BIT 35 (1995), 504–515.

STABILITY ANALYSIS OF NUMERICAL METHODS FOR SYSTEMS OF NEUTRAL DELAY-DIFFERENTIAL EQUATIONS *

GUANG-DA HU and TAKETOMO MITSUI

Graduate School of Human Informatics, Nagoya University, Nagoya 464-01, Japan
email: gdhu@torii.nuie.nagoya-u.ac.jp; a41794a@nucc.cc.nagoya-u.ac.jp

Abstract.

Stability analysis of some representative numerical methods for systems of neutral delay-differential equations (NDDEs) is considered. After the establishment of a sufficient condition of asymptotic stability for linear NDDEs, the stability regions of linear multistep, explicit Runge-Kutta and implicit A-stable Runge-Kutta methods are discussed when they are applied to asymptotically stable linear NDDEs. Some mentioning about the extension of the results for the multiple delay case is given.

AMS subject classification: 65L20, 65L06, 65Q05, 34K20, 34K40.

Key words: Stability region, linear stability analysis, numerical methods, neutral delay-differential equation, underlying numerical method for ODEs, sufficient condition of asymptotic stability.

1 Introduction.

Consider the systems of neutral delay-differential equations (NDDEs)

$$(1.1)\qquad \begin{cases} \dot{u}(t) = f\left(t, u(t), u(t-\tau), \dot{u}(t-\tau)\right), & t \geq 0, \\ u(t) = g(t), \quad -\tau \leq t \leq 0 \end{cases}$$

where f and g denote given vector-valued functions, τ is a given positive constant and $u(t)$ is the vector-valued unknown function to be solved for $t \geq 0$. Sufficient conditions have been known for the unique existence of the solution of (1.1) in the literature, *e.g.* [4]. Hereafter we assume the existence of unique exact solution of the system. However, since analytical solutions can be computed only in very restricted cases, many methods have been proposed for the numerical approximation of the problem (1.1).

It is the purpose of the present paper to investigate conditions for stability of numerical methods for the linear test equation of the type (1.1), *i.e.*

$$(1.2)\qquad \begin{cases} \dot{u}(t) = Lu(t) + Mu(t-\tau) + N\dot{u}(t-\tau), & t \geq 0 \\ u(t) = g(t), \quad -\tau \leq t \leq 0. \end{cases}$$

*Received January 1995. Revised June 1995.

Here L, M and N are constant complex-valued matrices. The solution of (1.1) is said to be (asymptotically) stable if the solution $u(t)$ tends to zero as $t \to \infty$. First we will state a sufficient condition for stability of the system (1.2) with respect to the triplet (L, M, N). Next we will examine the corresponding stability conditions of numerical solutions of the linear multistep and the Runge-Kutta methods for (1.2) under the assumption for (L, M, N). This means a *linear stability analysis* of the numerical methods for NDDEs.

An application of numerical method for NDDEs to the test equation (1.2) usually leads to a recurrence relation of fixed but arbitrarily high order. This difficulty may cause few known results on linear stability analysis of numerical methods for neutral equations. Two exceptions are those by Brayton and Willoughby [6], who analyzed linear stability properties of the θ-methods for (1.1) in the case of linear test equation (1.2) with symmetric real L, M and N, and by Bellen, Jackiewicz and Zennaro [4] who considered similar properties of one-step methods in the case when L, M and N reduce to scalar complex parameters. However, recently Kuang et al. [14] studied the stability of the θ-method for (1.2), and obtained a condition of the unconditional stability of the method.

In Zennaro [18], Liu and Spijker [16], and in't Hout and Spijker [10], some new techniques are introduced for stability analysis of numerical methods for *delay-differential* equations (DDEs) in their application to test equation

$$\begin{cases} \dot{u}(t) = au(t) + bu(t-\tau), & t \geq 0, \\ u(t) = g(t), & -\tau \leq t \leq 0, \end{cases} \tag{1.3}$$

where a and b are complex constants, g is a given initial function and $\tau \geq 0$. Similar studies on the stability of numerical methods for (1.3) can be found in many other literature, *e.g.* [2, 3, 5, 11, 17]. However, in't Hout [9] and Koto [13] recently treated stability analysis of θ- and implicit Runge-Kutta methods when a and b are replaced by constant complex-valued matrices in (1.3). But they did not deal with the test equation (1.2).

