A Characterization of Polyhedral Market Games

By L. J. Billera [1]) and R. E. Bixby [2])

Abstract: The class of games without side payments obtainable from markets having finitely many commodities and continuous concave utility functions is considered. It is first shown that each of these so-called market games is totally balanced, for a reasonable generalization of the idea of a balanced side payment game. It is then shown that among polyhedral games (i.e., games for which each ($V(S)$ is a polyhedron), this property characterizes the market games.

1. Introduction

The idea of obtaining an n-person game from an economic market is due to Shapley, and in Shapley and Shubik [1969], the authors characterize those games with side payments which are obtainable from markets with continuous concave utility functions. They show that the market games are the same as the totally balanced games (which in the side payment theory coincide with the games for which each subgame has a nonempty core). They also raise the question of characterizing market games without side payments.

In Billera and Bixby [1972], the authors showed that any "reasonable" set (i.e., a set of the form $C - R^n_+$, where $C \subset R^n$ is compact and convex) can be realized as the set of attainable utility outcomes for a market with at most $n(n-1)$ commodities, and continuous concave utility functions. Therefore, the question of characterizing market games without side payments is reduced to one of finding a condition relating the various sets $V(S)$ to one another. By generalizing the notion of balanced side payment games (as is discussed in Billera [1972]), one obtains a definition of balanced game which is stronger than that used by Scarf. In section 2, we prove that a market game (coming from a market with continuous concave utility functions) is always totally balanced in this sense. In section 3, we prove that for polyhedral games, the converse of this statement is also true, i.e., that totally balanced polyhedral games are always market games.

2. Market Games

Let $N = \{1,\ldots,n\}$ and, for $\emptyset \neq S \subseteq N$, let $R^S = \{x \in R^n \mid x_i = 0 \text{ for } i \notin S\}$. Further let $R^n_+ = [0,\infty)^n$ and $R^S_+ = R^S \cap R^n_+$. We will define an (*n-person co-*

[1]) Professor Louis J. Billera, Department of Operations Research, Cornell University, Ithaca, New York. Partially supported by the National Science Foundation under grants GK 29838 and GP 32314X.

[2]) Professor Robert E. Bixby, Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506. Partially supported by a National Science Foundation Graduate Fellowship and by National Science Foundation grant GK 29838 at Cornell University.

operative) game (*without side payments*) to be a function V which assigns to each $S \in 2^N = \{S \subseteq N \mid S \neq \emptyset\}$ a subset $V(S) \subseteq R^n$ of the form $V(S) = C_S - R^S_+$ where $C_S \subset R^S$ is nonempty compact and convex.

Let $I^m = [0,1]^m$ be the unit m-cube, where m is a nonnegative integer ($I^0 = \{0\}$). Suppose for each $i \in N$ we are given a concave, continuous function $u_i : I^m \to R$, and a point $\omega^i \in I^m$. (In what follows, we may assume, with no loss of generality, that $\sum_{i \in N} \omega^i \leqq e^m = (1,\ldots,1) \in I^m$.) The collection $\{(u_i, \omega^i) \mid i \in N\}$ will be called an *n-trader, m-commodity market* (see, e.g., SHAPLEY and SHUBIK [1969] or SCARF [1967]).

Given a market $\{(u_i, \omega^i) \mid i \in N\}$, we derive a game as follows. As in BILLERA and BIXBY [1972], we define the *attainable set* for $S \in 2^N$ to be

$$V(S) = \left\{x \in R^S \mid x_i \leqq u_i(y^i), y^i \in I^m, i \in S; \sum_{i \in S} y^i = \sum_{i \in S} \omega^i\right\}.$$

It follows from Theorem 2.3 of that paper that the function V so defined is a game, called the *market game* of the given market. We note here that a given market game may arise from more than one market, and "having the same game" is an equivalence on the set of all markets.

Given any game V, it also follows from the above mentioned theorem that for each $S \in 2^N$, there is a market such that $V(S)$ is the attainable set for S with respect to that market. The problem of characterizing market games is the problem of finding for which games can a single market give $V(S)$ for all S.

