Values of Mixed Games[1])

By SERGIU HART[2])

Abstract: AUMANN and SHAPLEY [1973] have investigated values of games in which all players are individually insignificant, i.e. form a non-atomic continuum, or "ocean". In this paper we treat games in which, in addition to such an ocean, there are also some "atoms", i.e. players who are individually significant. We define spaces of such games that are analogous to those investigated by AUMANN and SHAPLEY, and prove the existence of values on some of them. Unlike in the non-atomic case, we find that in general there are infinitely many values, corresponding to various ways in which the atoms can be imbedded in the ocean. The results generalize those of MILNOR and SHAPLEY [1961]. Precise statements will be found in Section 2.

1. Preliminaries

All the definitions and notations are as in AUMANN and SHAPLEY [1968].

Let $(I,\mathscr{C})$ be a measurable space (i.e., I is a set and $\mathscr{C}$ is a σ-field of subsets of I), which will be fixed throughout. We will assume (AUMANN and SHAPLEY [1968], assumption (2.1)) that:

(1.1) $(I,\mathscr{C})$ is isomorphic to $([0,1],\mathscr{B})$, where $\mathscr{B}$ is the σ-field of Borel sets on $[0,1]$ (i.e., there is a one-one mapping from I onto $[0,1]$ that is measurable in both directions).

A *set function* will always be a real-valued function v on $\mathscr{C}$ such that $v(\emptyset) = 0$.

The members of I are *players*, the members of $\mathscr{C}$ are *coalitions*, and the set functions are *games*.

A set function v is *monotonic* if $S \subset T$ implies $v(S) \leq v(T)$ for $S,T \in \mathscr{C}$. A set function is *of bounded variation* if it is the difference between two monotonic set functions. The space of all set functions of bounded variation is called BV. The subspace of BV consisting of all bounded, finitely additive set functions (i.e., the bounded, finitely additive signed measures on $(I,\mathscr{C})$) is denoted FA.

Let Q be any subspace of BV. The set of monotonic games in Q is denoted Q^+. A mapping of Q into BV is *positive* if it maps Q^+ into BV^+.

Let $\mathscr{G}$ denote the group of automorphisms of $(I,\mathscr{C})$ (i.e., one-one mappings of I onto itself that are measurable in both directions). Each $\theta \in \mathscr{G}$ induces a

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[2]) SERGIU HART, Department of Mathematical Sciences, Tel-Aviv University, Tel-Aviv, Israel.

linear mapping θ_* of BV onto itself, defined by $(\theta_* v)(S) = v(\theta S)$ for all $S \in \mathscr{C}$. A subspace Q of BV is called *symmetric* if $\theta_* Q = Q$ for all $\theta \in J$.

Let Q be a symmetric subspace of BV. A *value* on Q is a mapping Φ from Q into FA, which satisfies:

(1.2) Φ is linear

(1.3) Φ is positive

(1.4) $\Phi\theta_* = \theta_*\Phi$ for all $\theta \in \mathscr{J}$

(1.5) $(\Phi v)(I) = v(I)$ for all $v \in Q$.

On BV we define a norm called the *variation norm* by $\|v\| = \inf(u(I) + w(I))$ for all $v \in BV$, where the infimum ranges over all monotonic set functions u and w such that $v = u - w$. A *chain* Ω is a sequence of sets of the form:

$$\emptyset = S_0 \subset S_1 \subset \cdots \subset S_m = I.$$

The variation of a set function v over a chain Ω is

$$\|v\|_\Omega = \sum_{i=1}^{m} |v(S_i) - v(S_{i-1})|.$$

It can be proved (AUMANN and SHAPLEY [1968], Proposition 4.1) that

$$\|v\| = \sup \|v\|_\Omega,$$

where the supremum ranges over all chains Ω.

The space of all real-valued functions f of bounded variation on $[0,1]$ that obey $f(0) = 0$ and are continuous at 0 and 1 is denoted bv'. The subspace of bv' consisting of all left-(right-) continuous functions will be called lc' (rc'), and the subspace of bv' consisting of all continuous functions will be denoted c.

A *carrier* of a game v is a coalition I' such that $v(S) = v(S \cap I')$ for all $S \in \mathscr{C}$. A coalition S is *null* if its complement is a carrier, and a player s is *null* if $\{s\}$ is null. If all the players are null, the game is *non-atomic*. The subspace of BV consisting of all non-atomic measures (by "measure" we mean a completely additive, totally finite, signed scalar measure) is denoted NA. The subspace of BV consisting of all measures with a finite carrier will be denoted FC. All measures in BV that can be represented as the sum of two measures, one non-atomic and the other with a finite carrier, form a subspace called FL (i.e., $FL = NA + FC$).

