cases by a maximization over a convex combination of weighting factors

$$\min_{\mathbf{q}^k} \inf_{\mathbf{t}} \max_{\lambda^k} \left\{ \sum_{k=1}^{M} \lambda^k \left[\frac{1}{2} \sum_{i=1}^{m} \frac{\ell_i^2}{E_i} \frac{(q_i^k)^2}{t_i} \right] \right\} ,$$

$$\text{subject to:} \ \sum_{k=1}^{M} \lambda^k = 1 \, ; \quad \lambda^k \ge 0 \, , \quad k = 1, \dots, M \, ;$$

$$\mathbf{B}^T \mathbf{q}^k = \mathbf{p}^k \, , \quad k = 1, \dots, M \, ;$$

$$\sum_{i=1}^{m} t_i = V \, , \quad t_i > 0 \, , \quad i = 1, \dots, m \, , \quad (65)$$

we can derive the optimal values of the bar volumes as

$$t_i = \frac{1}{\sqrt{2A}} \sqrt{\frac{\ell_i^2}{E_i} \sum_{k=1}^{M} \lambda^k (q_i^k)^2} \, . \quad (66)$$

Here A is a Lagrange multiplier for the volume constraint, which is uniquely determined by this constraint. Inserting in (65) we obtain the following problem in the member forces only:

$$\min_{\mathbf{q}^k} \max_{\lambda^k} \left\{ \frac{1}{2V} \left\{ \sum_{i=1}^{m} \left[\frac{\ell_i}{\sqrt{E_i}} \sqrt{\sum_{k=1}^{M} \lambda^k (q_i^k)^2} \right] \right\}^2 \right\} ,$$

$$\text{subject to:} \ \sum_{k=1}^{M} \lambda^k = 1 \, ; \quad \lambda^k \ge 0 \, , \quad k = 1, \dots, M \, ,$$

$$\mathbf{B}^T \mathbf{q}^k = \mathbf{p}^k \, , \quad k = 1, \dots, M \, , \quad (67)$$

and the optimal bar volumes are given as

$$t_i = V \sqrt{\frac{\ell_i^2}{E_i} \sum_{k=1}^{M} \lambda^k (q_i^k)^2} \cdot \left[\sum_{i=1}^{m} \sqrt{\frac{\ell_i^2}{E_i} \sum_{k=1}^{M} \lambda^k (q_i^k)^2} \right]^{-1} . \quad (68)$$

For the single load case we recover the traditional linear programming formulation (44) in the disguised form

$$\min_{\mathbf{q}} \left\{ \frac{1}{2V} \left[\sum_{i=1}^{m} \left(\frac{\ell_i}{\sqrt{E_i}} |q_i| \right) \right]^2 \right\} ,$$

$$\text{subject to:} \ \mathbf{B}^T \mathbf{q} = \mathbf{p} \, . \quad (69)$$

Rescaling the objective function and taking the square root of the objective function results in (44). Note that we have again seen that the stress constraint values for the plastic topology problem should be chosen as $\sqrt{E_i}$. Also, as (69) was obtained by direct duality without rescaling, one can see that the optimal value Π of the optimal compliance will relate to the optimal value Ψ of the minimum weight plastic design problem as

$$\Pi = \frac{1}{V} \Psi^2 \, . \quad (70)$$

This relation has also been reported recently by Rozvany (1992).

Note that (69) is the natural formulation for the stress only reformulation of the minimum compliance problem stated in terms of stresses and the complementary energy. Problem (69) can be viewed as a corresponding equilibrium problem for a structure with a non-smooth, convex complementary energy. This energy arises from a truss for which the bars automatically adjust their sizing and connectivities with the purpose of minimizing the compliance of the currently applied loading. This is completely analogous to the situation for the displacement based problem and the similarly formulated continuum problems, as described, e.g. by Allaire and Kohn (1992); Jog *et al.* (1993).

7 Combined truss topology and geometry optimization

The "ultimate truss" should clearly be obtained by combining topology optimization with a possibility of optimizing simultaneously the positions of the nodal points (or for a FEM model, a possibility of optimizing at the same time the shape of the finite elements and the distribution of material to these elements). For the combined topology and geometry problem we have the simplest formulation as

$$\min_{\mathbf{u},\mathbf{a},\mathbf{x}} \mathbf{p}^T\mathbf{u}\,,$$

$$\text{subject to:}\quad \sum_{i=1}^{m} a_i\ell_i(\mathbf{x})\mathbf{A}_i(\mathbf{x})\mathbf{u} = \mathbf{p}\,,\quad \sum_{i=1}^{m} a_i\ell_i(\mathbf{x}) = V\,,$$

$$a_i \ge 0\,,\quad i = 1,\dots,m\,,$$

$$b_j^k \le x_j^k \le c_j^k\,,\quad j = 1,\dots,n\,,\quad k = 1, 2, (3)\,, \tag{71}$$