The organization of the present paper is as follows. In the following section we will derive a sufficient condition of the asymptotic stability of linear systems of NDDEs (1.2). Assuming the sufficient condition, we will obtain a new result on the stability region of the linear multistep method for (1.2) in Section 3, while Section 4 is devoted to the case of Runge-Kutta methods. We will introduce the matrix $Q(\xi) = (I - \xi N)^{-1}(L + \xi M)$, whose eigenvalues with ξ of unit magnitude are crucial for the stability criterion for numerical methods.

Furthermore, in Section 5, the results in the previous sections are extended to the multiple delay case by the introduction of a matrix which corresponds to $Q(\xi)$ in the single delay case.

2 Stability of linear systems of NDDEs.

We will consider a sufficient condition of asymptotic stability of linear systems of neutral delay-differential equations

$$\dot{u}(t) = Lu(t) + Mu(t-\tau) + N\dot{u}(t-\tau), \tag{2.1}$$

where L, M and N are d-dimensional constant complex-valued matrices, and $\tau > 0$.

THEOREM 2.1. *The system (1.2) is asymptotically stable if the conditions*

$$\Re\lambda_i[(I - \xi N)^{-1}(L + \xi M)] < 0 \tag{2.2}$$

for all i and $\xi \in \boldsymbol{C}$ such that $|\xi| \leq 1$, and

$$\rho(N) < 1 \tag{2.3}$$

hold. Here $\lambda_i(F)$ and $\rho(F)$ stand for the i-th eigenvalue and the spectral radius, respectively, of a complex-valued matrix F.

The Laplace transformation for Eq. (2.1) implies

$$\det[(sI - L) - (M + Ns)\exp(-\tau s)] = 0 \tag{2.4}$$

as its characteristic equation. Hence the stability analysis of (2.1) reduces to the root-locus problem for (2.4). From this viewpoint we need the following lemma for the proof of Theorem 2.1.

LEMMA 2.1. *Define the following bivariate polynomial*

$$P(s, z) = \det[(sI - L) - (M + Ns)z] \quad (s, z \in \boldsymbol{C}). \tag{2.5}$$

If the conditions

$$P(s, 0) \neq 0 \quad \textit{for } s \textit{ such as } \Re s \geq 0 \tag{2.6}$$

and

$$P(s, z) \neq 0 \quad \textit{for } (s, z) \textit{ such as } \Re s = 0 \textit{ and } |z| \leq 1 \tag{2.7}$$

hold, then we have

$$P(s, z) \neq 0 \quad \textit{for } (s, z) \textit{ such as } \Re s \geq 0 \textit{ and } |z| \leq 1. \tag{2.8}$$

PROOF: We perform the Möbius transformation

$$s = \frac{1 - w}{1 + w} \tag{2.9}$$

on $P(s, z)$. The transformation maps the right and left half planes of s to the inner and outer regions, respectively, of the unit circle of w, while the imaginary axis of s to the unit circle $|w| = 1$.

The equation $P(s, z) = 0$ yields an algebraic function $z = v(s)$ which has a number of branches (see *e.g.* [1]). If $v(s)$ has singularities, they can be removed with the usual detour similarity as in [12]. Hence $v(s)$ can be regarded as a holomorphic function on an appropriate Riemann surface.

Since Eq. (2.6) implies $z = v(s) = u(w) \neq 0$ whenever $|w| \leq 1$ holds, the minimum of $|u(w)|$ on the unit disk $|w| \leq 1$ attains on its boundary $|w| = 1$, i.e., $\Re s = 0$. On the other hand, Eq. (2.7) yields the inequality $\min |v(s)| =$

$\min|u(w)| > 1$ whenever $\Re s = 0$ and $|w| = 1$. Therefore our conclusion (2.8) holds. □