The closed subspace of BV spanned by the set functions of the form $f \circ \mu$, where $f \in bv'$ and $\mu \in NA^+$ is a probability measure (i.e., $\mu(I) = 1$), is denoted $bv'NA$. In the same manner will be defined $bv'FL$, $lc'FL$, $rc'FL$, cFL: the closed subspaces of BV spanned by the set functions $f \circ \omega$ where $f \in bv'$ (or lc', rc', c respectively) and $\omega \in FL^+$ is a probability measure. The subspace of $bv'NA$ spanned by all powers of measures in NA^+ is denoted pNA, and the subspace of $bv'FL$ (in fact, of cFL) spanned by all powers of measures in FL^+ will be called pFL.

2. Statement of the Results

In this paper, we will deal with spaces of *mixed games*, i.e. spaces of set functions defined with the aid of measures that have a finite number of atoms and a non-atomic part (measures in FL). We will show that on each of the spaces $bv'FL$, $lc'FL$, $rc'FL$, cFL, pFL, the number of values is infinite.

The values will be defined as follows (see definition (2.2)):

The value of the game v to an atom is equal to the "contribution" to v of the atom in a "random ordering" of all the players; the value of the game to the "ocean" (i.e., the non-atomic part – see MILNOR and SHAPLEY [1961]) is the remainder after substracting the values to all the atoms from $v(I)$ (remember the efficiency condition (1.5)), and is distributed proportionally to its measure (weight).

What is a "random ordering" of all the players? Let p be a continuous probability distribution on $(0,1)$. Think of the ocean as being uniformly[3]) spread along $(0,1)$. Place each atom at random in $(0,1)$, in accordance with the distribution p; the placements are assumed independent.

One can define a similar process for distributions p on $(0,1)$ that are not necessarily continuous. The ocean is spread on $(0,1)$ as before. As for the atoms, take n independent random variables $T_1, T_2, \ldots, T_n$, all identically distributed according to p, and arrange them in non-decreasing order:

$$0 < T^{(1)} \leq T^{(2)} \leq \cdots \leq T^{(n)} < 1 .$$

Choose an order on the atoms at random[4]), and insert them in the ocean in the order chosen, at the points $T^{(1)}, T^{(2)}, \ldots, T^{(n)}$.

We come now to the exact definitions.

For each positive integer i, let J_i be the set $\{1,2,\ldots,i\}$, and let $J_0 = \emptyset$.

Definition 2.1:

Let $f \in bv'$, let n be a positive integer, and let ξ be a measure on J_n with $0 \leq \xi(J_n) \leq 1$. For each $i \in J_n$ and for each $t \in (0,1)$ define:

$$\Delta(i,t,\xi,f) = f[t(1 - (J_n)) + \xi(J_i)] - f[t(1 - \xi(J_n)) + \xi(J_{i-1})] .$$

Let $v = f \circ \omega$, where $f \in bv'$, $\omega \in FL^+$ is a probability measure decomposing into measures λ in NA^+ and ξ in FC^+ (i.e., $\omega = \lambda + \xi$), and let J_n be a finite carrier of ξ. Then $\Delta(i,t,\xi,f)$ is the "contribution" to v of the atom i, on the assumption that the atoms "enter" in the order $1,2,\ldots,n$, and the measure of the part of the ocean preceding i is the fraction t of its total measure $\omega(I \backslash J_n)$ $(= 1 - \xi(J_n))$.

Definition 2.2:

Let $v = f \circ \omega$, where $f \in bv'$, $\omega \in FL^+$ is a probability measure, $\omega = \mu + \nu$ its decomposition, $\mu \in NA^+$ and $\nu \in FC^+$. Let N be a finite carrier of ν, and n the

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[3]) i.e., the weight of an oceanic set is proportional to its measure in $(0,1)$.

[4]) Each order with probability $1/n!$

number of its elements. Let Π be the set of all one-one mappings of N onto J_n (there are $n!$ such mappings). Let p be a probability measure on the Borel sets of $(0,1)$, and let $T_1, T_2, \ldots, T_n$ be n independent random variables, all identically distributed according to p. Let $0 < T^{(1)} \leq T^{(2)} \leq \cdots \leq T^{(n)} < 1$ be the order statistics

$$(\text{i.e., } T^{(1)} = \min(T_1, T_2, \ldots, T_n), \ldots, T^{(n)} = \max(T_1, T_2, \ldots, T_n)),$$

and

$$T = (T^{(1)}, T^{(2)}, \ldots, T^{(n)}).$$

Define a set function $\varphi_p v$ in FA by:

$$(\Phi_p v)(\{s\}) = E\left(\frac{1}{n!}\sum_{\pi \in \Pi} \Delta(\pi s, T^{(\pi s)}, v \circ \pi^{-1}, f)\right), \quad \text{for} \quad s \in N$$

$$(\Phi_p v)(S) = \alpha \cdot \mu(S), \qquad \text{for} \quad S \subset I \backslash N$$

where the expectation E is taken over the variable T, (see remark below), and $\alpha = \alpha_p(v)$ is independent of S, and is defined by:

$$\alpha_p(v) = \begin{cases} [f(1) - (\Phi_p v)(N)]/\mu(I), & \text{if} \quad \mu(I) > 0 \\ 0 & , \text{ if} \quad \mu(I) = 0. \end{cases}$$

Remark:

f is the difference of two monotonic real-valued functions ($f \in bv'$) hence measurable. Thus, the expression in the brackets is measurable, and also bounded (e.g., by $2 \sup_{t \in [0,1]} |f(t)|$), therefore the expectation exists.