We first derive a necessary condition. Given a game V on N and a nonempty subset $T \subseteq N$, we define the *subgame of V on T* to be the restriction of V to 2^T. We say a game V on N is *balanced* if

$$V(N) \supseteq \sum_{S \subseteq N} \delta_S V(S)$$

whenever $\delta_S \geqq 0$, $S \subseteq N$, are such that $\sum_{S \ni i} \delta_S = 1$ for each $i \in N$. The subgame of V on T is said to be balanced if it is balanced as a game on T, i.e.,

$$V(T) \supseteq \sum_{S \subseteq T} \delta_S V(S)$$

whenever $\delta_S \geqq 0$, $S \subseteq T$, and $\sum_{S \ni i} \delta_S = 1$ for each $i \in T$. The game V is said to be *totally balanced* if it and each of its subgames is balanced (see SHAPLEY and SHUBIK [1969]).

Theorem 2.1:

A market game is always totally balanced.

Proof:

Since any subgame of a market game V is again a market game, we need only show that V is balanced. So, let $x_S \in V(S)$ for each $S \in 2^N$ and suppose $\delta_S \geqq 0$, $S \subseteq N$, and $\sum_{S \ni i} \delta_S = 1$ for $i \in N$. We must show $x = \sum_{S \subseteq N} \delta_S x_S \in V(N)$.

Let $\{(u_i, \omega^i) \mid i \in N\}$ be a market which gives rise to V. For each $S \in 2^N$, by definition of $V(S)$, there are $y_S^i \in I^m$, $i \in S$, such that $\sum_{i \in S} y_S^i = \sum_{i \in S} \omega^i$ and $(x_S)_i \leqq u_i(y_S^i)$ for each $i \in S$. Define $z^i \in I^m$, $i \in N$, by $z^i = \sum_{S \ni i} \delta_S y_S^i$.

To show $x \in V(N)$, we will show first that $\sum_{i \in N} z^i = \sum_{i \in N} \omega^i$, and then that $x_i \leqq u_i(z^i)$ for $i \in N$. Now,

$$\sum_{i \in N} z^i = \sum_{i \in N} \sum_{S \ni i} \delta_S y_S^i = \sum_{S \subseteq N} \delta_S \sum_{i \in S} y_S^i = \sum_{i \in N} \omega^i \sum_{S \ni i} \delta_S = \sum_{i \in N} \omega^i .$$

Also, since u_i is concave,

$$u_i(z^i) = u_i\left(\sum_{S \ni i} \delta_S y_S^i\right) \geqq \sum_{S \ni i} \delta_S u_i(y_S^i) \geqq \sum_{S \ni i} \delta_S (x_S)_i = x_i .$$

Note that the notion of balanced game defined above is stronger than that used by SCARF [1967], and in SHAPLEY [1973] and BILLERA [1970]. An example is given in BILLERA [1972]. In order to assure that a game V (for which the sets $V(S)$ are not necessarily convex) is balanced in SCARF's sense, it is enough to assume that it arises from a market having quasi-concave u_i. In order to guarantee the property of balanced defined above, concave u_i are necessary. By working with the stronger balanced property and games with convex $V(S)$, we are losing some of the generality implicit in SCARF [1967], but we remain within the class of markets considered by SHAPLEY and SHUBIK. We suspect that some of this generality can be recovered without too much difficulty.

We conjecture that the converse of Theorem 2.1 is true, i.e., that the property of being totally balanced characterizes market games. In the next section we prove this is true for a special class of games (those for which each $V(S)$ is a polyhedron).

For $x \in R^n$, let $x(S)$ denote its projection onto R^S. If V is a game on N then let $V + x$ be the game for which $(V + x)(S) = V(S) + x(S)$. The following is an easy consequence of Lemma 2.2 in BILLERA and BIXBY [1972].

Proposition 2.2:

If V is a market game on N, then so is $V + x$ for any $x \in R^n$.

As a consequence of the above, we will assume that all games V have the property that for each S, $V(S) = C_S - R_+^S$ where C_S is a nonempty compact, convex subset of R_+^S such that $C_S \cap \{x \in R^S \mid x_i > 0 \text{ for } i \in S\} \neq \emptyset$.

