Oddness of the Number of Equilibrium Points: A New Proof

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Abstract: A new proof is offered for the theorem that, in "almost all" finite games, the number of equilibrium points is *finite* and *odd*. The proof is based on constructing a one-parameter family of games with logarithmic payoff functions, and studying the topological properties of the graph of a certain algebraic function, related to the graph of the set of equilibrium points for the games belonging to this family. In the last section of the paper, it is shown that, in the space of all games of a given size, those "exceptional" games which fail to satisfy the theorem (by having an even number or an infinity of equilibrium points) is a closed set of measure zero.

1. Introduction

WILSON [1971, Theorem 1 on p. 85] has shown that, apart from certain degenerate cases, in any finite game, the number of equilibrium points is *finite* and *odd*. The purpose of this paper is to offer a new proof for WILSON's theorem.

Let Γ be a finite noncooperative game. The k-th pure strategy of player $i\,(i = 1,\ldots,n)$ will be called a_i^k, whereas the set of all his K_i pure strategies will be called A_i. Let

$$K = \prod_{i=1}^{n} K_i\,. \tag{1}$$

We shall assume that the K possible n-tuples of pure strategies are numbered consecutively as $a^1,\ldots,a^m,\ldots,a^K$. Let

$$a^m = (a_1^{k_1},\ldots,a_i^{k_i},\ldots,a_n^{k_n})\,. \tag{2}$$

Then we shall write

$$a^m(i) = a_i^{k_i}\,, \tag{3}$$

to denote the pure strategy used by player i in the strategy n-tuple a^m. The set of all K possible pure-strategy n-tuples will be called A. We have $A = A_1\times\cdots\times A_n$.

Any mixed strategy of a given player $i\,(i = 1,\ldots,n)$ can be identified with a probability vector p_i of the form

$$p_i = (p_i^1,\ldots,p_i^k,\ldots,p_i^{K_i})\,, \tag{4}$$

where $p_i^1,\ldots,p_i^{K_i}$ are the probabilities that this mixed strategy assigns to his pure strategies $a_i^1,\ldots,a_i^{K_i}$. The set P_i of all mixed strategies available to player i is a

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simplex, consisting of all K_i-vectors satisfying the conditions

$$p_i^k \geqq 0\,, \quad \text{for} \quad k = 1,\ldots,K_i\,, \tag{5}$$

and

$$\sum_{k=1}^{K_i} p_i^k = 1\,. \tag{6}$$

The set $P = P_1 \times \cdots \times P_n$ of all possible n-tuples $p = (p_1,\ldots,p_n)$ of mixed strategies is a compact and convex polyhedron, and will be called the *strategy space* of game Γ. We shall write $p = (p_i, \overline{p_i})$, where $\overline{p_i} = (p_1,\ldots,p_{i-1},p_{i+1},\ldots,p_n)$ is the strategy $(n-1)$-tuple representing the mixed strategies of the $(n-1)$ players other than player i.

The set $C(p_i)$ of all pure strategies a_i^k to which the mixed strategy p_i assigns positive probabilities $p_i^k > 0$ is called the *carrier* of p_i. If the carrier $C(p_i)$ of a given strategy p_i contains only *one* pure strategy a_i^k, then p_i will be identified with this pure strategy $a_i^k = p_i$. On the other hand, if $C(p_i)$ contains *all* K_i pure strategies of player i, then p_i will be called a *complete(ly) mixed* strategy. Finally, if p_i is neither pure nor complete, then it will be called an *incomplete(ly) mixed* strategy.

For any strategy n-tuple $p = (p_1,\ldots,p_n)$, the *carrier* $C(p)$ of p will be defined as the union of the carriers of its component strategies, that is, as

$$C(p) = \bigcup_{i=1}^{n} C(p_i)\,. \tag{7}$$

Suppose that the i-th component of the pure-strategy n-tuple a^m is $a^m(i) = a_i^k$, and that a given mixed strategy p_i of player i assigns the probability p_i^k to this pure strategy a_i^k. Then, we shall write

$$q_i^m(p_i) = p_i^k\,. \tag{8}$$

Of course, if $p_i = a_i^k$ is a pure strategy, then we have

$$q_i^m(a_i^k) = 1\,, \quad \text{when} \quad a^m(i) = a_i^k\,, \tag{9}$$

but

$$q_i^m(a_i^k) = 0 \quad \text{when} \quad a^m(i) \neq a_i^k\,. \tag{10}$$

When the n players use the pure-strategy n-tuple a^m, then player i $(i = 1,\ldots,n)$ will receive the payoff

$$U_i(a^m) = u_i^m\,, \tag{11}$$

whereas if they use the mixed-strategy n-tuple $p = (p_1,\ldots,p_n)$, then his payoff will be

$$U_i(p) = \sum_{m=1}^{K} \left[\prod_{i=1}^{n} q_i^m(p_i) \right] u_i^m\,. \tag{12}$$

Let $\mathscr{I} = \mathscr{I}(n; K_1,\ldots,K_n)$ be the set of all n-person games in which players $1,\ldots,n$ have exactly $K_1,\ldots,K_n$ pure strategies, respectively. Thus, $\mathscr{I}$ is the set of all games of a given size. Each specific game Γ in $\mathscr{I}$ can be characterized by the (nK)-vector

$$u = (u_1^1,\ldots,u_1^{K_1};\ldots;u_i^1,\ldots,u_i^{K_i};\ldots;u_n^1,\ldots,u_n^{K_n}), \quad (13)$$

whose components $u_i^m = U_i(a^m)$ are the payoff to various players i for the different pure-strategy combinations a^m. We can identify each game Γ with its vector $u = u(\Gamma)$ of possible payoffs for pure-strategy combinations, and can regard the set $\mathscr{I}$ as an (nK)-dimensional Euclidean space $\mathscr{I} = \{u\}$.