For each such probability measure p, the function Φ_p defined here is a "candidate" for a value on the previously mentioned spaces of mixed games.

The value of MILNOR and SHAPLEY [1961] was obtained in the same manner, using the uniform distribution (for voting games, the contribution to v can be only 0 or 1, and the later if and only if the player is pivotal, i.e. he and his predecessors are a winning coalition, but his predecessors alone are a losing coalition).

A probability measure is called *continuous* if the corresponding distribution function is continuous, i.e. the probability of any single point is zero.

Now we are in position to state the theorems we are going to prove:

Theorem A:

Let p be a *continuous* probability measure on the Borel sets of $(0,1)$. Then there is a value Φ on $bv'FL$, such that $\Phi v = \Phi_p v$ for all v as in definition (3.2).

Theorem B:

Let p be a probability measure on the Borel sets of $(0,1)$. Then there is a value Φ on $lc'FL$ ($rc'FL$) such that $\Phi v = \Phi_p v$ for all $v = f \circ \omega$ as in definition (3.2), where $f \in lc'$ (rc', respectively).

From the trivial inclusions $pFL \subset cFL \subset lc'FL$, it follows that each value Φ of Theorem B is a value also on the spaces pFL and cFL.

3. The Main Lemma and its Proof

We need the following definition:

Definition 3.1:

Let $v = f \circ \omega$, where $f \in bv'$ and $\omega \in FL^+$ is a probability measure, $\omega = \mu + \nu$ its decomposition, $\mu \in NA^+$ and $\nu \in FC^+$. Let N be a finite carrier of ν, and n the number of its elements. Let π be a one-one mapping of N onto J_n. Let $\tau = (\tau_1, \tau_2, \dots, \tau_n)$, where $0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n < 1$. Define a set function $\psi_{\pi,\tau} v$ in FA by:

$$(\psi_{\pi,\tau} v)(\{s\}) = \Delta(\pi s, \tau_{\pi s}, \nu \circ \pi^{-1}, f), \quad \text{for} \quad s \in N$$

$$(\psi_{\pi,\tau} v)(S) = \alpha \cdot \mu(S), \qquad \text{for} \quad S \subset I \backslash N$$

where $\alpha = \alpha_{\pi,\tau}(v)$ is independent of S, and is defined by:

$$\alpha_{\pi,\tau}(v) = \begin{cases} [f(1) - (\psi_{\pi,\tau} v)(N)]/\mu(I), & \text{if} \quad \mu(I) > 0 \\ 0 & , \text{if} \quad \mu(I) = 0 . \end{cases}$$

Recalling definition (2.2) of $\Phi_p v$, it is clear that:

$$\Phi_p v = E\left(\frac{1}{n!} \sum_{\pi \in \Pi}' \psi_{\pi,T} v\right)$$

where Π and the random variable T are defined there.

The crucial point in the proof of the stated theorems is the following main lemma:

Main Lemma:

Let $v = \sum_{k=1}^{n} f_k \circ \omega_k$, where $f_1, f_2, \dots, f_m \in bv'$ and $\omega_1, \omega_2, \dots, \omega_m \in FL^+$ are probability measures, decomposing: $\omega_k = \mu_k + \nu_k$, $\mu_k \in NA^+$ and $\nu_k \in FC^+$ for all $k (1 \leq k \leq m)$. Let N be a finite carrier of all ν_k (e.g., if N_k is a finite carrier of ν_k, then N is the union of all N_k), and n the number of its elements. Let π be a one-one mapping of N onto J_n and let $\tau = (\tau_1, \tau_2, \dots, \tau_n)$, where $0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n < 1$. We assume that:

(3.2) for all k, f_k is continuous at the points:

$$\tau_i \mu_k(I) + \nu_k \circ \pi^{-1}(J_i), \qquad \tau_i \mu_k(I) + \nu_k \circ \pi^{-1}(J_{i-1})$$

for all $i (1 \leq i \leq n)$.

$$\text{Then:} \left\| \sum_{k=1}^{m} \psi_{\pi,\tau}(f_k \circ \omega_k) \right\| \leq \|v\| .$$

Proof of the Main Lemma:

For each $k, \psi_{\pi,\tau}(f_k \circ \omega_k) \in FA$, hence the sum is also a member of FA. Decomposing I into its atoms and its non-atomic part, we get:

$$\left\| \sum_{k=1}^{m} \psi_{\pi,\tau}(f_k \circ \omega_k) \right\| = \sum_{s \in N} \left| \sum_{k=1}^{m} \psi_{\pi,\tau}(f_k \circ \omega_k)(\{s\}) \right| + \left\| \sum_{k=1}^{m} \alpha_{\pi,\tau}(f_k \circ \omega_k) \cdot \mu_k \right\| . \quad (3.3)$$