3. Polyhedral Games

We will say a game V is *polyhedral* if for each $S \in 2^N$, the set C_S is a compact polyhedron in R_+^S (or if polyhedral C_S can be found). The following will prove useful in our consideration of balanced polyhedral games.

Definition

Let V be a game on N and let $\overline{V(N)}$ denote the Pareto surface of $V(N)$, i.e., the set of maximal elements with respect to the usual partial order on R^n (see

BILLERA and BIXBY [1972]). We say $x \in \overline{V(N)}$ is in the *strong core* of V if for each $S \in 2^N$ and each $y \in V(S)$ there exists $z \in \overline{V(N)}$ such that $z(N \backslash S) = x(N \backslash S)$ and $z(S) \geqq y$.

We note here that the strong core is a subset of the core [SCARF, 1967; SHAPLEY, 1973], while it in some sense generalizes the idea of the core of a game with side payments [SHAPLEY and SHUBIK, 1969].

Lemma 3.2:

Suppose $\pi \in R^n$ and $\pi > 0$. Let V be a balanced game on N with

$$V(N) = \{x \in R^n_+ \mid \langle x,\pi \rangle \leqq 1\} - R^n_+ .$$

Then V has a nonempty strong core.

Proof:

For each $S \in 2^N$, define $v(S) = \max\{\langle x,\pi \rangle \mid x \in V(S)\}$, and suppose $v(S)$ is attained by $y_S \in V(S) \cap R^S_+$ (recall the remark following Proposition 2.2). Hence $v(S) \geqq 0$ for all S. Since V is balanced, it follows that for any $\delta_S \geqq 0$, $S \in 2^N$, such that $\sum_{S \ni i} \delta_S = 1$ for $i \in N$, we must have $\sum_{S \subseteq N} \delta_S y_S \in V(N)$, i.e.,

$$1 \geqq \langle \sum \delta_S y_S, \pi \rangle = \sum \delta_S \langle y_S, \pi \rangle = \sum \delta_S v(S) .$$

Hence by linear programming duality, there exists an $\bar{x} \in R^n$ such that $\langle \pi,\bar{x} \rangle = 1$ and $\langle \pi,\bar{x}(S) \rangle \geqq v(S)$ for each $S \in 2^N$ (this is essentially the proof of the SHAPLEY-BONDAREVA theorem on the core). Since $v(\{i\}) \geqq 0$ for $i \in N$, we have $\bar{x} \geqq 0$, and hence $\bar{x} \in \overline{V(N)}$.

To show $\bar{x}$ is in the strong core of V, let $S \in 2^N$ and $y \in V(S)$. Assume without loss of generality that $y \geqq 0$. Choose $j \in S$ and define $z \in R^n$ by $z(N \backslash S) = \bar{x}(N \backslash S)$, $z(S \backslash \{j\}) = y(S \backslash \{j\})$ and $z_j = (1/\pi_j)(1 - \langle \pi, z(N \backslash j) \rangle)$. Clearly $\langle \pi, z \rangle = 1$ and $z_i \geqq y_i$ for $i \in S \backslash \{j\}$. If $y_j > z_j$, then

$$\begin{aligned} \langle z(S),\pi \rangle < \langle y,\pi \rangle &\leqq v(S) \\ &\leqq \langle \bar{x}(S),\pi \rangle \\ &= 1 - \langle z(N \backslash S),\pi \rangle \\ &= \langle z(S),\pi \rangle , \end{aligned}$$

and so $y_j \leqq z_j$, which shows $z(S) \geqq y$, and $z \geqq 0$. Hence $z \in \overline{V(N)}$, which proves $\bar{x}$ is in the strong core.

We comment here that Lemma 3.2 is, in general, false when $V(N)$ is not of this special form. In fact, examples can be found of balanced games with empty strong cores, and games with nonempty strong cores which are not balanced.