Let $\bar{\mathscr{S}}(\mathscr{I})$ be the set of all games Γ in $\mathscr{I}$ for which a given mathematical statement $\mathscr{S}$ is false. We shall say that statement $\mathscr{S}$ is true for *almost all* games if, for every possible set $\mathscr{I}$ of games of a particular size, this set $\bar{\mathscr{S}}(\mathscr{I})$ is a *closed* set of *measure zero* within the relevant set $\mathscr{I}$, regarded as an (nK)-dimensional Euclidean space. (Concerning the closure requirement for $\bar{\mathscr{S}}(\mathscr{I})$, see DEBREU [1970, p. 387].)

2. Logarithmic Games

Let Λ be an n-person noncooperative game, where the n players have the same simplexes $P_1,\ldots,P_n$ they have in game Γ as strategy spaces, but where the payoff function L_i of each player $i(i = 1,\ldots,n)$ is of the form[2])

$$L_i(p) = L_i(p_i) = \sum_{k=1}^{K_i} \log p_i\,. \quad (14)$$

Thus, Λ is a "degenerate" game, in which each player's payoff L_i depends only on his own strategy p_i, and does not depend on the other players' strategies $p_j, j \neq i$.

Finally, we define a one-parameter family of games $\{\Lambda^*(t)\}$, with $0 \leqq t \leqq 1$. In any particular game $\Lambda^*(t)$ with a specific value of the parameter t, the payoff function of player $i(i = 1,\ldots,n)$ is

$$L^*(p,t) = (1 - t)\,U_i(p) + t\,L_i(p_i)\,. \quad (15)$$

Obviously, $\Lambda^*(0) = \Gamma$, whereas $\Lambda^*(1) = \Lambda$. All games $\Lambda^*(t)$ with $0 < t \leqq 1$ will be called *logarithmic* games. Γ will sometimes be called the *original* game, while Λ will be called the *pure* logarithmic game.

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[2]) Since the payoff function L_i are logarithmic functions (instead of being multilinear functions in the probabilities p_i^k as is the case in ordinary finite games), this game Λ – as well as the games $\Lambda^*(t)$ to be defined below – are best regarded as being *infinite* games in which the *pure* strategy of every player i consists in choosing a specific point p_i from the simplex P_i, which makes each p_i a pure strategy, rather than a mixed strategy, in game Λ (or $\Lambda^*(t)$). But, for convenience, we shall go on calling any given strategy p_i a (complete or incomplete) *mixed* strategy – even in discussing the infinite games Λ and $\Lambda^*(t)$ – if p_i would represent a (complete or incomplete) mixed strategy in the finite game Γ. This terminology will not give rise to any confusion because, in analyzing these infinite games, we shall never consider mixed strategies having the nature of probability mixtures of two or more strategies p_i.

3. Equilibrium Points

A given strategy p_i of player i will be called a *best reply* to a strategy combination $\bar{p}_i$ used by the other $(n-1)$ players in game $\Lambda^*(t)$ if

$$L^*(p_i,\bar{p}_i,t) \geqq L^*(p_i',\bar{p}_i,t) \quad \text{for all} \quad p_i' \in P_i. \tag{16}$$

A given strategy n-tuple $p = (p_1,\ldots,p_n)$ is an *equilibrium point* [NASH, 1951] in game $\Lambda^*(t)$ if every component p_i of p is a best reply to the corresponding strategy combination $\bar{p}_i$ of the other $(n-1)$ players.

An equilibrium point p is called *strong*[3]) if all n components p_i of p satisfy (16) with the strong inequality sign $>$ for all $p_i' \neq p_i$. That is, p is a strong equilibrium point if every player's equilibrium strategy p_i is his *only* best reply to the other players' strategy combination $\bar{p}_i$. An equilibrium point is called *weak* if it is not strong.

An equilibrium point p is quasi-strong[4]) if no player i has pure-strategy best replies to $\bar{p}_i$ other than the pure strategies belonging to the carrier $C(p_i)$ of his equilibrium strategy p_i. An equilibrium point that is not even quasi-strong is called *extra-weak*.

A given game Γ itself will be called *quasi-strong* if *all* its equilibrium points are quasi-strong; and it will be called *extra-weak* if at least *one* of its equilibrium points is extra-weak.

In the original game $\Gamma = \Lambda^*(0)$, a best reply p_i to any given strategy combination $\bar{p}_i$ may be a pure strategy or may be a mixed strategy. (It can be a mixed strategy only if *all* pure strategies a_i^k in its carrier $C(p_i)$ are themselves best replies to $\bar{p}_i$.) In contrast, in a logarithmic game $\Lambda^*(t)$ with $t > 0$, only a *complete* mixed strategy can be a best reply. This is so because, in view of (12), (14), and (15), any player i will obtain an infinite negative payoff $L_i^* = -\infty$ if he uses a pure or an incompletely mixed strategy, but will always obtain a finite payoff $L_i^* > -\infty$ if he uses a completely mixed strategy. Consequently, in these logarithmic games, all equilibrium points will be in completely mixed strategies.

In the original game Γ, in general, the mathematical conditions characterizing an equilibrium point $p = (p_1,\ldots,p_n)$ will be partly equations and partly inequalities. The former will be of the form:

$$U_i(a_i^k,\bar{p}_i) = U_i(a_i^{k'},\bar{p}_i) \quad \text{if} \quad a_i^k, a_i^{k'} \in C(p_i); \tag{17}$$

whereas the latter will be of the form:

$$U_i(a_i^k,\bar{p}_i) \geqq U_i(a_i^{k'},\bar{p}_i) \quad \text{if} \quad a_i^k \in C(p_i) \quad \text{while} \quad a_i^{k'} \notin C(p_i). \tag{18}$$

Only in the special case where all n equilibrium strategies $p_1,\ldots,p_n$ are *pure*

[3]) I am using the term "strong equilibrium point" in a different sense from AUMANN's [1959, p. 300].

[4]) Many of the concepts used in this paper were first introduced in HARSANYI [1973].

strategies, will all conditions characterizing p be inequalities of form (18); and only in the special case where all n equilibrium strategies are *complete* mixed strategies, will all these conditions be equations of form (17).