PROOF of Theorem 2.1. As a special case of $\xi = 0$ of the condition of (2.2) of the Theorem, the inequality

$$\Re\lambda_i(L) < 0$$

holds. This means (2.6). The condition (2.3) of the Theorem implies

$$\det[I - zN] \neq 0 \quad \text{whenever} \quad |z| \leq 1.$$

Thus we obtain

$$\begin{aligned} P(s,z) &= \det[I - zN]\det[sI - (I - zN)^{-1}(L + zM)] \\ &= \det[I - zN]\prod_{i=1}^{d}\left(s - \lambda_i[(I - zN)^{-1}(L + zM)]\right). \end{aligned}$$

Henceforth we have the following inequality.

$$P(s,z) \neq 0 \quad \text{for} \quad \Re s = 0 \quad \text{and} \quad |z| \leq 1,$$

which is nothing but the condition (2.7). Together with the condition (2.2) we obtain the inequality (2.8). Recall that the characteristic equation of the system (2.1) is

$$P(s, \exp(-s\tau)) = 0.$$

The conditions of the Theorem have been shown to yield

$$P(s, \exp(-s\tau)) \neq 0 \quad \text{for} \quad \Re s \geq 0 \quad \text{and} \quad |\exp(-s\tau)| \leq 1.$$

It is trivial to show $|\exp(-s\tau)| \leq 1$ for $\Re s \geq 0$ and $\tau > 0$. Thus we have

$$P(s, \exp(-s\tau)) \neq 0 \quad \text{for} \quad \Re s \geq 0,$$

whose contraposition reads that the equation $P(s, \exp(-s\tau)) = 0$ implies $\Re s < 0$. □

COROLLARY 2.1. *The two assumptions in Theorem 2.1 are equivalent to the following two conditions:*

$$\Re\lambda_i[(I - \xi N)^{-1}(L + \xi M)] < 0 \quad \textit{for all} \quad i \quad \textit{and} \quad \xi \in \boldsymbol{C} \quad \textit{such as} \quad |\xi| = 1$$

and

$$\rho(N) < 1.$$

PROOF: It is an immediate consequence that the assumptions of the Theorem 2.1 imply the above two conditions. Assume the above two assumptions hold. Since the real part of every eigenvalue

$$\Re\lambda_i[(I - zN)^{-1}(L + zM)]$$

is a harmonic function whenever $|z| \leq 1$, we can show that the maximum of $\Re\lambda_i[(I - zN)^{-1}(L + zM)]$ over $|z| \leq 1$ attains at z such as $|z| = 1$, which completes the proof. □

Assume that the two conditions of Theorem 2.1 are satisfied and introduce the matrix

$$Q(\xi) = (I - \xi N)^{-1}(L + \xi M). \tag{2.10}$$

Then we have the inequality

$$\Re\lambda_i(Q(\xi)) < 0 \quad \text{for all} \quad i \quad \text{and} \quad \xi \in \boldsymbol{C}; |\xi| \leq 1.$$

3 Stability region of linear multistep methods.

For initial value problem of ODEs

$$\dot{y}(t) = f(t, y(t)) \quad t \geq 0 \qquad \text{and} \qquad y(0) = y_0,$$

a linear k-step method is given in a standard form as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{3.1}$$

where h stands for the stepsize and α_j, β_j are the formula parameters. The characteristic polynomial of the method (3.1) is

$$\pi(z; \hat{h}) = \sum_{j=0}^{k} (\alpha_j - \hat{h}\beta_j) z^j, \tag{3.2}$$

where $\hat{h} = h\lambda$ and λ is a complex parameter with negative real part. $\pi(z; \hat{h})$ is also called the stability polynomial [15] of the method (3.1).

The linear multistep (LM in short) method (3.1) is said to be absolutely stable for a certain h in ODE-sense if all the roots z_i of $\pi(z; \hat{h})$ are less than unity in magnitude for $\hat{h} = h\lambda$. Furthermore, a region $\mathcal{R}_{LM}$ in the complex $\hat{h}$-plane is said to be the *region of absolute stability* if for all $\hat{h} \in \mathcal{R}_{LM}$ the method is absolutely stable. More restrictedly, the method is said to be A-stable if $\mathcal{R}_{LM}$ includes the left half-plane of $\hat{h}$.