Since π is a one-one mapping of N onto J_n, the first term of the sum is:

$$\sum_{i=1}^{n} \left| \sum_{k=1}^{m} \Delta(i, \tau_i, v_k \circ \pi^{-1}, f_k) \right| . \tag{*}$$

Let

$$\hat{\mu}_k(S) = \begin{cases} \mu_k(S)/\mu_k(I), & \text{if} \quad \mu_k(I) > 0 \\ 0 & , \text{if} \quad \mu_k(I) = 0 \end{cases}$$

for all $S \subset I \backslash N$. Then $\hat{\mu}_k$ is either a non-atomic probability measure on $I \backslash N$, or is identically zero. The second term in the right side of (3.3) thus becomes:

$$\left\| \sum_{k=1}^{m} \left[f_k(1) - \sum_{i=1}^{n} \Delta(i, \tau_i, v_k \circ \pi^{-1}, f_k) \right] \cdot \hat{\mu}_k \right\| . \tag{**}$$

We will define new functions $\hat{f}_k (1 \leq k \leq m)$ on $[0,1]$ by:

$$\hat{f}_k(t) = f_k[t \mu_k(I) + v_k \circ \pi^{-1}(J(t))] - \sum_{i \in J(t)} \Delta(i, \tau_i, v_k \circ \pi^{-1}, f_k) \tag{3.4}$$

for $t \in [0,1]$, where $J(t) = \{i \in J_n | \tau_i < t\}$.

It is clear that $\hat{f}_k$ is a real-valued function of bounded variation on $[0,1]$ since $f_k \in bv'$, and the sum $\sum_{i \in J(t)} \Delta(i)$ is a jump function). Furthermore, $\hat{f}_k(0) = 0$ and $\hat{f}_k$ is continuous at 0 and 1 ($0 < \tau_i < 1$ for all i), hence $\hat{f}_k \in bv'$ for all k.

Let $w = \sum_{k=1}^{m} \hat{f}_k \circ \mu_k = \sum_k{}' \hat{f}_k \circ \mu_k$, where $\sum_k{}'$ denotes the sum over all such k such that $\mu_k(I) > 0$ (i.e., $\mu_k \neq 0$), thus $w \in bv'NA$. Let Φ denote the unique value on $bv'NA$ (AUMANN and SHAPLEY [1968], Theorem A), then:

$$\Phi w = \sum_k{}' \hat{f}_k(1) \cdot \hat{\mu}_k = \sum_{k=1}^{m} \hat{f}_k(1) \cdot \hat{\mu}_k = \sum_{k=1}^{m} \left[f_k(1) - \sum_{i=1}^{n} \Delta(i, \tau_i, v_k \circ \pi^{-1}, f_k) \right] \cdot \hat{\mu}_k$$

Recalling (*) and (**), we get from (3.3):

$$\left\| \sum_{k=1}^{m} \psi_{\pi,\tau}(f_k \circ \omega_k) \right\| = \sum_{i=1}^{n} \left| \sum_{k=1}^{m} \Delta(i, \tau_i, v_k \circ \pi^{-1}, f_k) \right| + \| \Phi w \| . \tag{3.5}$$

For each k, let $f_k + h_k$ be its unique decomposition into an absolutely continuous function g_k and a singular function h_k (with respect to the Lebesgue measure – cf. AUMANN and SHAPLEY [1968], Chapter 8). Let $w_1 = \sum_{k=1}^{m} g_k \circ \hat{\mu}_k$ and $w_2 = \sum_{k=1}^{m} h_k \circ \hat{\mu}_k$, then $w = w_1 + w_2$.

Before we go on with the proof, we have to bring some results from AUMANN and SHAPLEY [1968].

Lemma I:

Let v in pNA be such that there exists μ, f and U as follows: μ is a vector of non-atomic measures with range R, f is a real-valued function defined on R and continuously differentiable there, U is a convex neighborhood in R of the diagonal $[0, \mu(I)]$, and

$$v(S) = f(\mu(S)) \qquad \text{whenever} \qquad \mu(S) \in U .$$

Then, given $\varepsilon > 0$, for any m large enough there is a set $S^+ \subset I$ and a chain Ω given by:

$$\emptyset = S_0 \subset S_1 \subset \cdots \subset S_{2m} = I$$

such that:

(a) $\mu(S_{2j}) = \frac{j}{m}\mu(I)$. for $0 \leq j \leq m$,

(b) $\mu(S_{2j+1}) = \frac{j}{m}\mu(I) + \frac{1}{m}\mu(S^+)$, for $0 \leq j \leq m-1$,

and (c) $\|v\|_\Omega \geq \|\Phi v\| - \varepsilon$

where Φ denotes the unique value on pNA.

Proof:

This follows from the proof of Proposition 7.6 in AUMANN and SHAPLEY [1968]: the chain Ω satisfying (a) and (b) was obtained there, and (c) is implied by (7.8)–(7.10).