An equilibrium point p of game Γ will be *quasi-strong* if and only if, for every player i, and for every strategy a_i^k in the carrier $C(p_i)$ of i's equilibrium strategy, and for every strategy $a_i^{k'}$ *not* in this carrier, condition (18) is satisfied with the *strong* inequality sign $>$.

In contrast to equilibrium points in the finite game Γ, every equilibrium point p in any logarithmic game $\Lambda^*(t)$ with $t > 0$, is always characterized by *equations* of the following form:

$$\left(\frac{\partial L_i^*(p,t)}{\partial p_i^k}\right)_{\sum_i p_i^k = 1} = 0\,, \qquad \begin{array}{ll} \text{for} & k = 1,\ldots,K_i - 1 \\ \text{and for} & i = 1,\ldots,n\,. \end{array} \tag{19}$$

Here each partial derivative $\partial L_i^*/\partial p_i^k$ must be evaluated at the equilibrium point p itself. Of course, these eqs. (19) express only the first-order conditions for maximizing the payoff function L_i^* with respect to the vector p_i. But, since each function L_i^* is strictly concave in p_i, the second-order conditions are always satisfied, so that the eqs. (19) are both necessary and sufficient conditions for maximization.

The functions L_i^* can also be written as

$$L_i^*(p,t) = (1 - t)\sum_{k=1}^{K_i} p_i^k\, U_i(a_i^k,\bar{p}_i) + \sum_{k=1}^{K_i} \log p_i^k\,. \tag{20}$$

Therefore, using the fact that

$$p_i^1 = 1 - \sum_{k=2}^{K_i} p_i^k\,, \tag{21}$$

we can write (19) in the form

$$(1 - t)\left[U_i(a_i^k,\bar{p}_i) - U_i(a_i^1,\bar{p}_i)\right] + \frac{t}{p_i^k} - \frac{t}{p_i^1} = 0\,, \tag{22}$$

or, equivalently, in the form

$$\begin{array}{c} (1 - t)p_i^1 p_i^k\left[U_i(a_i^k,\bar{p}_i) - U_i(a_i^1,\bar{p}_i)\right] + t(p_i^1 - p_i^k) = 0\,, \\ \text{for} \quad i = 1,\ldots,n; \\ \text{and for} \quad k = 2,\ldots,K_i\,. \end{array} \tag{23}$$

The number of equations of form (23) is:

$$K^* = \sum_{i=1}^{n} (K_i - 1) = \sum_{i=1}^{n} K_i - n\,. \tag{24}$$

Thus, together with the n equations of form (6), we have all together

$$K^{**} = K^* + n = \sum_{i=1}^{n} K_i \tag{25}$$

independent equations for characterizing each equilibrium point p, which is the same as the number of the variables p_i^k determined by these equations.

In view of (12), all equations of form (23) are algebraic equations in the variables p_i^k and in the parameter t. All equations of form (6) are likewise algebraic. Let S be the set of all $(K^{**} + 1)$-vectors (t,p) satisfying the K^{**} equation of forms (6) and (23). Clearly, S will be typically a one-dimensional algebraic variety, i.e., an algebraic curve. (In degenerate cases, however, S may also contain zero-dimensional subsets, i.e., isolated points, and/or subsets of more than one dimension, i.e., algebraic surfaces of various dimensionality.)

Let T be the set of all vectors (t,p) satisfying, not only the K^{**} equations of forms (6) and (23), but also the K^{**} inequalities of form (5). Clearly, T is simply that part of the algebraic variety S which lies within the compact and convex cylinder (polyhedron) $R = P \times I$, where $I = [0,1]$ is the closed unit interval. Since T is the locus of all solutions (t,p) to the simultaneous equations and inequalities (5), (6), and (23), T will be called the *solution graph* for the latter.

For any point (t,p), t will be called its *first coordinate*. Within the cylinder R, the strategy space P of any specific game $\Lambda^*(t)$ is represented by the set R^t of all points (t,p) in R whose first coordinate is the relevant t value.

For any game $\Lambda^*(t)$, let E^t be the set of all points (t,p) in R^t such that p is an equilibrium point of $\Lambda^*(t)$. Finally, let T^t be the intersection of R^t with the solution graph T. We can now state:

Lemma 1:

For all t with $0 < t \leqq 1, E^t = T^t$. In contrast, for $t = 0$, in general, we have only $E^0 \subsetneq T^0$.

Proof:

For all t with $0 < t \leqq 1$, conditions (5), (6), and (23) are sufficient and necessary conditions for any given point p to be an equilibrium point of game $\Lambda^*(t)$. On the other hand, for $t = 0$, it is easy to verify that all equilibrium points p of the game $\Lambda^*(0) \mp \Gamma$ satisfy all these conditions but, in general, so will also some strategy combinations p that are *not* equilibrium points of Γ. For example, all these conditions will be satisfied by any pure-strategy n-tuple $p = a^m$, whether it is an equilibrium point of game Γ or not.

4. Some Topological Properties of the Solution Graph T

Consider the mapping $\mu: t \to T^t$. The Jacobian of this mapping, as evaluated at any given point (p,t) of T^t, can be written as

$$J(t,p) = \frac{\partial(F_1^1,\ldots,F_i^k,\ldots,F_n^{K_n})}{\partial(p_1^1,\ldots,p_i^k,\ldots,p_n^{K_n})}, \qquad i = 1,\ldots,n; \quad \text{and, for each } i, \quad k = 1,\ldots,K_i. \tag{26}$$

Here

$$F_i^1(t,p) = \sum_{i=1}^{n} p_i^k - 1, \qquad i = 1,\ldots,n; \tag{27}$$

whereas

$$F_i^k(t,p) = (1-t)p_i^1 p_i^k[U_i(a_i^k,\bar{p}_i) - U_i(a_i^1,\bar{p}_i)] + t(p_i^1 - p_i^k), \qquad i = 1,\ldots,n; \quad \text{and, for each } i, \quad k = 2,\ldots,K_i. \tag{28}$$

For points of the form $(t,p) = (0,p)$ in set T^0, the functions $F_i^k (k \neq 1)$ take the following simpler form:

$$F_i^k(0,p) = p_i^1 p_i^k[U_i(a_i^k,\bar{p}_i) - U_i(a_i^1,\bar{p}_i)], \qquad i = 1,\ldots,n; \quad \text{and, for each } i, \quad k = 2,\ldots,K_i. \tag{29}$$

Any equilibrium point p of the original game Γ will be called regular if $J(0,p) \neq 0$; and will be called *irregular* if $J(0,p) = 0$. A given game Γ itself will be called *regular* if *all* of its equilibrium points are regular; and it will be called *irregular* if at least *one* of its equilibrium points is irregular.