which is just problem (8) rewritten as a problem depending also on the nodal positions x_j, $j = 1, \dots, n$. The nodal positions are restricted to lie within certain bounds that should be chosen to make the resultant trusses realizable. Because the member volumes are dependent on the nodal positions, we have here reverted to the cross-sectional areas of the individual bars as design variables. Problem (71) can be solved as a combined problem considering the problem either as a combined analysis and design problem or as a standard structural optimization problem, which can be solved through an adjoint method in the areas and nodal positions only (if small lower bounds on the cross sectional areas are applied). An alternative solution procedure is to apply a multilevel approach to the combined problem, treating, e.g. the topology problem as the inner problem. Because of the size of the topology problem, earlier work has usually involved some form of heuristics to speed up the very significant amount of computations involved (Kirsch 1989; Topping 1992). By combining the effective truss topology design methods described above with appropriate tools from non-smooth optimization the multilevel approach can be put in a solid mathematical framework.

For a fixed set of nodal positions we choose here the form (29) of the topology design problem and thus write (71) as a two-level problem,

$$\max_{\substack{\mathbf{x}\\ \mathbf{b}\le\mathbf{x}\le\mathbf{c}}} \left\{ \min_{\mathbf{u}} \left\{ \max_{i=1,\dots,m} \left[\frac{V}{2}\mathbf{u}^T\mathbf{A}_i(\mathbf{x})\mathbf{u} - \mathbf{p}^T\mathbf{u} \right] \right\} \right\}. \tag{72}$$

The inner topology problem in the displacements $\mathbf{u}$ can effectively be solved (for fixed $\mathbf{x}$) by one of the methods mentioned in the previous sections. The main part remaining is then, of course, the minimization of the so-called master function,

$$F(\mathbf{x}) := \min_{\mathbf{u}} \left\{ \max_{i=1,\dots,m} \left[\frac{V}{2}\mathbf{u}^T\mathbf{A}_i(\mathbf{x})\mathbf{u} - \mathbf{p}^T\mathbf{u} \right] \right\}, \tag{73}$$

on the outer level. The number of variables (the nodal positions) in this outer problem will usually be moderate. However, there are two decisive drawbacks. There is no reason for F to be convex and F is not differentiable everywhere. Hence we cannot expect to find more than local minima of F and we have to work with codes from nonsmooth optimization [e.g. bundle methods (Schramm and Zowe 1992)]. These codes require that for each iterate $\mathbf{x}$ we can compute a so-called sub-gradient as a substitute for the gradient. Using tools from nonsmooth calculus it is easily seen that this causes no difficulties for the above min-max function F. We add that it is straightforward to show that each local minimizer $\mathbf{x}^*$ of F together with the associated $\mathbf{t}^*$ and $\mathbf{u}^*$, which solve the topology problem for the fixed nodal positions $\mathbf{x}^*$, gives a local minimizer $(\mathbf{u}^*, \mathbf{a}^*, \mathbf{x}^*)$ [with $a_i^* = t_i^*/\ell_i(x_i^*)$] for problem (71).

The two-level approach becomes especially attractive if we consider the single load truss topology problem for which the member stiffness matrices are dyadic products. Then [compare (43) from Section 5] $F(\mathbf{x})$ reduces to the parametrized linear programming problem

$$F(\mathbf{x}) = \min_{\mathbf{u}} \left[-\mathbf{p}^T\mathbf{u} \,\middle|\, -1 \le \sqrt{\frac{VE_i}{2}}\, \frac{\mathbf{b}_i(\mathbf{x})^T\mathbf{u}}{\ell_i(\mathbf{x})} \le 1\,,\; i = 1,\dots,m \right]. \tag{74}$$

The sub-gradient in this case is basically the derivative with respect to $\mathbf{x}$ of the Lagrange function for this LP-problem. Hence we get a sub-gradient "for free" when solving (43) for a given set of nodal positions $\mathbf{x}$. For details we refer to the paper by Ben-Tal *et al.* (1993).

8 Numerical results

The availability of efficient methods to solve large (sparse) LP problems makes it natural to solve the single load truss topology design problem using the LP formulations (43) or (44)-(46). For problems with multiple loads and/or bounded bar areas, for the reinforcement problem as well as for the FEM case, we cannot obtain a linear programming formulation of the problem and we are forced to solve problems of the type (8)-(10), (12), (27)-(32) or (39) directly. Problems (8)-(10) generalize most easily to more general design situations involving stress and displacement constraints but it is large scale and non-convex. Problems (27)-(32) and (39) are convex and have the size of the degrees of freedom of the ground structure; (27)-(32) are non-differentiable and unconstrained and (39) is differentiable, but at the cost of a high number of constraints. The algorithm presented for solving problem (25) is a specialized algorithm and it has been implemented to take advantage of the sparsity of the matrices $\mathbf{A}_i$. General purpose algorithms for min-max optimization or non-differentiable optimization can also be employed, but comparison is difficult for problem sizes where sparsity plays an important role; also most general purpose methods have enormous computer storage requirements. Likewise, problem (39) can be solved by general purpose algorithms (SQP etc.), but again sparsity and the fact that the number of variables is much lower than the number of constraints should be utilized. Of the wide range of algorithms we have tried out we have found that the Penalty/Barrier/Multiplier (PBM) method gives the best performance as a general purpose method for both single load and multiple load worst-case design. This statement is generally true, but for certain problems with special geometry of ground structure and/or optimal topology other algorithms may be just as efficient. It is our experience that the truss topology design problem is a very challenging mathematical programming problem with structure and properties that are a test for even the best of algorithm.