Lemma 3.3:

Suppose $\pi \in R^n$ and $\pi > 0$. Let V be a game on N with

$$V(N) = \{x \in R^n_+ \mid \langle x,\pi \rangle \leqq 1\} - R^n_+ ,$$

and suppose V has a nonempty strong core. Then there is a market game $\hat{V}$ on N (coming from a 1-commodity market) such that

$$\hat{V}(S) \supseteq V(S) \quad \text{for} \quad S \in 2^N, \quad \text{and} \tag{3.3.1}$$

$$\hat{V}(N) = V(N). \tag{3.3.2}$$

Proof:

For each $i \in N$, let $u_i : I \to R$ be given by $u_i(x) = x/\pi_i$ for $x \in I$, and let $\omega^i = \pi_i y_i \in I$ where y is a point in the strong core of V. Note $\sum_{i \in N} \omega^i = 1$. Let $\hat{V}$ be the market game given by the 1-commodity market $\{(u_i, \omega^i) \mid i \in N\}$.

For any $S \in 2^N$, let $x_S \in V(S)$. Since y is in the strong core of V, we have $z \in R^n_+$ such that $\langle z, \pi \rangle = 1$, $z(N \backslash S) = y(N \backslash S)$ and $z(S) \geqq x_S$. But

$$\begin{aligned}\sum_{i \in S} z_i \pi_i &= 1 - \langle z(N \backslash S), \pi \rangle \\ &= 1 - \langle y(N \backslash S), \pi \rangle = \sum_{i \in S} \omega^i,\end{aligned}$$

and so $x_S \in \hat{V}(S)$ since for each $i \in S$ we have $u_i(z_i \pi_i) = z_i \geqq x_i$. Thus $V(S) \subseteq \hat{V}(S)$.

Now suppose $x \in \hat{V}(N)$. Then there are $a^i \in I$, $i \in N$, so that $\sum_{i \in N} a^i = \sum_{i \in N} \omega^i = 1$, and $x_i \leqq u_i(a^i) = a^i/\pi_i$. Let $\bar{x} = (a_1/\pi_1, \ldots, a_n/\pi_n)$. We have $\bar{x} \geqq 0$ and $\langle \pi, \bar{x} \rangle = 1$ which implies $\bar{x} \in V(N)$. Since $x \leqq \bar{x}$, we must also have $x \in V(N)$, proving (3.3.2).

Suppose Γ is any index set and $\{V_\gamma \mid \gamma \in \Gamma\}$ is a collection of games, all on the same set N. We define the intersection of the collection of games, $\bigcap_\gamma V_\gamma$, by $\left(\bigcap_\gamma V_\gamma\right)(S) = \bigcap_\gamma V_\gamma(S)$. It is easy to verify that $\bigcap_\gamma V_\gamma$ is a game if either Γ is finite or if each V_γ satisfies the convention made following Proposition 2.2.

Proposition 3.4:

The intersection of any finite collection of market games is a market game.

Proof:

It is enough to show that the intersection of two market games is a market game. For $\gamma = 1,2$, let V_γ be the market game on N given by the n-trader, m_γ-commodity market $\{(u_i^\gamma, \omega_\gamma^i) \mid i \in N\}$. It follows as in Lemma 2.2 of Billera and Bixby [1972] that $V_1 \cap V_2$ is the market game given by the n-trader, $(m_1 + m_2)$-commodity market $\{(u_i^1 \wedge u_i^2, \omega_1^i \oplus \omega_2^i) \mid i \in N\}$ where $\omega_1^i \oplus \omega_2^i = (\omega_1^i, \omega_2^i)$ and for $(x,y) \in I^{m_1+m_2}$, $(u_i^1 \wedge u_i^2)(x,y) = u_i^1(x) \wedge u_i^2(y)$ (where $a \wedge b = \min(a,b)$).

Theorem 3.5:

Let A be a finite index set, and for each $\alpha \in A$, let $\pi^\alpha \in R^n$ and $\pi^\alpha > 0$. Suppose V is a balanced game on N with

$$V(N) = \{x \in R^n_+ \mid \langle \pi^\alpha, x \rangle \leqq 1 \text{ for all } \alpha \in A\} - R^n_+ .$$

Then there is a market game $\hat{V}$ on N (coming from a $|A|$-commodity market) such that

$$\hat{V}(S) \supseteq V(S) \quad \text{for} \quad S \in 2^N, \quad \text{and} \tag{3.5.1}$$

$$\hat{V}(N) = V(N). \tag{3.5.2}$$

Proof:

For each $\alpha \in A$, let V_α be the game on N with $V_\alpha(S) = V(S)$ for $S \neq N$, and

$$V_\alpha(N) = \{x \in R^n_+ \mid \langle \pi^\alpha, x \rangle \leqq 1\} - R^n_+ .$$

Since V is balanced, so is each V_α, and hence by Lemma 3.2, each V_α has a non-empty strong core. It follows from Lemma 3.3, that for each $\alpha \in A$, there is a (1-commodity) market game $\hat{V}_\alpha$ such that $\hat{V}_\alpha(S) \supseteq V_\alpha(S)$ for each $S \in 2^N$, and $\hat{V}_\alpha(N) = V_\alpha(N)$. Setting $\hat{V} = \bigcap_\alpha \hat{V}_\alpha$, it follows from Proposition 3.4 that $\hat{V}$ is a ($|A|$-commodity) market game. $\hat{V}$ clearly satisfies (3.5.1) and (3.5.2).

We note here that the market giving the game $\hat{V}$ in Lemma 3.3, and hence that in Theorem 3.5, has the property that the u_i are nondecreasing (in each variable).

In order for us to treat the case of a general polyhedral game, we need a version of Theorem 3.5 which allows the π^α to have some zero coordinates.

Theorem 3.6:

The conclusion of Theorem 3.5 remains valid if we change the conditions on the π^α from $\pi^\alpha > 0$ to $\pi^\alpha \geqq 0$, $\sum_{\alpha \in A} \pi^\alpha > 0$.

Proof:

Let $M = \max\left\{ \sum_{i \in N} x_i \mid x \in V(N) \right\}$. For each $j = 1,2,\ldots$, define

$$\pi^{\alpha,j} = \frac{j}{M+j}\pi^\alpha + \frac{1}{M+j}e^n .$$

We have, then, that $\pi^{\alpha,j} > 0$ for each j, and $\pi^{\alpha,j} \to \pi^\alpha$.

For each j, define the game V^j by $V^j(S) = V(S)$, for $S \neq N$, and

$$V^j(N) = \{x \in R^n_+ \mid \langle x, \pi^{\alpha,j} \rangle \leqq 1, \forall \alpha \in A\} - R^n_+ .$$

Suppose $x \geqq 0$ and $x \in V(N)$. Then for each j and each $\alpha \in A$, we have

$$\langle x, \pi^{\alpha,j} \rangle = \frac{j}{M+j}\langle x, \pi^\alpha \rangle + \frac{1}{M+j}\sum_{i \in N} x_i \leqq 1 .$$

It follows that $V(N) \subseteq V^j(N)$ for each j, and hence each V^j is balanced. Therefore all the V^j satisfy the hypothesis of Theorem 3.5, and there exists for each j a market game $\hat{V}^j$ satisfying

$$\hat{V}^j(S) \supseteq V^j(S) = V(S) \quad \text{for} \quad S \neq N, \quad \text{and} \tag{3.6.1}$$

$$\hat{V}^j(N) = V^j(N) \supseteq V(N) . \tag{3.6.2}$$

Let $\{(u^j_i, \omega^i_j) \mid i \in N\}$ be the $|A|$-commodity market giving rise to $\hat{V}^j$, as guaranteed by Theorem 3.5. Implicit in the proof of that theorem is the fact that each u^j_i is given by $u^j_i(y) = \bigwedge_\alpha (y_\alpha / \pi^{\alpha,j}_i)$ for $y \in I^{|A|}$, and $\sum_{i \in N} \omega^i_j = e^{|A|}$. For each $i \in N$, let $A_i = \{\alpha \in A \mid \pi^\alpha_i > 0\}$ (which is nonempty since $\sum_{\alpha \in A} \pi^\alpha > 0$), and let u_i be given by $u_i(y) = \bigwedge_{\alpha \in A_i} (y_\alpha/\pi_i^\alpha)$ for $y \in I^{|A|}$. Assume without loss of generality that for each $i \in N$, $\omega_j^i \to \omega^i \in I^{|A|}$. It is clear that $\{(u_i, \omega^i) \mid i \in N\}$ is a market; let $\hat{V}$ be its market game. We claim $\hat{V}$ satisfies the conclusions of the theorem, i.e., (3.5.1) and (3.5.2).