We shall now state two lemmas, based on well-known facts in algebraic geometry.

Lemma 2:

Let (x^0,x^*) be an arc of an algebraic curve S in a ν-dimensional Euclidean space X^ν, connecting the two points $x^0 = (x_1^0,\ldots,x_\nu^0)$ and $x^* = (x_1^*,\ldots,x_\nu^*) \neq x^0$. Then, this arc (x^0,x^*) can be uniquely continued analytically beyond point x^* (and beyond point x^0).

Proof:

If x^* is not a singular point, then the possibility of analytic continuation follows from the Implicit Function Theorem. On the other hand, if x^* is a singular point, then this possibility follows from PUISEUX's Theorem [VAN DER WAERDEN, 1939, Theorem 14]. By this theorem, if x^* is a point of some branch S^* of a given algebraic curve S, then, whether x^* is a singular point or not, in some neighborhood $N(x^*)$ of x^*, the coordinates x_i of any point $x = (x_1,\ldots,x_\nu)$ of this branch S^* can be represented by ν convergent power series $\pi_i(y)$ in an auxiliary parameter y, so that we can write $x_i = \pi_i(y)$ for a suitably chosen value of $y (i = 1,\ldots,\nu)$. Moreover, we can select the ν functions π_i in such a way that x^* itself will correspond to $y = 0$ (so that $x_i^* = \pi_i(0)$, for $i = 1,\ldots,\nu$), and in such a way that all other points

x of the arc (x^0, x^*) will correspond to negative values of y. Then, by assigning positive values to y, we can analytically continue the arc (x^0, x^*) beyond x^*. Even though we can choose the v functions π_i in many different ways, all choices will yield the same curve as the analytic continuation of (x^0, x^*).

Corollary:

Let S be an algebraic curve, and x be an arbitrary point. Then, the number of arcs belonging to S and originating from x is always even (possibly zero). These arcs always uniquely partition themselves into pairs, so that the two arcs belonging to the same pair are analytic continuations of each other, and are not analytic continuations of any other arc originating from x.

Lemma 3:

Let (x^0, x^*) be an arc of an algebraic curve S. Suppose that (x^0, x^*) lies wholly within a given compact and convex set R with a nonempty interior, and that x^0 is a boundary point of R whereas x^* is an interior point of R. Then, by analytically continuing (x^0, x^*) far enough beyond x^*, we shall once more eventually reach a boundary point x^{00} of R.

Proof:

Let S^* be the curve we obtain if we continue (x^0, x^*) beyond x^* as far as possible without leaving set R. For each coordinate x_i, let

$$m_i = \inf_{x \in S^*} x_i \quad \text{and} \quad m^i = \sup_{x \in S^*} x_i . \tag{30}$$

Since S^* is not an isolated point, at least for one coordinate x_i, its variation on S^*, $\Delta_i = m^i - m_i$, must be positive. On the other hand, since S^* is an arc of an algebraic curve, it can be divided up into a *finite* number of segments $\alpha^1, \ldots, \alpha^\mu, \ldots, \alpha^M$, such that, as we move away from x^0 along any given segment α^μ, this coordinate x_i is either strictly increasing or is strictly decreasing. Let us assume that, starting from x^0 and moving along S^*, we reach these segments in the order they have been listed. Now, first suppose that, along the *last* segment α^M, x_i *increases*. Then, since R is a compact set, x_i must reach a local maximum at some point x^{00} of α^M. Obviously, this point x^{00} can only be the endpoint of α^M furthest away from x^0. Moreover, it can only be a *boundary* maximum point for x_i because, if it were an interior maximum point, then α^M could not be the last segment of S^*. Therefore, this point x^{00} must be a boundary point of R. By the same token, if x_i *decreases* along α^M, then the endpoint x^{00} of α^M must be a local boundary minimum point for x_i and, therefore, it must be boundary point of R. Thus, in either case, S^* will eventually reach a boundary point x^{00} of R.

Let $\bar{P}$ be the boundary of the strategy space P. Thus, $\bar{P}$ is the set of all strategy n-tuples $p = (p_1, \ldots, p_n)$ having at least one pure or incompletely mixed strategy p_i as a component. Let $I^0 = (0,1)$ be the open unit interval. Let B be the set

$B = \bar{P} \times I^0$. Clearly, the boundary hypersurface $\bar{R}$ of cylinder R is made up of the three disjoint sets P^0, B, and P^1.

Lemma 4:

Let $\bar{T}$ be the intersection of the solution graph T and of the boundary hypersurface $\bar{R}$ of cylinder R. Let (t,p) be any nonisolated point of $\bar{T}$. Then, *either* $(t,p) = (1,\tilde{p})$, where $\tilde{p}$ is the unique equilibrium point of the pure logarithmic game $\Lambda^*(1)$; *or* $(t,p) = (0,p^*)$, where p^* is an equilibrium point of the original game $\Lambda^*(0) = \Gamma$.

Proof:

For any t with $0 < t \leqq 1$, the vector p characterizing any given point (t,p) of $\bar{T}$ must be an equilibrium point of the game $\Lambda^*(t)$, because (t,p) is a point of the solution graph T. Therefore, (t,p) cannot belong to set B, since the logarithmic games $\Lambda^*(t)$ with $0 < t \leqq 1$ have no equilibrium point using a pure or an incompletely mixed strategy p_i as equilibrium strategy. Hence, if $t > 0$, then (t,p) can only be a point belonging to set P^1, which is possible only if $(t,p) = (1,\tilde{p})$.