Let the method (3.1) be applied to a system of NDDEs with $h = \tau/m$ where m is a positive integer. For the application to the linear test equation (1.2), by introducing

$$\dot{u}(t) = v(t)$$

the system (1.2) can be written

$$\dot{u}(t) = v(t), \quad v(t) = Lu(t) + Mu(t - \tau) + Nv(t - \tau). \tag{3.3}$$

A characterization of the region of absolute stability in NDDE-sense can be given in the following.

THEOREM 3.1. *If*

(i) the assumptions of Theorem 2.1 hold,

(ii) $h\lambda_i(Q(\xi)) \in \mathcal{R}_{LM}$ *for all* i *and* ξ *such as* $|\xi| = 1$,

(iii) $h = \tau/m$, *where* m *is a positive integer,*

then the linear multistep method applied to NDDEs (1.2) is asymptotically stable.

PROOF: Consider the method (3.1) be applied to (3.3), we have

$$\sum_{j=0}^{k} \alpha_j u_{n+j} = h \sum_{j=0}^{k} \beta_j v_{n+j} \tag{3.4}$$

and

$$v_{n+j} = Lu_{n+j} + Mu_{n-m+j} + Nv_{n-m+j}. \tag{3.5}$$

The former yields

$$hN \sum_{j=0}^{k} \beta_j v_{n-m+j} = N \sum_{j=0}^{k} \alpha_j u_{n-m+j}. \tag{3.6}$$

At the same time we obtain

$$\sum_{j=0}^{k} \alpha_j u_{n+j} = h \sum_{j=0}^{k} \beta_j (Lu_{n+j} + Mu_{n-m+j}) + hN \sum_{j=0}^{k} \beta_j v_{n-m+j},$$

which, together with (3.6), implies

$$\sum_{j=0}^{k} \alpha_j u_{n+j} = h \sum_{j=0}^{k} \beta_j (Lu_{n+j} + Mu_{n-m+j}) + N \sum_{j=0}^{k} \alpha_j u_{n-m+j}.$$

The characteristic polynomial of the above difference formula is

$$P_{LM}(z) \equiv \det[I \sum_{j=0}^{k} \alpha_j z^j - Nz^{-m} \sum_{j=0}^{k} \alpha_j z^j - h \sum_{j=0}^{k} \beta_j (Lz^j + Mz^{-m+j})].$$

Assume that $|z| \geq 1$ is a root of $P_{LM}(z)$. Since $\det[I - z^{-m}N] \neq 0$ holds, by virtue of Schur's unitary upper-triangularization theorem we can calculate as

$$\begin{aligned} P_{LM}(z) &= \det[I - z^{-m}N] \det \left[\sum_{j=0}^{k} \left(\alpha_j I - h\beta_j Q(z^{-m})\right) z^j \right] \\ &= \det[I - z^{-m}N] \prod_{i=1}^{d} \pi(z; h\sigma_i), \end{aligned}$$

where σ_i is an eigenvalue of $Q(z^{-m})$. From the definition of $Q(\xi)$, we obtain $\Re\lambda_i(Q(z^{-m})) < 0$ whenever $|z| \geq 1$. The assumption of the Theorem, together with Corollary 2.1, implies $h\lambda_i(Q(z^{-m})) \in \mathcal{R}_{LM}$ for $|z| \geq 1$. Hence the definition of $\pi(z;\hat{h})$ implies the inequality

$$P_{LM}(z) \neq 0 \quad \text{for} \quad z \in \boldsymbol{C}; |z| \geq 1,$$

which completes the proof. □

The Theorem tells that the stability region of a linear multistep method applied to NDDEs is governed by the eigenvalues of $Q(\xi)$ with ξ of unit magnitude, and that if all the eigenvalues multiplied by the stepsize fall into the stability region of the LM for ODEs, then LM is asymptotically stable for NDDEs.