Lemma II:

Let $g_1, g_2, \ldots, g_l$ be singular functions in bv', let $v_1, v_2, \ldots, v_l$ be pairwise different probability measures in NA, and let $u \varepsilon AC$. Then

$$\left\| u + \sum_{p=1}^{l} g_p \circ v_p \right\| = \|u\| + \sum_{p=1}^{l} \|g_p\|.$$

Proof:

This is exactly Proposition 8.17 in AUMANN and SHAPLEY [1968], revised version.

Lemma III:

Let $v = u + \sum_{p=1}^{l} g_p \circ v_p$, where $g_1, g_2, \ldots, g_l$ are singular functions in bv', $v_1, v_2, \ldots, v_l$ are pairwise different probability measures in NA, and $u \varepsilon AC$.

Let Λ be the subchain $S_1 \subset S \subset S_2$, and let $\delta > 0$.

Then there is a set S_t such that:

(a) $S_1 \subset S_t \subset S_2$,

(b) for all p $(1 \leq p \leq l)$, g_p is continuous at $v_p(S_t)$, and

(c) $\left| \|u\|_\Lambda - \|u\|_{\Lambda_t} \right| < \delta$,

where Λ_t is the new subchain $S_1 \subset S_t \subset S_2$.

Proof:

The proof is exactly like the first part of the proof of Proposition 8.17 in AUMANN and SHAPLEY [1968], revised version.

Let v_0 in NA be such that $u \ll v_0$, and let $\xi = (v_0, v_1, \ldots, v_l)$. By Lyapunov's theorem applied to $I = S \backslash S_1$ we may, for each t in $[0,1]$, find a set S_t such that $S_1 \subset S_t \subset S \subset S_2$, and

$$\xi(S_t) = t\xi(S_1) + (1-t)\xi(S).$$

Then as $t \to 0$, we have $v_0(S \backslash S_t) \to 0$, and hence $|u(S) - u(S_t)| \to 0$. So if t is chosen sufficiently small, (c) is satisfied.

On the other hand, the g_p can have only denumerably many jumps; so by choosing t appropriately, we can see to it that (c) holds and that the g_p have no jumps at $v_p(S_t)$.

We return now to the proof of the Main Lemma.

Lemma 3.6:

Given $\varepsilon > 0$, there is a chain Ω on $I \backslash N$ with the following properties:

(i) $\|w_1\|_\Omega > \|\Phi w_1\| - \varepsilon$,

(ii) for each $i (1 \leq i \leq n)$ there is a member S of the chain Ω such that $\mu_k(S) = \tau_i \mu_k(I)$ for all $k (1 \leq k \leq m)$.

Proof:

For each k, g_k is absolutely continuous, hence $w_1 = \sum_{k=1}^{m} g_k \circ \hat{\mu}_k$ is in pNA (AUMANN and SHAPLEY [1968], Theorem C). By definition of pNA, there are $\lambda_1, \lambda_2, \ldots, \lambda_r$ probability non-atomic measures, $n_1, n_2, \ldots, n_r$ positive integers, and $a_1, a_2, \ldots, a_r$ real numbers, such that:

$$\left\| w_1 - \sum_{q=1}^{r} a_q \lambda_q^{n_q} \right\| < \frac{\varepsilon}{4}.$$

Without loss of generality, $\{\hat{\mu}_k | \hat{\mu}_k \neq 0\} \subset \{\lambda_1, \lambda_2, \ldots, \lambda_r)$ (we can add them to the sum with coefficient zero).

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be the vector measure, and R its range ($R \subset [0,1]^r$). We define a real-valued function f on R by:

$$f(X_1, X_2, \ldots, X_r) = \sum_{q=1}^{r} a_q X_q^{n_q} \quad \text{for} \quad (X_1, X_2, \ldots, X_r) \in R$$

then f is continuously differentiable on R, and

$$\|w_1 - f \cdot \lambda\| < \frac{\varepsilon}{4}.$$

We make use now of Lemma I (for λ instead of μ, and all R as the neighborhood U of the diagonal) to get for any m large enough a chain Ω: $\emptyset = S_0 \subset S_1 \subset \ldots \subset S_{2m} = I \backslash N$ such that

(a) $\lambda(S_{2j}) = \dfrac{j}{m} \cdot e \qquad$ for $\quad 0 \leq j \leq m$

(b) $\lambda(S_{2j+1}) = \dfrac{j}{m} \cdot e + \dfrac{1}{m} \cdot \lambda(S^+)$, for $\quad 0 \leq j \leq m-1$, and

(c) $\|f \cdot \lambda\|_\Omega \geq \|\Phi(f \cdot \lambda)\| - \dfrac{\varepsilon}{4}$, where $e = (1,1,\ldots,1) = \lambda(I\backslash N)$, $S^+ \subset I\backslash N$ and Φ denotes the unique value on pNA (hence $\|\Phi\| = 1$).

Let m be large enough so that the next two conditions will also be satisfied: (d) between any two different consecutive τ_i's there is a number of the form $\frac{j}{m}(1 \le j \le m-1)$, and

(e) $\frac{1}{m}\cdot\left(2n\sum_{q=1}^{r}|a_q|\cdot n_q\right) < \frac{\varepsilon}{4}$.