On the other hand, if $t = 0$, then $(t,p) = (0,p)$ is a point belonging to set P^0. As (t,p) is a nonisolated point of T, it is a limit point of some convergent point sequence $(t^1,p^1),\ldots,(t^j,p^j),\ldots$, where each p^j is an equilibrium point in game $\Lambda^*(t^j)$, with $t^j > 0$. Consequently, p itself is an equilibrium point in game $\Lambda^*(0) = \Gamma$, because the correspondence $\mu^*: t \to E^t$ is upper semi-continuous (where E^t is the set mentioned in Lemma 1). This completes the proof.

Lemma 5:

The point $(t,p) = (1,p)$, corresponding to the unique equilibrium point $\tilde{p}$ of the pure logarithmic game $\Lambda^*(1) = \Lambda$ is always a nonsingular point of the graph T, and is the endpoint of exactly one branch $\beta(\tilde{p})$ of T.

Proof:

As is easy to verify, $J(1,\tilde{p}) \neq 0$. Consequently, $(1,\tilde{p})$ is nonsingular and, by the Implicit Function Theorem, it lies on exactly one branch $\alpha(\tilde{p})$ of T.

Lemma 6:

Let Γ be a *regular* and *quasi-strong* game. Then, any point $(t,p) = (0,p)$ corresponding to an equilibrium point p of game $\Lambda^*(0) = \Gamma$ is always a nonsingular point of the graph T, and is the endpoint of exactly one branch $\beta(p)$ of T.

Proof:

Since Γ is regular, we have $J(0,p) \neq 0$. Hence, if p is an interior point of the strategy space P, then the present lemma can be established by the same reasoning as was used in the proof of Lemma 5. However, if p is a boundary point of P, then this reasoning shows only that $(0,p)$ lies on exactly one branch $\beta(p)$ of the algebraic variety S. In order to prove the lemma, we have to show also that $\beta(p)$ belongs to the graph T, i.e., that it lies within cylinder R. In other words, we have to show that $\beta(p)$ goes from $(0,p)$ towards the interior of R, which is equivalent to showing that, for any zero component $p_i^k = 0$ of the vector p, the total derivative $\mathrm{d}p_i^k/\mathrm{d}t$ is positive at the point $(0,p)$. Now, by differentiating eq. (23) with respect to t, and then setting $t = p_i^k = 0$, we obtain

$$p_i^1\left[U_i(a_i^k,\bar{p}_i) - U_i(a_i^1,\bar{p}_i)\right] + p_i^1 = 0\,. \tag{31}$$

Since the numbering of player i's pure strategies is arbitrary, without loss of generality we can assume that

$$p_i^1 > 0\,. \tag{32}$$

On the other hand, since $p_i^1 > 0$ and $p_i^k = 0$, we have $p_i^1 \in C(p_i)$ but $p_i^k \notin C(p_i)$. Since p is a quasi-strong equilibrium point, condition (18) must be satisfied by a *strong* inequality sign if we set $a_i^{k'} = a_i^1$. Therefore,

$$U_i(a_i^k,\bar{p}_i) - U_i(a_i^1,\bar{p}_i) < 0\,. \tag{33}$$

But (31), (32), and (33) together imply that $\mathrm{d}p_i^k/\mathrm{d}t > 0$, as desired.

In what follows, when we say that two points are "connected", we shall mean that they are connected by some branch α of the solution graph T.

Theorem 1:

Let Γ be a *regular* and *quasi-strong* finite game. Then, the number of equilibrium points in Γ is finite. Moreover, there exists exactly one distinguished equilibrium point p^* in Γ such that the corresponding point $(0,p^*)$ is connected with the point $(1,\tilde{p})$, associated with the unique equilibrium point $\tilde{p}$ of the pure logarithmic game $\Lambda^*(1) = \Lambda$. All other equilibrium points of Γ form pairs, such that the two equilibrium points belonging to the same pair are connected with each other and with no other equilibrium point. Therefore, the number of equilibrium points in Γ is odd.

Proof:

By Lemma 6, every equilibrium point p of Γ lies on some branch $\beta(p)$ of T. But T, being the intersection of an algebraic variety S and of a compact and convex set R, can have only a finite number of branches. Moreover, on any given branch β, there can be at most two equilibrium points, corresponding to the two endpoints of β. Therefore, the number of equilibrium points in Γ is finite.

By Lemma 5, there exists a unique branch $\alpha(p)$ of T, originating from the point $(1,\tilde{p})$. By Lemmas 1 and 2, this branch $\alpha(\tilde{p})$ must lead to a boundary point x^{00} of R. As $J(1,\tilde{p}) \neq 0$, we must have $x^{00} \neq (1,\tilde{p})$, because otherwise T would have two local branches originating from $(1,\tilde{p})$, contrary to the Implicit Function Theorem. Consequently, by Lemma 4, $x^{00} = (0,p^*)$, where p^* is an equilibrium point – called the *distinguished* equilibrium point – of game Γ.

Finally, let $p \neq p^*$ be any equilibrium point of Γ, other than the distinguished equilibrium point p^*. By Lemma 6, there exists a unique branch $\beta(p)$ of T, originating from the point $(0, p)$. By an argument similar to the one used in the last paragraph, it can be shown that $\beta(p)$ must lead to another boundary point x^{00} of R, with $x^{00} = (0, p')$, where $p' \neq p$ and $\neq p^*$ is another equilibrium point of Γ. Hence, all equilibrium points of Γ, other than the distinguished equilibrium point $\tilde{p}$, are pairwise connected. But this means that the number of these latter equilibrium points is *even*, which makes the total number of equilibrium points in Γ *odd*.