4 Stability region of Runge-Kutta methods.

Linear stability analysis and regions of stability are derived for a natural extension of the Runge-Kutta (RK) method to NDDEs. We consider an application of the s-stage Runge-Kutta method in ODE-case to (1.2). As in the previous section, we employ the stepsize h as an integral fraction of τ and the step-points $t_n = nh$ $(n = 0, 1, \ldots)$. Denoting the stage values of RK formula by $K_{n,i}$, we can obtain the RK scheme for (3.3) as follows.

$$\text{(4.1)} \quad K_{n,i} = hL\Big(u_n + \sum_{j=1}^{s} a_{ij}K_{n,j}\Big) + hM\Big(u_{n-m} + \sum_{j=1}^{s} b_j(c_i)K_{n-m,j}\Big) + N\Big(\sum_{j=1}^{s} c_{ij}K_{n-m,j}\Big),$$

$$\text{(4.2)} \quad u_{n+1} = u_n + \sum_{i=1}^{s} b_i K_{n,i}.$$

Here a_{ij}, b_i and c_i stand for the parameters of the underlying Runge-Kutta method, whereas $b_i(\theta)$ and c_{ij} are given below. We employ the matrix notations

$$A = (a_{ij}), \qquad B = (b_j(c_i)), \qquad \text{and} \qquad C = (c_{ij}) \quad (1 \leq i, j \leq s),$$

where, $b_j(\theta), j = 1, \cdots, s$ are polynomials which define the continuous extension of the RK method and $c_{ij} = \dfrac{db_j}{d\theta}(c_i)$ (see [4, 18]). If the conditions $A = B$ and $C = I$ hold, we call it the *natural Runge-Kutta method* for NDDEs. Moreover we introduce the following notation.

$$\boldsymbol{b} = (b_1, b_2, \cdots, b_s)^T.$$

In the sequel, we only consider the natural RK case.

The scheme in (4.1) and (4.2) then becomes

$$K_{n,i} = hL(u_n + \sum_{j=1}^{s} a_{ij}K_{n,j}) + hM(u_{n-m} + \sum_{j=1}^{s} a_{ij}K_{n-m,j}) + NK_{n-m,i} \tag{4.3}$$

and

$$u_{n+1} = u_n + \sum_{i=1}^{s} b_i K_{n,i}. \tag{4.4}$$

The stability function of the underlying RK method is given by

$$r(\hat{h}) = 1 + \hat{h}\boldsymbol{b}^T(I - \hat{h}A)^{-1}\boldsymbol{e} = \frac{\det[I - \hat{h}(A - \boldsymbol{e}\boldsymbol{b}^T)]}{\det(I - \hat{h}A)}$$

in ODE-case (see [15]), where $\boldsymbol{e} = (1, 1, \ldots, 1)^T$, $\hat{h} = h\lambda$ and $\Re\lambda < 0$. The region

$$\mathcal{R}_{RK} = \left\{\hat{h} \in \boldsymbol{C}; |r(\hat{h})| < 1\right\} \tag{4.5}$$

is called the region of absolute stability of the RK method. The A-stability of the RK method is similarly introduced as in LM case.

The following is a sufficient condition of stability of explicit natural RK for NDDEs.

THEOREM 4.1. *Assume that*

(i) the assumptions of Theorem 2.1 hold;

(ii) $h\lambda_i(Q(\xi)) \in \mathcal{R}_{RK}, i = 1, 2, \cdots, d$ *for* $|\xi| = 1$;

(iii) the underlying RK method is explicit and $A = B, C = I$;

(iv) $h = \tau/m$ *where* m *is a positive integer.*

Then, the natural RK scheme in (4.3) and (4.4) for (1.2) is asymptotically stable.

PROOF: The equalities (4.3) and (4.4) can be rewritten through the Kronecker product as follows.

$$\begin{bmatrix} I_{sd} - h(A \otimes L) & 0 \\ -\boldsymbol{b}^T \otimes I_d & I_d \end{bmatrix} \begin{bmatrix} \boldsymbol{K}_n \\ u_{n+1} \end{bmatrix} - \begin{bmatrix} 0 & h(\boldsymbol{e} \otimes L) \\ 0 & I_d \end{bmatrix} \begin{bmatrix} \boldsymbol{K}_{n-1} \\ u_n \end{bmatrix}$$
$$- \begin{bmatrix} h(A \otimes M) + I_s \otimes N & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{K}_{n-m} \\ u_{n-m+1} \end{bmatrix}$$
$$- \begin{bmatrix} 0 & h(\boldsymbol{e} \otimes M) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{K}_{n-m-1} \\ u_{n-m} \end{bmatrix} = 0,$$

where $\boldsymbol{K}_n = (K_{n,1}, K_{n,2}, \ldots, K_{n,s})^T$. The characteristic equation of the above difference equation turns out to