Table 1 contains typical run-time results for solving single and multiple load truss topology design problems by the Penalty/Barrier/Multiplier (PBM) method. For the single load case the problem solved was the formulation (41) and for the multiple load, worst case design formulation the problem solved was the following reformulation of problem (62)

Table 1. Typical performance of the Penalty/Barrier/Multiplier method for truss topology design. The ground structures include all non-overlapping connections. For details, see text

Example	Degrees of freedom (number of variables)	Number of bars	Number of load cases	Number of Newton steps	CPU	see Fig.
13 × 13	334 (334)	8744	1	34	4′	4A
13 × 13	334 (334)	8744	1	34	4′	4B
13 × 13	334 (1002)	8744	3	58	2 h	4B
3 × 33	194 (582)	2818	3	70	35′	5
21 × 11	458 (458)	16290	1	37	11′	7A
21 × 11	458 (1374)	16290	3	69	6 h 4′	7B

$$\inf_{\substack{s^k \geq 0, \mathbf{x}^k, \tau \\ \sum_{k=1}^{M} (s^k)^2 = 1}} \left(V\tau - \sum_{k=1}^{M} s^k \mathbf{p}^{k^T} \mathbf{x}^k \right),$$

$$\text{subject to :} \quad \sum_{k=1}^{M} \mathbf{x}^{k^T} \mathbf{A}_i \mathbf{x}^k - 2\tau \leq 0, \quad i = 1, \ldots, m. \quad (75)$$

This problem is derived from (62) by the transformation $s^k = \sqrt{\lambda^k}$, $\mathbf{x}^k = \mathbf{u}^k/\sqrt{\lambda^k}$ of variables. For a truss with N degrees of freedom, m potential bars and M load cases, problem (41) has N variables and m constraints, while problem (75) has $NM + M + 1$ variables and m non-linear constraints. The main computational effort in applying the PBM method is the minimization of the unconstrained penalty/barrier function (Step 1 in the algorithm as described in Section 5). This is done using a Newton method. Therefore the number of Newton steps reported in Table 1 reflects well the number of main iterations; note that each Newton step corresponds to solving a linear system of equations, which for the single load case is comparable in size to the linear system solved for one full equilibrium analysis step of the "Optimality Criteria Method". For all problems the starting point was $\tau = 0$, $u^k = 0$ and $\lambda^k = 1/M$ for $k = 1, \ldots, M$. The algorithm was stopped when 6 digits of accuracy of the objective function was obtained. The Newton step was performed using the routine EOLBF from the NAG Library. Computations were executed on a SUN-4-s.3 Mega-flops computer. The examples used for the table are all for ground structures in a rectangular domain in the plane with a fairly regular layout of nodal points. As potential connections all non-overlapping connections were used.

9 Examples

We have chosen to illustrate some prominent features of truss topology design for single loads, for multiple loads and for the case of reinforcement as well as self-weight problems. The main purpose is to illustrate the effect of various modelling choices on the geometry of the lay-outs. Space does not permit an exhaustive discussion on this subject, as there are many features that influence the final designs, such as the choice of nodal points as well as the geometry and connectivities of the ground structure, the geometry of the loadings, the geometry of the supports, etc. Also, for the combined geometry and topology optimization, the allowed movements of the nodal points play an important role. More examples of the efficiency of large scale truss topology optimization methods can be found in the work of, for example, Achtziger (1992, 1993); Achtziger *et al.* (1992); Ben-Tal and Bendsøe (1992); Ben-Tal, Kocvara and Zowe (1993); Kocvara and Zowe (1992); Ringertz (1985, 1989); and Zhou and Rozvany (1991).

In all but one of the examples the ground structures consist of all possible connections (as in Fig. 1D) or of only connections to the neighbouring points (as in Fig. 1B). For problems with all possible connections, the possibility of redundant, overlapping bar members entering the ground structure was avoided by removing overlapping bars, in the sense that for any two nodal points the straight line connection between the two points always consists of the connection through eventual other nodal points lying on the line connecting the two points. For problems with self-weight overlapping bars do not represent a redundancy, as a connection through an extra nodal point introduces the possibility of self-weight loads at such extra nodes. This underlines the weakness in the modelling of self-weight loads in trusses.