Note:

The proof of Theorem 1 shows that, for any game $\Lambda^*(t)$ with $0 \leqq t \leqq 1$, the set Q^t of all equilibrium points in $\Lambda^*(t)$ is *nonempty*. This is so because branch $\alpha(\tilde{p})$ of graph T connects the two points $(1, \tilde{p})$ and $(0, p^*)$. Therefore, $\alpha(\tilde{p})$ intersects every set R^t with $0 \leqq t \leqq 1$ at some point (t, p^t). As is easy to verify, the strategy n-tuple p^t defining this point must be an equilibrium point of game $\Lambda^*(t)$.

5. Three "Almost All" Theorems

Within a given set $\mathscr{I} = \mathscr{I}(n; K_1, \ldots, K_n)$ of games of a particular size, let $\mathscr{F}(C^*)$ be the set of all games Γ that have at least one equilibrium point p with a specified set $C^* = C(p)$ as its carrier. There are only a *finite* number of different sets $\mathscr{F}(C^*)$ in $\mathscr{I}$ because, for all games Γ in $\mathscr{I}$, the number of possible carrier sets C^* is finite. This is so because any set C^* is a subset of the finite set

$$A^* = \bigcup_{i=1}^{n} A_i, \tag{34}$$

consisting of the set of all K^{**} pure strategies a_i^k for the n players in each game Γ, where

$$K^{**} = \sum_{i=1}^{n} K_i. \tag{35}$$

(Of course, two sets $\mathscr{F}(C^*)$ corresponding to different carrier sets C^* will in general overlap.) We can now state the following theorem.

Theorem 2:

Almost all finite games are quasi-strong.

Proof:

Let $\bar{\mathscr{F}}(C^*)$ be the set of all games Γ in $\mathscr{I}$ that have at least one extra-weak equilibrium point p with the set $C^* = C(p)$ as its carrier. Obviously, $\bar{\mathscr{F}}(C^*) \subset \mathscr{F}(C^*)$. Let $\bar{\bar{\mathscr{F}}}(C^*) = \mathscr{F}(C^*) - \bar{\mathscr{F}}(C^*)$. Thus, all games Γ in $\bar{\bar{\mathscr{F}}}(C^*)$ have the property that they contain one or more equilibrium points p with the set $C^* = C(p)$ as their carrier set, and all these equilibrium points p are quasi-strong.

All games Γ in a given set $\mathscr{F}(C^*)$ are characterized by the fact that their defining vector $u = u(\Gamma)$ satisfies a finite number of algebraic equations and algebraic weak inequalities, of forms (17) and (18), in which the functions U_i are defined by (12). Thus, if we regard the set $\mathscr{I}$ as an (nK)-dimensional Euclidean space $\mathscr{I} = \{u\}$, then each set $\mathscr{F}(C^*)$ will correspond to a subset of $\mathscr{I}$, bounded by pieces of a finite number of algebraic hypersurfaces. In view of (12) and (18), these bounding hypersurfaces are multilinear, i.e., they are hyperboloids. Within each set $\mathscr{F}(C^*)$, all games Γ belonging to $\bar{\bar{\mathscr{F}}}(C^*)$ are characterized by the fact that their defining vectors $u = u(\Gamma)$ satisfy *all* the inequalities of form (18) used in defining this set $\mathscr{F}(C^*)$, with a *strong in*equality sign $>$. In contrast, all games Γ belonging to $\bar{\mathscr{F}}(C^*)$ have a defining vector $u = u(\Gamma)$ satisfying one or more of these weak inequalities with an *equality* sign $=$. Therefore, all games in $\bar{\bar{\mathscr{F}}}(C^*)$ correspond to interior points u of $\mathscr{F}(C^*)$, whereas all games in $\bar{\mathscr{F}}(C^*)$ correspond to boundary points of $\mathscr{F}(C^*)$. Hence, as a subset of the (nK)-dimensional Euclidean space $\mathscr{I}$, $\bar{\mathscr{F}}(C^*)$ consists of pieces of a finite number of hyperboloids of at most $(nK - 1)$ dimensions. Consequently, each set $\bar{\mathscr{F}}(C^*)$ is a set of measure zero in $\mathscr{I}$.

Let $\bar{\mathscr{F}}^*$ be the set of all extra-weak games in $\mathscr{I}$. $\bar{\mathscr{F}}$ is the union of all sets $\bar{\mathscr{F}}(C^*)$, corresponding to various possible carrier sets C^*. Thus, $\bar{\mathscr{F}}^*$ is a union of a finite number of sets of measure zero in $\mathscr{I}$. Therefore, $\bar{\mathscr{F}}^*$ itself is also a set of *measure zero* in $\mathscr{I}$.

Next, we shall show that $\bar{\mathscr{F}}^*$ is a *closed* set. Let $\Gamma^1, \Gamma^2, \ldots$ be a sequence of extra-weak games in $\mathscr{I}$, with the defining vectors $u^1 = u(\Gamma^1), u^2 = u(\Gamma^2), \ldots$. Suppose that the sequence $u^1, u^2, \ldots$ converges to a given vector u^0. Let Γ^0 be the game corresponding to this vector $u^0 = u(\Gamma^0)$. We have to show that Γ^0 is likewise an extra-weak game.

Since the games $\Gamma^j (j = 1, 2, \ldots)$ are extra-weak, each vector u^j satisfies one or more inequalities of form (18), with an equality sign. Yet, there are only a finite number of inequalities of this form. Therefore, at least one of these inequalities – let us call it inequality (18)* – will be satisfied by infinitely many vectors u^j, with an equality sign. As the sequence of these latter vectors, being a subsequence of the original sequence $\{u^j\}$, converges to u^0, this vector u^0 itself will also satisfy (18)* with an equality sign, which makes the corresponding game Γ^0 extra-weak, as desired. This completes the proof of Theorem 2.