$$\det P_{RK}(z) = 0$$

where

$$P_{RK}(z) = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},$$

$$T_1 = z^{m+1}[I_{sd} - h(A \otimes L)] - z[h(A \otimes M) + I_s \otimes N],$$

$$T_2 = -z^m h\boldsymbol{e} \otimes (L + z^{-m}M), \quad T_3 = -z^{m+1}(\boldsymbol{b}^T \otimes (I_d)),$$

$$T_4 = (z^{m+1} - z^m)I_d.$$

To prove the Theorem, it suffices to show the implication

$$\det P_{RK}(z) = 0 \quad \Rightarrow \quad |z| < 1.$$

We will first prove

$$\det T_1 \neq 0 \quad \text{for} \quad |z| \geq 1.$$

A calculation yields

$$\begin{aligned} T_1 &= z^{m+1}[(I_{sd} - h(A \otimes L)) - z^{-m}(h(A \otimes M) + I_s \otimes N)] \\ &= z^{m+1}[(I_s \otimes (I_d - z^{-m}N)) - A \otimes h(L + z^{-m}M)]. \end{aligned}$$

Due to the condition (2.3) of Theorem 2.1, we have

$$\det[I_d - z^{-m}N] \neq 0 \quad \text{for} \quad z \in \boldsymbol{C}; |z| \geq 1,$$

which derives

$$T_1 = z^{m+1}(I_s \otimes (I_d - z^{-m}N))[I_{sd} - (I_s \otimes (I_d - z^{-m}N))^{-1}(A \otimes h(L + z^{-m}M))].$$

The definition of $Q(\xi)$, together with Schur's unitary upper-triangularization theorem, enables us to obtain

$$\begin{aligned} (4.6)\ T_1 &= z^{m+1}(I_s \otimes (I_d - z^{-m}N))(I_{sd} - hA \otimes Q(z^{-m})) \\ \det T_1 &= \left\{\det(z^{m+1}I_s)\right\}^s \left\{\det(I_d - z^{-m}N)\right\}^d \prod_{i=1}^{s} \det(I_d - h\alpha_i Q(z^{-m})), \end{aligned}$$

where $\alpha_i = \lambda_i(A) \quad (i = 1, 2, \ldots, s)$. Since the RK method is explicit, in fact all α_i's vanish. Hence $\det T_1$ cannot vanish for z such as $|z| \geq 1$. Then we obtain the identity

$$\det P_{RK}(z) = \det T_1 \cdot \det(T_4 - T_3 T_1^{-1} T_2).$$

Thus it is sufficient to show

$$\det(T_4 - T_3 T_1^{-1} T_2) \neq 0 \quad \text{for} \quad z \in \boldsymbol{C}; |z| \geq 1. \tag{4.7}$$

Assume $z(|z| \geq 1)$ is a root of the left-hand side of (4.7). Eq. (4.7) and the definitions of T_2, T_3 and T_4 imply

$$\begin{aligned} &T_4 - T_3 T_1^{-1} T_2 \\ &= z^m[zI_d - I_d - (\boldsymbol{b}^T \otimes I_d)(I_{sd} - hA \otimes Q(z^{-m}))^{-1} \\ &\times (I_s \otimes (I_d - z^{-m}N)^{-1})(h\boldsymbol{e} \otimes (L + z^{-m}M))] \\ &= z^m[zI_d - I_d - (\boldsymbol{b}^T \otimes I_d)(I_{sd} - hA \otimes Q(z^{-m}))^{-1}(h\boldsymbol{e} \otimes Q(z^{-m}))] \\ &= z^m[zI_d - I_d - (\boldsymbol{b}^T \otimes Q(z^{-m}))(I_{sd} - hA \otimes Q(z^{-m}))^{-1}(h\boldsymbol{e} \otimes I_d)] \end{aligned}$$