For the truss topology problem with a single loading case it is possible to generate a catalogue of optimal topologies. Problem (4) is made up of expressions that are element wise linear in all variables, except geometric data. Thus, for a specific choice of ground structure geometry and load vector direction, the optimal topology needs only to be computed for one set of assigned values of Young's modulus E, volume V, load size, and one geometric scale; for any other values of these variables, the optimal values of the design variables t, the deformation u and the compliance $p^T u$ can be derived by a simple scaling; the non-dimensional parameter

$$\Phi = \frac{(p^T u) V E}{\|p\|^2 L^2}, \quad (76)$$

is a constant for optimal topologies generated with equivalent topologies of the ground structure, with L being a measure of scale. A similar non-dimensional parameter can be devised for the multiple load case, the case of self-weight loads, etc., but here the catalogue will depend on a further range of parameters, such as the ratios between the sizes of the different applied loads.

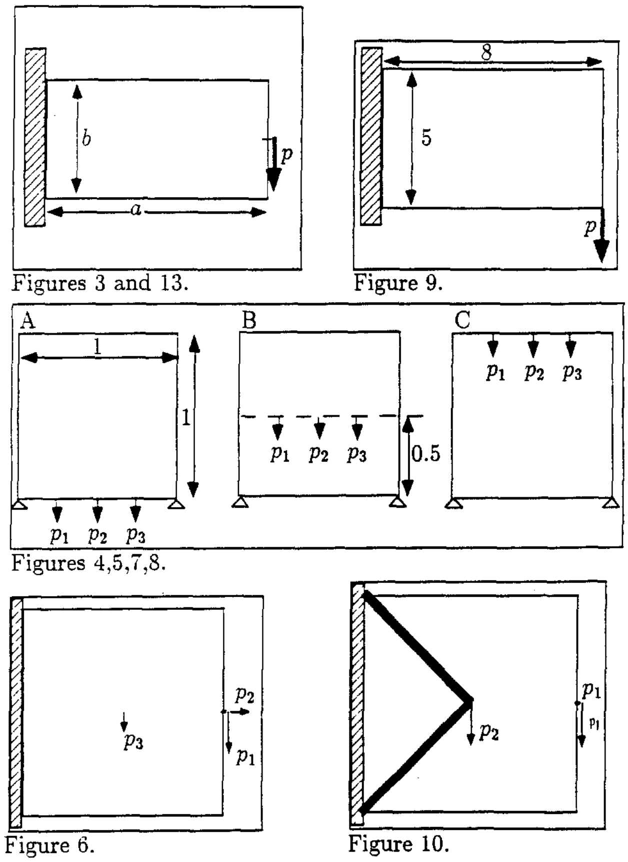

Fig. 2. The ground structure geometry, loading and support conditions used for most of the examples

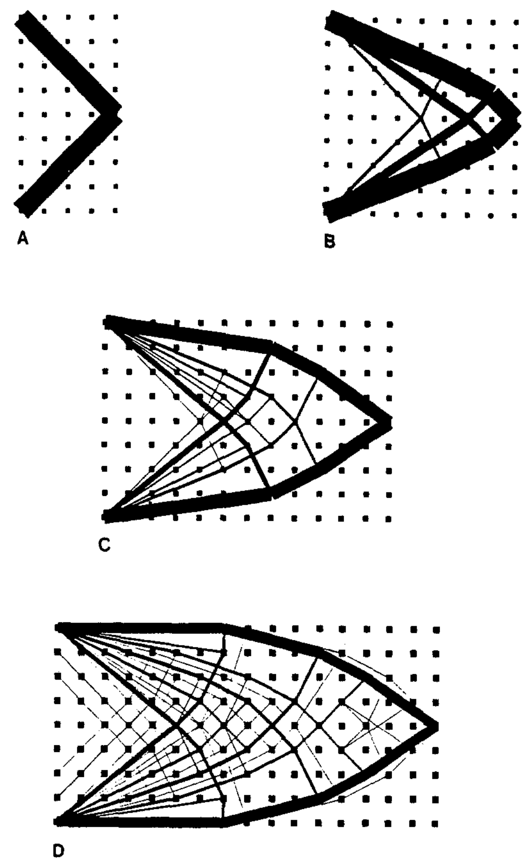

Fig. 3. The influence of the ground structure geometry on the optimal topology. Optimal truss topologies for transmitting a single vertical force to a vertical line of supports (see Fig. 2). The ground structures consist of all possible non-overlapping connection between the nodal points of a regular mesh in a rectangle of varying aspect ratios $R = a/b$. **A**: 632 potential bars for 5 by 9 nodes in a rectangle with $R = 0.5$. Optimal non-dimensional compliance $\Phi = 4.000$. **B**: 2040 potential bars, 9 by 9 nodes, $R = 1.0$, $\Phi = 5.975$. **C**: 4216 potential bars, 13 by 9 nodes, $R = 1.5$, $\Phi = 9.1676$. **D**: 7180 potential bars, 17 by 9 nodes, $R = 2.0$, $\Phi = 12.5756$. See also Fig. 13