Let p be an equilibrium point in game Γ belonging to set $\mathscr{I}$, with the carrier $C^* = C(p) = \bigcup_i C(p_i)$. Thus $\Gamma \in \mathscr{F}(C^*)$. Suppose the carriers $C(p_1), \ldots, C(p_n)$ of the equilibrium strategies $p_1, \ldots, p_n$ consist of exactly $\gamma_1, \ldots, \gamma_n$ pure strategies, respectively. In studying games Γ in set $\mathscr{F}(C^*)$, we shall adopt the following notational convention, which, of course, involves no loss of generality:

> The pure strategies a_i^k of each player $i (i = 1, \ldots, n)$ have been renumbered in such a way that the carrier $C(p_i)$ of his equilibrium strategy p_i now contains his *first* γ_i pure strategies $a_i^1, \ldots, a_i^{\gamma_i}$. (36)

We can fully characterize each equilibrium strategy p_i by the $(\gamma_i - 1)$ probability numbers $p_i^2, p_i^3, \ldots, p_i^{\gamma_i}$, since we have

$$p_i^1 = 1 - \sum_{j=2}^{\gamma_i} p_i^j, \tag{37}$$

and

$$p_i^k = 0, \quad \text{for} \quad k = \gamma_i + 1, \ldots, K_i. \tag{38}$$

Let π_i be the probability vector

$$\pi_i = (p_i^2, \ldots, p_i^{\gamma_i}), \quad \text{for} \quad i = 1, \ldots, n. \tag{39}$$

Thus, π_i is a subvector of the probability vector p_i.

Let Π_i $(i = 1, \ldots, n)$ be the set of all $(\gamma_i - 1)$-vectors satisfying the two conditions

$$p_i^k > 0, \quad k = 2, \ldots, \gamma_i; \tag{40}$$

and

$$\sum_{j=2}^{\gamma_i} < 1. \tag{41}$$

Let π be the composite vector

$$\pi = (\pi_1, \ldots, \pi_n). \tag{42}$$

Thus, π is a vector consisting of γ^* probability numbers p_i^k, where

$$\Gamma^* = \sum_{i=1}^{n} (\gamma_i - 1) = \sum_{i=1}^{n} \gamma_i - n. \tag{43}$$

Clearly, π is a subvector of the probability vector p.

Let Π be the set of all γ^*-vectors π whose subvectors $\pi_1, \ldots, \pi_n$ satisfy conditions (40) and (41). Clearly, $\Pi = \Pi_1 \times \cdots \times \Pi_n$.

We now define

$$m^*(1,k) = k - 1, \quad \text{for} \quad k = 2, \ldots, \gamma_1; \tag{44}$$

and

$$\begin{aligned} m^*(i,k) &= \sum_{j=1}^{i-1} (\gamma_j - 1) + (k - 1) \\ &= \sum_{j=1}^{i-1} \gamma_j - i + k, \quad \text{for} \quad i = 2, \ldots, n;\ k = 2, \ldots, \gamma_i. \end{aligned} \tag{45}$$

In addition to (36), we now introduce the following further notational convention, which again involves no loss of generality:

> The pure-strategy n-tuples a^m of the game have been re-numbered in such a way that the first γ^* pure-strategy n-tuples $a^1, \ldots, a^{\gamma^*}$ will now have the following form. For any m with $1 \leqq m \leqq \gamma^*$, let i and k be the unique pair of numbers satisfying $m^*(i,k) = m$.

Then

$$a^m = (a_1^1, \ldots, a_{i-1}^1, a_i^k, a_{i+1}^1, \ldots, a_m^1). \tag{46}$$

Thus, we can write

$$u_i^m = u^{m*(i,k)} = U_i(a_1^1,\dots,a_{i-1}^1,a_i^k,a_{i+1}^1,\dots,a_n^1),\ \text{for}\ i = 1,\dots,n;\ k = 2,\dots,\gamma_i. \tag{47}$$

Let u^* be the vector formed of those γ^* components u_i^m of vector u which can be written in form (47). Let u^{**} be the vector formed of the remaining $(nK - \gamma^*)$ components of u. Hence

$$u = (u^*,u^{**}). \tag{48}$$

The set of all possible vectors u^* is a γ^*-dimensional Euclidean space, to be denoted as $\mathscr{I}^* = \{u^*\}$; whereas the set of all possible vectors u^{**} is an $(nK - \gamma^*)$-dimensional Euclidean space, to be denoted as $\mathscr{I}^{**} = \{u^{**}\}$. Clearly, $\mathscr{I}^* \times \mathscr{I}^{**} = \mathscr{I}$.

Since p is an equilibrium point in game Γ, it must satisfy condition (17). This condition can also be written as

$$p_i^1 p_i^k[U_i(a_i^k,\bar{p}_i) - U_i(a_i^1,\bar{p}_i)] = 0, \quad \text{for} \quad i = 1,\dots,n;\ k = 2,\dots,\gamma_i. \tag{49}$$

Since p_i^1 and $p_i^k > 0$, (49) is equivalent to (17). In view of (12), this condition can also be written as

$$u^{m^*} = \sum_{\substack{m \neq m^* \\ m \in M}} \frac{[q_i^m(a_i^k) \prod_{j \neq i} q_j^m(\bar{p}_j)]\, u_i^m}{\prod_{j \neq i} q_j^{m^*}(p_j)} \tag{50}$$

$$- \sum_{m \in M} \frac{[q_i^m(a_i^1) \prod_{j \neq i} q_j^m(\bar{p}_j)]\, u_i^m}{\prod_{j \neq i} q_j^{m^*}(p_j)}, \quad \text{for} \quad i = 1,\dots,n;\ k = 2,\dots,\gamma_i.$$

Here $M = \{1,2,\dots,K\}$ and $m^* = m^*(i,k)$. It is permissible to write eq. (17) [or (49)] in form (50) because, by (46), we have $q_j^{m^*}(p_j) = p_j^1$ for all $j \neq i$, and $p_j^1 > 0$ since $a_j^1 \in C(p_j)$.