Then, for each τ_i there is an integer j such that $\frac{j}{m} \le \tau_i < \frac{j+1}{m}$, and no different τ_i satisfies this (follows from (d)).

By (a), $\lambda(S_{2j}) = \frac{j}{m}\cdot e$, $\lambda(S_{2j+2}) = \frac{j+1}{m}\cdot e$, hence there is a set S^* whose measure is $\lambda(S^*) = \tau_i\cdot e$, and $S_{2j} \subset S^* \subset S_{2j+2}$ (this follows from the convexity of the range of λ by Lyapunov's theorem [Lyapunov, 1940], or directly from Aumann and Shapley [1968], Lemma 5.4). Now we replace S_{2j+1} in the chain Ω by S^*. Doing this for each τ_i (for equal τ_i's only once) we get a new chain Ω^* on $I\backslash N$. Clearly Ω^* satisfies (ii): if $\hat{\mu}_k \neq 0$, then $\hat{\mu}_k(S^*) = \tau_i$, or $\mu_k(S) = \tau_i\cdot\mu_k(I)$ (recall that $\hat{\mu}_k = \lambda_q$ for some q), and the same is true for $\hat{\mu}_k = 0$ (i.e., $\mu_k = 0$).

Replacing S_{2j+1} by S^*, the change in the variation of $f\circ\lambda$ will be

$$\begin{aligned}
&\big|\,\|f\circ\lambda\|\{S_{2j}\subset S_{2j+1}\subset S_{2j+2}\} - \|f\circ\lambda\|\{S_{2j}\subset S^*\subset S_{2j+2}\}\big| \\
&\qquad \le 2|(f\circ\lambda)(S_{2j+1}) - (f\circ\lambda)(S^*)| = 2\left|\sum_{q=1}^{r} a_q[\lambda_q^{n_q}(S_{2j+1}) - \lambda_q^{n_q}(S^*)]\right| \\
&\qquad \le 2\sum_{q=1}^{r}|a_q|\cdot n_q\cdot|\lambda_q(S_{2j+1}) - \lambda_q(S^*)| \le 2\sum_{q=1}^{r}|a_q|\cdot n_q\cdot\frac{1}{m}
\end{aligned}$$

(because $\frac{j}{m} = \lambda_q(S_{2j}) \le \lambda_q(S_{2j+1}) \le \lambda_q(S_{2j+2}) = \frac{j+1}{m}$ and the same holds for $\lambda_q(S^*)$). The number of such changes from Ω to Ω^* is at most n, hence:

$$\big|\,\|f\circ\lambda\|_\Omega - \|f\circ\lambda\|_{\Omega^*}\big| \le n\cdot 2\sum_{q=1}^{r}|a_q|\cdot n_q\cdot\frac{1}{m} < \frac{\varepsilon}{4}$$

(the last inequality is (e)). From (c) we get:

$$\|f\circ\lambda\|_{\Omega^*} > \|\Phi(f\circ\lambda)\| - \frac{\varepsilon}{2}$$

and finally:

$$\begin{aligned}
\|w_1\|_{\Omega^*} &\ge \|f\circ\lambda\|_{\Omega^*} - \|w_1 - f\circ\lambda\|_{\Omega^*} > \left(\|\Phi(f\circ\lambda)\| - \frac{\varepsilon}{2}\right) - \frac{\varepsilon}{4} \\
&\ge \|\Phi w_1\| - \|\Phi w_1 - \Phi(f\circ\lambda)\| - \frac{3\varepsilon}{4} \\
&\ge \|\Phi w_1\| - \|\Phi\|\cdot\|w_1 - f\circ\lambda\| - \frac{3\varepsilon}{4} > \|\Phi w_1\| - \varepsilon.
\end{aligned}$$

Thus Ω^* satisfies also (i), and the lemma is proved.

Lemma 3.7:

Given $\varepsilon > 0$, there is a chain Ω on $I \backslash N$ with the following properties:
(i) $\|w\|_\Omega > \|\Phi w\| - \varepsilon$,
(ii) for each $i (1 \leq i \leq n)$ there is a member S of the chain Ω such that $\mu_k(S) = \tau_i \mu_k(I)$ for all $k (1 \leq k \leq m)$. (Note that (ii) is the same as (ii) in lemma (3.6)).

Proof:

Let Ω be the chain obtained in lemma 3.6 for $\frac{\varepsilon}{2}$; then

$$\|w_1\|_\Omega > \|\Phi w_1\| - \frac{\varepsilon}{2} \quad \text{(property (i))}.$$

First, we assume that all non-zero measures $\hat{\mu}_k$ are pairwise different. Then

$$w = w_1 + \sum_k{}' h_k \cdot \hat{\mu}_k$$

satisfies all the assumptions of Lemma III. We apply it to each S in Ω such that some h_k has a jump at $\hat{\mu}_k(S)$, to get a new chain Ω_t (still on $I \backslash N$), such that

$$\|w_1\|_{\Omega_t} > \|\Phi w_1\| - \frac{\varepsilon}{2} \quad \text{(we took } \delta\text{'s small enough)}, \qquad (*)$$

and h_k is continuous at $\hat{\mu}_k(S)$ for all S in the chain Ω_t and for all k (with $\hat{\mu}_k \neq 0$).