which gives

$$\det[T_4 - T_3T_1^{-1}T_2] = z^{md}\det[zI_d - r(hQ(z^{-m}))].$$

This means that the root z should satisfy

$$\det[zI_d - r(hQ(z^{-m}))] = 0.$$

Since $\Re\lambda_i(Q(z^{-m})) < 0$ for $|z| \geq 1$ and $\lambda_i(A) = 0,\ (i = 1, 2, \dots, s)$, $\det(I_s - h\sigma A)$ cannot vanish whenever σ is in a neighborhood of $\lambda_i(Q(z^{-m}))$ $(i = 1, 2, \dots, d)$. Thus $r(h\sigma)$ is holomorphic with respect to σ in the neighborhood of $\lambda_i(Q(z^{-m}))$. From the spectral mapping theorem, we have the identity

$$\lambda_i[r(hQ(z^{-m}))] = r[\lambda_i(hQ(z^{-m}))].$$

Henceforth from the condition (ii) of the Theorem, we can conclude

$$|r[\lambda_i(hQ(z^{-m}))]| < 1 \quad \Rightarrow \quad |\lambda_i[r(hQ(z^{-m}))]| < 1,$$

which means

$$\det[zI_d - r(hQ(z^{-m}))] = 0 \quad \Rightarrow \quad |z| < 1.$$

This contradicts the assumption $|z| \geq 1$ and the proof is completed. □

Similarly to Theorem 3.1 for LM, the above Theorem characterizes the region of absolute stability of an explicit RK for NDDEs. Again, the eigenvalues of $Q(\xi)$ with ξ of unit magnitude governs the stability of a natural explicit RK. Now we discuss the stability of implicit Runge-Kutta method and can attain the following result.

THEOREM 4.2. *Assume that*

(i) the assumptions of Theorem 2.1 hold;

(ii) the underlying RK is A-stable for ODEs;

(iii) $A = B, C = I, \Re\lambda_i(A) \geq 0$ *for* $i = 1, 2, \dots, s$;

(iv) $h = \tau/m$ *where* m *is a positive integer.*

Then, the resulting difference system of (4.3) and (4.4) corresponding to (1.2) is asymptotically stable.

PROOF: The proof can be carried out similarly to that of Theorem 4.1. Note that the assumption $\Re\lambda_i(A) \geq 0$ ensures $\det T_1 \neq 0$ for $|z| \geq 1$. Thus the calculations on the determinant for $P_{RK}(z)$ are again available, and the definition of A-stability implies the conclusion.

Roughly speaking, a natural implicit RK is stable for NDDEs when the underlying RK for ODEs is A-stable and all the eigenvalues of the coefficient matrix A have nonnegative real parts, provided the eigenvalues of $Q(\xi)$ with ξ of unit magnitude are of negative real parts. An A-stable implicit RK with $\Re\lambda_i(A) \geq 0$ really exists. Remember, for instance, the two-stage fourth order Butcher-Kuntzmann formula.

5 Stability region of NDDEs with multiple delays.

Results in the previous sections are readily extended to the multiple delay case. Consider the neutral delay-differential equations

$$\dot{u}(t) = Lu(t) + \sum_{j=1}^{J}[M_j u(t - j\tau) + N_j \dot{u}(t - j\tau)] \tag{5.1}$$

where L, M_j and N_j are constant complex matrices, J is a positive integer and $\tau > 0$.

The following is an extension of Theorem 2.1.

THEOREM 5.1. *The system (5.1) is asymptotically stable if the conditions*

$$\Re\lambda_i[(I - \sum_{j=1}^{J}\xi^j N_j)^{-1}(L + \sum_{j=1}^{J}\xi^j M_j)] < 0, \quad \text{for all} \quad i \quad \text{and} \quad \xi \in \boldsymbol{C}; |\xi| \leq 1$$

and

$$\det[I - \sum_{j=1}^{J}\xi^j N_j] \neq 0, \quad \text{for} \quad \xi \in \boldsymbol{C}; |\xi| \leq 1.$$

hold.

The proof of the Theorem is carried out in parallel to those of Theorem 2.1. For NDDEs (5.1), the matrix

$$\mathcal{Q}(\xi) = (I - \sum_{j=1}^{J}\xi^j N_j)^{-1}(L + \sum_{j=1}^{J}\xi^j M_j)$$

plays the role of $Q(\xi)$ in the proofs of Corollary 2.1. Theorems 3.1, 4.1 and 4.2 lead us to similar results as these statements for the applications of LM and RK to NDDEs (5.1).

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