Note that each quantity $u_i^{m^*}$ for a specific value of $m^* = m^*(i,k)$ occurs, with a nonzero coefficient, only in *one* equation of form (50) (where it occurs on the left-hand side). This is so because, by (10) and (46), for any $k' \neq k$, we have $q_i^{m^*}(a_i^{k'}) = 0$. Therefore, if we know the γ^* components p_i^k of vector π, and know the $(nK - \gamma^*)$ components u_i^m of vector u^{**}, then we can compute each one of the γ^* components $u_i^{m^*}$ of vector u^* separately, from the relevant equation of form (50). Consequently, the γ^* equations of form (50) define a mapping $\rho\colon (\pi,u^{**}) \to u^*$ from set $\Pi \times \mathscr{I}^{**}$ to set $\mathscr{I}^*$. This mapping ρ is continously differentiable because, by (40), for each point π in Π we have $p_i^k > 0$ for $k = 2,\dots,\gamma_i$; so that the denominators on the right-hand side of (50) never vanish within Π.

We can use this mapping ρ to define another mapping $\rho^*\colon (\pi,u^{**}) \to (u^*,u^{**}) = u$, where $u^* = \rho(\pi,u^{**})$. This mapping ρ^* is from set $\Pi \times \mathscr{I}^*$ to set $\mathscr{I}^* \times \mathscr{I}^{***} = \mathscr{I}$; and it is continuously differentiable since ρ is.

We can now state the following theorem:

Theorem 3:

Almost all finite games are regular.

Proof:

Instead of using the γ^* equations of form (50), we can also use the equivalent γ^* equations of form (49), in order to define the mappings ρ and ρ^*. But if we do so, then the Jacobian of mapping ρ^* can be written as

$$J^*(\pi, u^{**}) = \frac{\partial(F_i^2, \ldots, F_i^k, \ldots, F_n^{\gamma_n})}{\partial(p_i^2, \ldots, p_i^k, \ldots, p_n^{\gamma_n})}, \quad i = 1, \ldots, n; \quad \text{and, for each} \quad k = 2, \ldots, \gamma_n, \tag{51}$$

where the F_i^k's are the functions $F_i^k = F_i^k(0,p)$ defined by (29). This means that $J^*(\pi, u^{**})$ is a subdeterminant of the Jacobian determinant $J(0,p)$ defined by (26, (27), and (29); it is that particular subdeterminant that we obtain if, for each player i, we cross out the rows and the columns corresponding to $k = 1$, and to $k = \gamma_i + 1, \ldots, K_i$. It is easy to verify that, owing to the special form of the functions F_i^1 ($i = 1, \ldots, n$) as defined by (27), and owing to the fact that $p_i^{\gamma_i+1} = \cdots = p_i^{K_i} = 0$, this crossing out of these rows and columns does not change the value of the original determinant $J(0,p)$. Hence, $J^*(\pi, u^{**}) = J(0,p)$ if π is the subvector of p defined by (39) and (42).

Let $\mathscr{E}(C^*)$ be the set of all games Γ in $\mathscr{I}$ having at least one *irregular* equilibrium point p with set $C^* = C(p)$ as its carrier set. Equivalently, $\mathscr{E}(C^*)$ can also be defined as the set of all vectors $u = \rho^*(\pi, u^{**})$ corresponding to those points (π, u^{**}) in set $(\Pi \times \mathscr{I}^{**})$ at which the Jacobian $J^*(\pi, u^{**}) = J(0,p)$ vanishes. By SARD's Theorem [SARD, 1942], this set $\mathscr{E}(C^*)$ is a set of measure zero in the (nK)-dimensional Euclidean space $\mathscr{I}$.

Let $\mathscr{E}^*$ be the set of all games Γ in $\mathscr{I}$ having at least one irregular equilibrium point p, regardless of what its carrier $C^* = C(p)$ is. Thus, $\mathscr{E}^*$ is simply the set of all irregular games in $\mathscr{I}$. $\mathscr{E}^*$ is the union of a finite number of sets $\mathscr{E}(C^*)$, corresponding to different carrier sets C^*. Since each set $\mathscr{E}(C^*)$ is a set of *measure zero* in $\mathscr{I}$, their union $\mathscr{E}^*$ will also have this property.

Next, we shall show that $\mathscr{E}^*$ is a *closed* set. Let $\Gamma^1, \Gamma^2, \ldots$ be a sequence of irregular games, with the defining vectors $u^1 = u(\Gamma^1), u^2 = u(\Gamma^2), \ldots$. Suppose that the sequence $u^1, u^2, \ldots$ converges to a given vector u^0. Let Γ^0 be the game corresponding to $u^0 = u(\Gamma^0)$. We have to show that Γ^0 is likewise an irregular game.

Let $p^1, p^2, \ldots$ be a sequence of strategy n-tuples, such that p^j ($j = 1, 2, \ldots$) is an irregular equilibrium point in game Γ^j. All these points p^j lie in the compact set P. Consequently, the sequence $\{p^j\}$ must contain a convergent subsequence. Suppose the latter consists of the points $p^{j_1}, p^{j_2}, \ldots$, and that it converges to some point p^0 in P. Then:

(1) This point p^0 will be an *equilibrium point* of game Γ^0. This is so because the set $Q(\Gamma)$ of all equilibrium points in any given game is an upper semi-continuous set function of the defining vectors $u = u(\Gamma)$ of Γ, i.e., of the payoffs u_i^m of Γ.

(2) This point p^0 will be an *irregular* equilibrium point of game Γ^0. This is so because $J(0,p^{j_1}) = J(0,p^{j_2}) = \cdots = 0$ since $p^{j_1}, p^{j_2}, \ldots$ are irregular equilibrium points. Consequently, $J(0,p^0) = 0$ since p^0 is the limit of the sequence $p^{j_1}, p^{j_2}, \ldots$, and since $J(0,p)$ is a continuous function of p.

Consequently, p^0 is an irregular equilibrium point in Γ^0 and, therefore, Γ^0 itself is an irregular game, as desired. This completes the proof of Theorem 3.

Theorems 1, 2, and 3 directly imply:

Theorem 4:

In almost all finite games, the number of equilibrium points is finite and odd.

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Received August, 1973