A very important feature of the truss topology method is the prediction of Michell frame type lay-outs in certain cases, if such a structure is nature's best topology with the given loads, supports and ground structure, as illustrated in Figs. 3 and 13. These figures illustrate the varied topologies that can be created for the simple problem of transmitting a single vertical load to a vertical line of supports, through ground structures of rectangular lay-out of different aspect ratios. The range of topologies goes from the optimal two-bar truss with two bars at $\pm 45°$ to long slender Michell frame lay-outs which at a global scale behaves like a sandwich beam in bending. The transition from "true" trusses to Michell truss continua for this setting has been studied by analytical means by Lewinski, Zhou and Rozvany (1993, 1994). Note that in these examples (as in all cases) we clearly see that the topology optimization not only predicts the optimal lay-out of the structures, but also finds the optimal use of the prescribed possible support conditions.

It was mentioned earlier that truss topology compliance optimization under a single loading condition leads to statically determinate solutions, but the resultant structures can in many situations be mechanisms, which are stable under the applied loading. This feature can in most cases be avoided by designing the truss for multiple loading cases, either in the weighted average formulation or in the worst case, min-max formulation. Figure 4 shows the difference between treating three nodal loads as one, combined load, or as three independent load cases. Note that we through the multiple load formulation avoid the mechanisms, at the expense of much more complicated topologies. Figure 4 also illustrates the differences that occur due to the relative position of the possible supports and the applied loads. In Fig. 5. we show, for a similar load and support condition in a different ground structure, the (small) difference between multiple load designs achieved through the weighted average formulation and the min-max formulation. That multiple loading conditions can also simplify the lay-out of the optimal topology is illustrated in Fig. 6. In all our examples with multiple load, worst case design, the nature of the applied loads is such that all loads have compliance value at the maximal value. This is usually not the case for problems where the optimal structure for one of the applied loads can carry the other loads.

Examples of large truss topology optimization results are shown in Figs. 7 and 8, again illustrating the effect of multiple loads versus single loads, as well as illustrating the very important effect of the type of possible supports. One of the

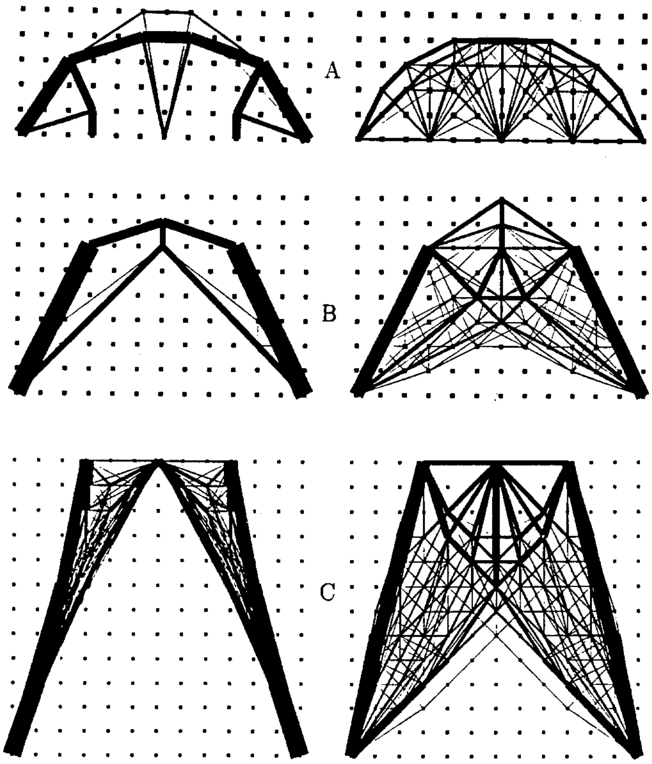

Fig. 4. The difference between multiple load and single load case problems. Optimal truss topologies for transmitting three vertical forces to two fixed supports. Various positions of the loads are considered and the trusses are optimized with the loads treated as a single load as well as three individual load cases for a min-max, worst case design situation. The ground structures consist of all 8744 possible non-overlapping connections between the nodal points of a regular 13 by 13 mesh in a square domain. The loads are vertical unit loads at three equidistant nodes along the lower line of nodes (**A**), across the middle of the ground structure (**B**) and at the top of the ground structure (**C**) (see also Fig. 2). The left-hand column shows the single load results, the right-hand column the multiple load, worst case results. In (**A**) and (**B**) we do not show the uppermost rows of nodes, as these are not part of the optimal structure. A slight asymmetry of the ground structure is reflected in the optimal truss topologies