Now we make use of the assumption (3.2): the continuity of f_k at the mentioned points implies the continuity of $\hat{f}_k$ at τ_i (see definition of $\hat{f}_k$ – (3.4)), hence the continuity of h_k at τ_i (g_k is absolutely continuous, thus continuous, and $h_k = \hat{f}_k - g_k$).

The chain Ω satisfies (ii), i.e., for each τ_i there is a member S of Ω such that $\hat{\mu}_k(S) = \tau_i$ if $\hat{\mu}_k \neq 0$. Therefore, all h_k are continuous at $\hat{\mu}_k(S)$ for all such S, and from the construction of Ω_t it is clear that they will be also members of the new chain Ω_t (they need no replacement). Hence, Ω_t satisfies too (ii).

Let Ω_t: $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_r = I \backslash N$, let $I^j = S_j \backslash S_{j-1}$ and

$$\mathscr{C}^j = \{S \cap I^j \mid S \in \mathscr{C}\}.$$

As in the mentioned proof, we define w^j on $(I^j, \mathscr{C}^j)$ by:

$$w^j(S) = w(S \cup S_{j-1}) - w(S_{j-1})$$

for $S \in \mathscr{C}^j$ and in the same manner we define w_1^j; the functions h_k^j are defined for $t \in [0, \hat{\mu}_k(I^j)]$ as follows:

$$h_k^j(t) = h_k(t + \hat{\mu}_k(S_{j-1})) - h_k(\hat{\mu}_k(S_{j-1})).$$

Obviously $w^j = w_1^j + \sum_k{}' h_k^j \cdot \hat{\mu}_k$ for all j and: $\sum_{j=1}^{r} \|h_k^j\| = \|h_k\|$ for all k.

Applying Lemma II to w^j, and using the inequality $\|u\| \geq |u(I)|$, we get

$$\|w^j\| \geq |w_1(S_j) - w_1(S_{j-1})| + \sum_k{}' \|h_k^j\|$$

(this is possible because h_k^j is continuous at 0 and $\hat{\mu}_k(I^j)$, which follows from the construction of Ω_t). Let Ω_j be a chain on I^j such that

$$\|w^j\|_{\Omega_j} \geq |w_1(S_j) - w_1(S_{j-1})| + \sum_k{}' \|h_k^j\| - \frac{\varepsilon}{2^{j+1}}. \qquad (**)$$

Let Ω^* be the chain (on $I \backslash N$) obtained by inserting between any two consecutive members of Ω_t, S_{j-1} and S_j, all the coalitions of the form $S_{j-1} \cup T$ where T ranges over Ω_j. Then, by definition of w^j, if follows that:

$$\|w\|_{\Omega^*} = \sum_{j=1}^{r} \|w^j\|_{\Omega_j},$$

hence we get by summing (**) over all $1 \leq j \leq r$

$$\|w\|_{\Omega^*} \geq \sum_{j=1}^{r} |w_1(S_j) - w_1(S_{j-1})| + \sum_{j=1}^{r} \sum_k{}' \|h_k^j\| - \frac{\varepsilon}{2} = \|w_1\|_{\Omega_t} + \sum_k{}' \|h_k\| - \frac{\varepsilon}{2}.$$

But $\|w_2\| = \sum_k{}' \|h_k\|$ (Lemma III), and let Φ be the unique value on $bv'NA$ (whose restriction on pNA is the unique value there and $\|\Phi\| = 1$ – cf. AUMANN and SHAPLEY [1968]), then we get (recall (*))

$$\begin{aligned} \|w\|_{\Omega^*} &> \left(\|\Phi w_1\| - \frac{\varepsilon}{2}\right) + \|w_2\| - \frac{\varepsilon}{2} \\ &\geq \|\Phi w_1\| + \|\Phi w_2\| - \varepsilon \geq \|\Phi(w_1 + w_2)\| - \varepsilon = \|\Phi w\| - \varepsilon. \end{aligned}$$

Thus the chain Ω^* satisfies (i). Being a refinement of Ω_t, it satisfies also (ii), and we proved the lemma in the case that all non-zero $\hat{\mu}_k$ are pairwise different.

In the general case, we may group terms in w_2 (e.g., if $\hat{\mu}_1 = \hat{\mu}_2 \neq 0$, we will write $(h_1 + h_2) \circ \hat{\mu}_1$ instead of $h_1 \circ \hat{\mu}_1 + h_2 \circ \hat{\mu}_2$) to get a new representation

$$w = w_1 + \sum_k{}'' h_k^* \circ \hat{\mu}_k$$

where the $\hat{\mu}_k$ in $\sum''$ are pairwise different, and each h_k^* is the sum of some h_k's, hence also singular and continuous at all the points τ_i. Using the result in the previous case, we get the chain satisfying (i) and (ii) (both properties are independent of the representation of w_2), and the lemma is proved.