Suppose $x \in \hat{V}(N) \cap R_+^n$. Then there exist $y^i \in I^{|A|}$, $i \in N$, such that $\sum_{i \in N} y^i = \sum_{i \in N} \omega^i = e^{|A|}$ and $x_i \leqq u_i(y^i) = \bigwedge_{\alpha \in A_i} (y_\alpha^i/\pi_i^\alpha)$ for each $i \in N$. For $\alpha \in A$ we have

$$\begin{aligned}\langle \pi^\alpha, x \rangle &\leqq \sum_{i \in N} \pi_i^\alpha u_i(y^i) \\ &\leqq \sum_{i \in N} y_\alpha^i = 1,\end{aligned}$$

and hence $x \in V(N)$. Thus we have shown that $\hat{V}(N) \subseteq V(N)$.

Now let $S \in 2^N$ and suppose $x \in V(S)$. By (3.6.1) and (3.6.2), $x \in \hat{V}^j(S)$ for every j. Therefore for $j = 1, 2, \ldots$, and for each $i \in S$, there exists $y_j^i \in I^{|A|}$ such that $\sum_{i \in S} y_j^i = \sum_{i \in S} \omega_j^i$ and $x_i \leqq u_i^j(y_j^i)$. Assume without loss of generality that $y_j^i \to y^i \in I^{|A|}$ for each $i \in N$. Thus $\sum_{i \in S} y^i = \lim_{j \to \infty} \sum_{i \in S} \omega_j^i = \sum_{i \in S} \omega^i$. For each $i \in N$ and j, put $\bar{u}_i^j(y) = \bigwedge_{\alpha \in A_i} (y_\alpha/\pi_i^{\alpha,j})$ for $y \in I^{|A|}$. Then for each i and j, $x_i \leqq \bar{u}_i^j(y_j^i)$ and $\bar{u}_i^j \to u_i$ uniformly on $I^{|A|}$ (this follows easily by induction on $|A_i|$). Since u_i is continuous and $y_j^i \to y^i$, it can be shown that $\bar{u}_i^j(y_j^i) \to u_i(y^i)$ and hence $x_i \leqq u_i(y^i)$. Thus $x \in \hat{V}(S)$ and we have $V(S) \subseteq \hat{V}(S)$. This concludes the proof.

We comment here that the previous proof is a special case of a more general limit theorem which can be proved. Essentially, what this theorem would say is that given a sequence of market games which come from a sequence of markets having utilities which are uniformly bounded below, and having a bounded number of commodities, then any limit point of this sequence (in the Hausdorff metric) is again a market game. In fact, it is the necessity for a bound on the number of commodities which prevents this limit theorem and the results of this paper from proving the converse to Theorem 2.1 for general V.

Lemma 3.7:

Let V be a game on N. Suppose the restriction of V to some subset $T \subseteq N$ is a market game on T. Then there exists a market game $\hat{V}$ on N such that

$$\hat{V}(R) \supseteq V(R) \quad \text{for all} \quad R \in 2^N, \quad \text{and} \tag{3.7.1}$$

$$\hat{V}(R) = V(R) \quad \text{for all} \quad R \in 2^T. \tag{3.7.2}$$

Proof:

Suppose the m-commodity market $\{(u_i, \omega^i) \mid i \in T\}$ generates the restriction of V to T. Let $M = \max\{x_i \mid x \in V(S), S \in 2^N; i \in N\}$ and $t = |T|$. Let $\hat{V}$ be the market game on N generated by the $(m+1)$-commodity market $\{(\bar{u}_i, \bar{\omega}^i) \mid i \in N\}$ where $\bar{\omega}^i \in I^{m+1}$ is given by

$$\bar{\omega}^i = \begin{cases} (\omega^i, 0) & i \in T \\ (0, 1) & i \notin T, \end{cases}$$

and $\bar{u}^i : I^{m+1} \to R$ is given by

$$\bar{u}^i(y,z) = \begin{cases} u_i(y) + (t+1)Mz & i \in T \\ (t+1)Mz & i \notin T, \end{cases}$$

for $(y,z) \in I^{m+1}$. It is easy to see that $\hat{V}$ satisfies (3.7.1) and (3.7.2).