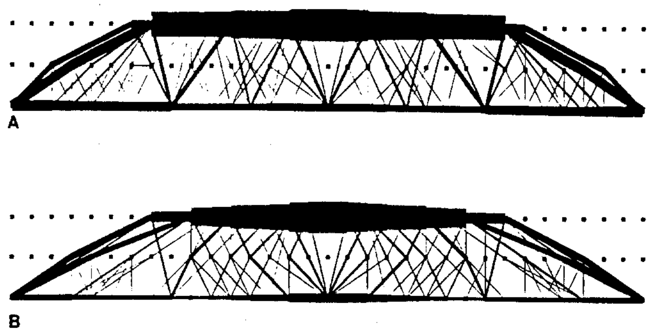

Fig. 5. The difference (and similarity) between multiple load case treated in the weighted average formulation (equal weights) (**A**) and treated in the worst case min-max formulation (**B**). Optimal truss topologies for transmitting three vertical forces to two fixed supports as in Fig. 4A (cf. Fig. 2), but for a long slender rectangular ground structure of aspect ratio 16 (like a long span bridge), with 33 by 3 equidistant nodes and all 2818 possible non-overlapping connections. In the figures, the vertical scale has been distorted in order to be able to show the results

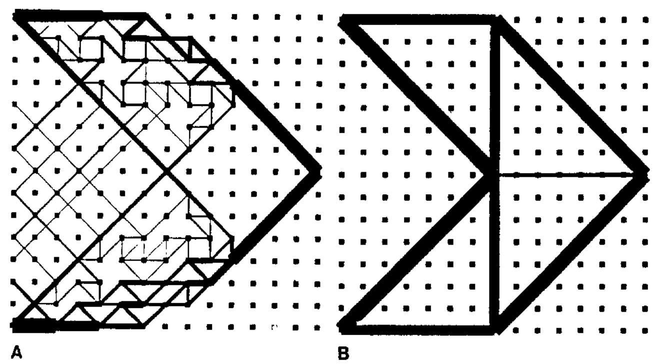

Fig. 6. A case where the introduction of multiple loads simplify the optimal lay-out. Also an example of the optimal topology for a ground structure with only neighbouring nodes in a square, regular 15 by 15 lay- out being connected (see Fig. 1B); this results in only 788 potential bars. All nodes at the left-hand side are potential supports (see Fig. 2). **A**: The optimal topology for a single vertical load at the mid right hand node. **B**: The optimal topology for three loadings cases including the load of the single load problem. The load of the single load example is twice as large as the two other loads, one of which is a horizontal load at the mid right hand node while the last load is a vertical load at the centre of the ground structure. This is for a weighted average formulation with equal weights. The compliance for the load case number 1 increases by only 2.58%, as compared to **A**, which is optimal for this load only

interesting features of topology design is that the extreme freedom of the design setting immediately reveals any weakness or misinterpretation of support and loading conditions, thus underlining the efficiency of topology design methods as an interactive tool in the initial steps of a design process.

Finally, in Figs. 9 and 10 we show examples of truss topology design with self-weight loads included in the formulation. In Fig. 11 we show a optimal topology for a complicated lay-out of the ground structure and in Fig. 12 we show the results of a combined topology and geometry design for a three-dimensional truss.

Note that in all illustrations we have chosen to show the individual bars, so that the plotted areas are proportional to the stiffness (i.e. area) of the bars. If we instead show the thicknesses as being proportional to the square root of the areas of the bars, corresponding to the dimensions that would be seen in a truss with circular cross-sections, we would obtain a more realistic plot but the visual impression of the distribution of stiffness will be distorted as illustrated in Fig. 13.

10 Conclusions

We have given a survey of formulations, problem structure and algorithms used for optimal truss topology design formulated in terms of weight and stiffness (compliance), or weight and strength. The relation to topology problems for continuum structures is also outlined. The truss topology problems exhibit an algebraic structure that allows for the generation of a sequence of equivalent problem statements in terms of displacements only, forces only, member volumes only and combinations thereof. All of these formulations give

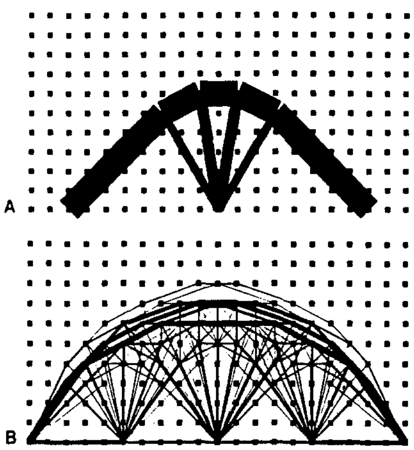

Fig. 7. A detailed study of the load and support situation of Fig. 4A. The ground structure is restricted to a rectangular domain of aspect ratio 2, and with the number of nodes increased to a 21 by 11 layout, with 16290 possible non-overlapping connections. As a single load problem we treat the case of only the mid- span load applied (**A**), while the multiple load situation covers the three loads of Fig. 4, in the min-max formulation (**B**). Note that the supports have been moved in by two nodes from each vertical side, in order to identify an eventual restriction of having the supports at the extreme points of the ground structure

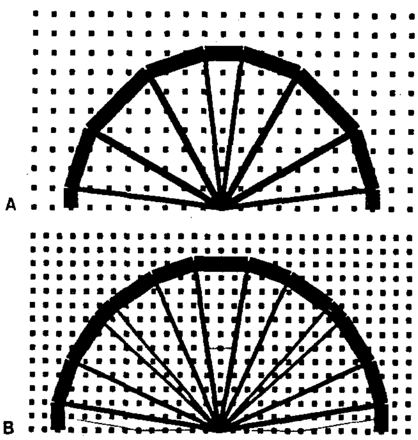

Fig. 8. The influence of the boundary conditions. A detailed study of the situation in Fig. 7A (single load case), but with the right-hand support changed from a fixed support to a "rolling" support which restricts movements in the vertical direction only. In (**A**) we use the ground structure of Fig. 7, in (**B**) the number of nodes is increased to a 29 by 15 lay-out with all 57770 possible non-overlapping connections

very valuable physical insight and the topology design problem can be viewed as an equilibrium problem for an optimal truss with design independent potential or complementary energy. Finally, through a limited number of examples various features of topology optimized truss structures have been indicated.

Acknowledgements

The work presented in this paper received support from the German-Israeli Foundation for Scientific Research and Development (A.B-T and JZ), the German Research Council (JZ) and the Danish Technical Research Council (Programme of Research on Computer-Aided Design) (MPB).

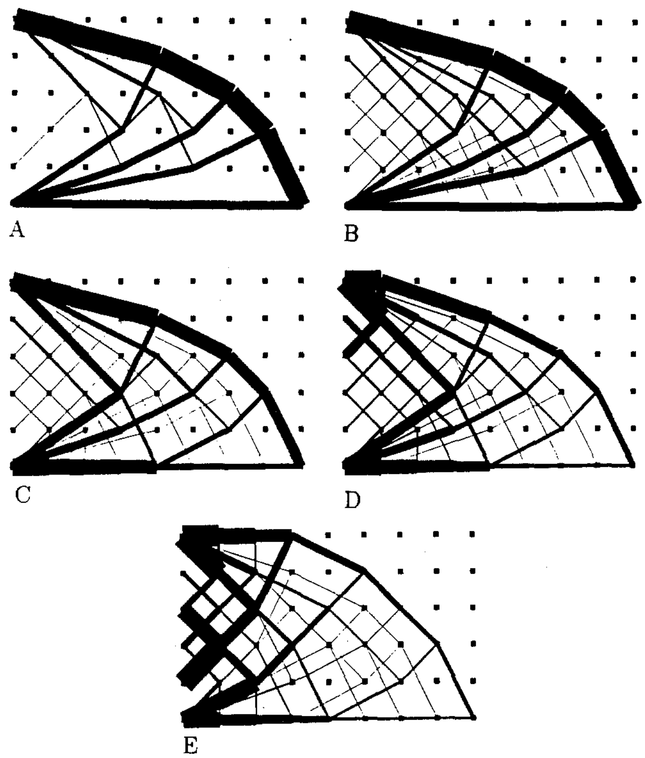

Fig. 9. The effect of self-weight loads. Optimal truss topologies for transmitting a single vertical force to a vertical line of supports (see Fig. 2). The figures show the variation of the resulting topologies for increasing specific self-weight loads, corresponding to increasing real lengths of the structures. In (**A**) self- weight is ignored, in (**B**) moderate self-weight is present, increased by 15 times to the design (**C**) and again 2 times more to the design (**D**). These designs are obtained for a 9 by 6 equidistant nodal lay-out in a rectangular domain of aspect ratio 1.6, and all 919 possible *non-overlapping* connections. If **ALL** 1431 possible connections are used the design (**D**) is modified to the design (**E**)

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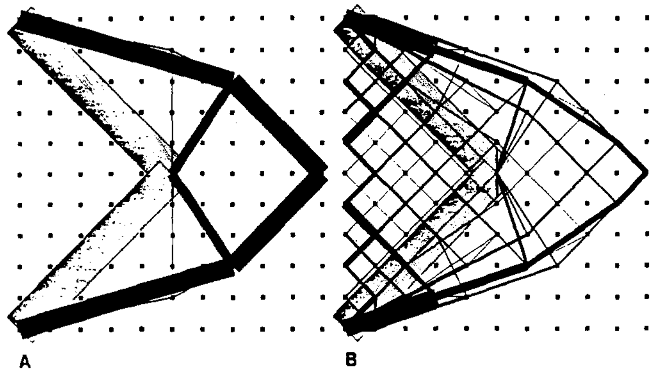

Fig. 10. A reinforcement example. Optimal truss topologies for transmitting two vertical forces to a vertical line of supports (see Fig. 2). The given structure is the optimal two-bar truss for carrying the load case number two at the center of the ground structure. We then seek the optimal reinforcement of this structure, with the purpose of carrying load number one as well. The reinforcement problem is treated as a weighted average, multiple load problem, without (**A**) or with self-weight taken into account (**B**). The ground structure for reinforcement is the 11 by 11 equidistant nodal lay-out in a square, with all 4492 possible non-overlapping connections