We are now in a position to characterize the market games among the polyhedral games. First, we can view the set $V(S)$ as a subset of an $|S|$-dimensional space by ignoring the zero coordinates corresponding to $N \backslash S$. Thus it is enough to show that for a polyhedral game V, the set $V(N)$ can be expressed in the form required by Theorem 3.6.

By our convention, $V(N) = C_N - R^n_+$ where $C_N \subseteq R^n_+$ is a polyhedron containing a point $q > 0$. Therefore $V(N)$ is a polyhedron and $q \in V(N)$. Suppose

$$V(N) = \{x \in R^n \mid \langle x, \rho^\alpha \rangle \leqq b^\alpha, \alpha \in A\}$$

where A is a finite index set and $\rho^\alpha \in R^n$, $b^\alpha \in R$ for each $\alpha \in A$. First, $\rho^\alpha \in R^n_+$ for each α since $V(N)$ is unbounded in all directions contained in $-R^n_+$. Since $q \in V(N)$, we must have $b^\alpha > 0$ for each α. Thus we have

$$V(N) = \{x \in R^n \mid \langle x, \pi^\alpha \rangle \leqq 1, \alpha \in A\}$$

where $\pi^\alpha = (1/b^\alpha)\rho^\alpha$. Finally, letting $P = V(N) \cap R^n_+$, we have $V(N) = P - R^n_+$, and we have obtained the required expression for $V(N)$. That $\sum_{\alpha \in A} \pi^\alpha > 0$ follows from the fact that C_N, and hence P, is bounded.

Theorem 3.8:

A polyhedral game is a market game if and only if it is totally balanced.

Proof:

Necessity is Theorem 2.1. To show sufficiency let V be a totally balanced polyhedral game. From Theorem 3.6 and Lemma 3.7 it follows that for each $T \in 2^N$, there is a market game V_T having the properties that $V_T(S) \supseteq V(S)$ for each $S \in 2^N$, and $V_T(T) = V(T)$. But then $V = \bigcap_{T \in 2^N} V_T$ is a market game by Proposition 3.4.

We comment that the methods used here essentially realize the game V with a market having u_i's which are continuous, nondecreasing, piecewise-linear concave functions on all of R^m. Also the number of commodities $m = k + 2^n - 2$, where k is the total number of "faces" π^α needed to describe each of the sets $V(S)$. Clearly, these methods do not work for non-polyhedral games unless one is willing to admit a countably infinite dimensional commodity space, I^∞. Even in such a case, it is not clear that the resulting functions u_i are continuous for any reasonable topology on I^∞, i.e. one for which I^∞ is a compact subset of a linear topological space (see SHAPLEY [1973]). This direction is being investigated (see BILLERA [1973]).

In any case, we can say that the market games are a dense subset (say, with respect to the Hausdorff metric) of the totally balanced games. If one could find a method to bound the number of commodities needed to realize an n-person polyhedral market game (say, as was done for attainable sets in BILLERA and BIXBY [1972]), then by using the aforementioned limit theorem, one could prove Theorem 3.8 for all games. Work is continuing along these lines as well.

References

BILLERA, L. J.: Some theorems on the core of an n-person game without side payments. Siam J. Appl. Math. **18**, 567–579, 1970.

–: A note on a kernel and the core for games without side payments. Technical Report No. 152, Dept. of O. R., Cornell U., June 1972.

–: On games without side payments arising from a general class of markets. Technical Report No. 184, Dept. of O. R., Cornell U., June 1973.

– and R. E. BIXBY: A characterization of Pareto surfaces. Technical Report No. 162, Dept. of O. R., Cornell U., December 1972, (to appear in Proc. Amer. Math. Soc.).

SCARF, H.: The core of an n-person game. Econometrica **35**, 50–69, 1967.

SHAPLEY, L. S.: On balanced games without side payments. Mathematical Programming, T. C. Hu and S. M. Robinson, Editors, Academic Press, New York, 1973.

– and M. SHUBIK: On market games. Journal of Economic Theory **1**, 9–25, 1969.

Received May, 1973.