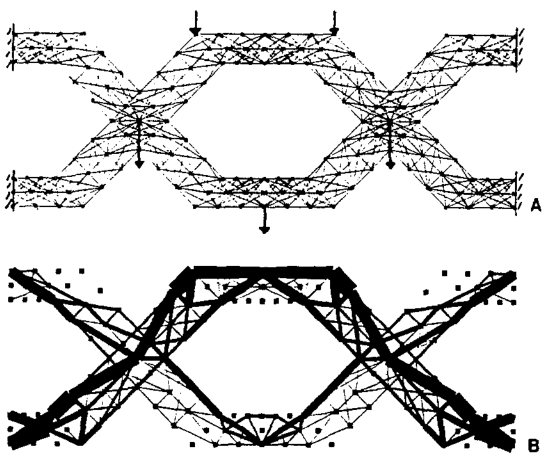

Fig. 11. An example of a complicated ground structure geometry, with 156 nodal points and 660 potential bars. The ground structure, supports and five loads are shown in (**A**). The resulting topology for a weighted average, multiple load problem formulation is shown in (**B**). The ground structure was generated by an interactive CAD-based programme developed by Ole Smith, Mathematical Institute, The Technical University of Denmark

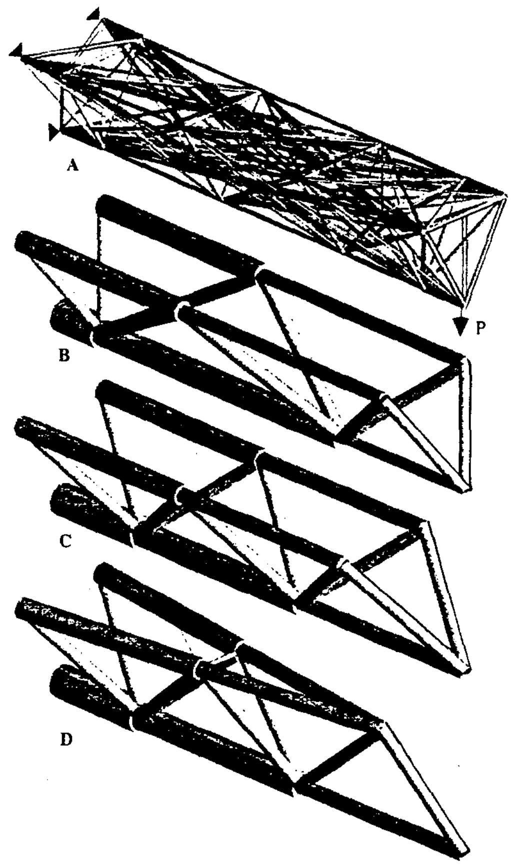

Fig. 12. An example of a 3-D topology and geometry optimization for a beam carrying a single load. In (**A**) we show the ground structure of nodal points and potential bars. Note that the ground structure has non- equidistant nodal point positions along the length axis of the "beam". In (**B**) we see the optimal topology for the fixed nodal lay-out of the ground structure, in (**C**) a combined geometry and topology optimization with nodal positions restricted to moved along the length axis of the "beam". Finally, in (**D**) the result of a combined geometry and topology optimization with totally free nodal positions is shown. The compliance values of the optimized designs are 6.944, 6.563 and 6.326, respectively

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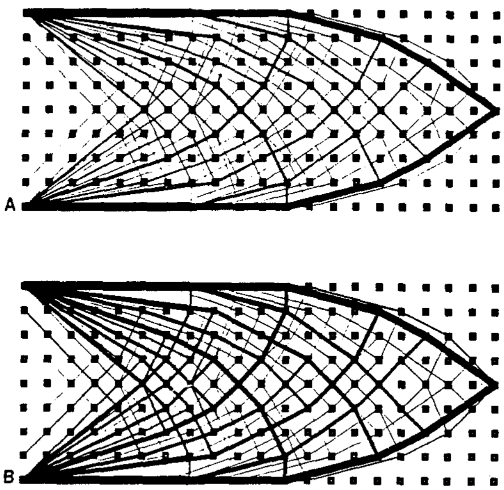

Fig. 13. The visual difference between illustrating the optimal designs with bar thicknesses representing stiffness (**A**) and representing the diameter of circular cross-sections of individual bars (**B**). Optimal truss topologies for transmitting a single vertical force to a vertical line of supports (see Figs. 2 and 3). The ground structures consist of all 10940 possible non-overlapping connection between the nodal points of a regular mesh in a rectangle with aspect ratio $R = 2.5$. The optimal non-dimensional compliance is $\Phi = 16.4929$

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Received May 14, 1993
Revised manuscript received Feb. 28, 1994

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