We return now to the *proof of the Main Lemma.*

Let Ω: $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_r = I \backslash N$ be the chain obtained in lemma 3.7 for a given $\varepsilon > 0$.

We will build a chain $\tilde{\Omega}$ on I in the following manner: for each $i (1 \leq i \leq n)$, let $j = j(i) (1 \leq j < r)$ be the index such that $\mu_k(S_{j(i)}) = \tau_i \mu_k(I)$ for all k (such j exists because Ω satisfies (ii)). We will also denote $j(0) = 0$ and $j(n+1) = r$. Let $\tilde{\Omega}$ be the chain

$$\emptyset = U_0 \subset U_1 \subset \cdots \subset U_{r+m} = I$$

where

$$U_q = S_{q-i} \cup \pi^{-1}(J_i) \quad \text{for} \quad j(i) + i \leq q < j(i+1) + (i+1) (0 \leq i \leq n).$$

The meaning of this definition is that to each S in Ω coming between $S_{j(i)}$ and $S_{j(i+1)}$ corresponds $S \cup \pi^{-1}(J_i)$ in Ω; if $S = S_{j(i)}$, then both $S_{j(i)} \cup \pi^{-1}(J_{i-1})$ and $S_{j(i)} \cup \pi^{-1}(J_i)$ are members of $\tilde{\Omega}$.

Therefore, the variation of $v = \sum_{k=1}^{m} f_k \circ \omega_k$ over the chain $\tilde{\Omega}$ can be decomposed into the variation over the links $\{S_{j(i)} \cup \pi^{-1}(J_{i-1}) \subset S_{j(i)} \cup \pi^{-1}(J_i)\}$ (the links where the atoms are added) and the variation over the rest of the chain. The first one is:

$$\sum_{i=1}^{n} \left| v(S_{j(i)} \cup \pi^{-1}(J_i)) - v(S_{j(i)} \cup \pi^{-1}(J_{i-1}) \right| = \sum_{i=1}^{n} \Big| \sum_{k=1}^{m} [f_k(\mu_k(S_{j(i)}) + v_k \circ \pi^{-1}(J_i)) - f_k(\mu_k(S_{j(i)}) + v_k \circ \pi^{-1}(J_{i-1}))] \Big| .$$

By definition of $j(i)$, $\mu_k(S_{j(i)}) = \tau_i \mu_k(I)$ for all k, hence we get

$$\sum_{i=1}^{n} \left| \sum_{k=1}^{m} \Delta(i, \tau_i, v_k \circ \pi^{-1}, f_k) \right|$$

The second variation is then

$$\sum_{i=0}^{n} \sum_{q=j(i)+1}^{j(i+1)} \left| v(S_q \cup \pi^{-1}(J_i)) - v(S_{q-1} \cup \pi^{-1}(J_i)) \right| = \sum_{i=0}^{n} \sum_{q=j(i)+1}^{j(i+1)} \Big| \sum_{k=1}^{m} [f_k(\mu_k(S_q) + v_k \circ \pi^{-1}(J_i)) - f_k(\mu_k(S_{q-1}) + v_k \circ \pi^{-1}(J_i))] \Big| .$$

When $j(i) + 1 \le q \le j(i+1)$, both $\hat{\mu}_k(S_q)$ and $\hat{\mu}_k(S_{q-1})$ are between $\tau_i [= \hat{\mu}_k(S_{j(i)})]$ and $\tau_{i+1} [= \hat{\mu}_k(S_{j(i+1)})]$, for all non-zero $\hat{\mu}_k$. Recalling the definition of $\hat{f}_k$ (3.4), the expression in the brackets [ ] is exactly

$$[\hat{f}_k(\hat{\mu}_k(S_q)) - \hat{f}_k(\hat{\mu}_k(S_{q-1}))]$$

(because the sum $\sum \Delta(i)$ to be subtracted is the same). For $\hat{\mu}_k = 0$, hence $\mu_k = 0$, we get zero, hence finally

$$\sum_{i=0}^{n} \sum_{q=j(i)+1}^{j(i+1)} \left| \sum_k{}' (\hat{f}_k \circ \hat{\mu}_k)(S_q) - \sum_k{}' (\hat{f}_k \circ \hat{\mu}_k)(S_{q-1}) \right| = \sum_{q=1}^{r} |w(S_q) - w(S_{q-1})| = \|w\|_{\Omega} > \|\Phi w\| - \varepsilon$$

(the last inequality holds because Ω satisfies (i) in lemma 3.7).

Adding the two variations we get from (3.5):

$$\|v\|_{\tilde{\Omega}} > \sum_{i=1}^{n} \left| \sum_{k=1}^{m} \Delta(i, \tau_i, v_k \circ \pi^{-1}, f_k) \right| + \|\Phi w\| - \varepsilon = \left\| \sum_{k=1}^{m} \psi_{\pi,\tau}(f_k \circ \omega_k) \right\| - \varepsilon .$$

But $\|v\| \ge \|v\|_{\tilde{\Omega}}$ hence:

$$\|v\| \ge \left\| \sum_{k=1}^{m} \psi_{\pi,\tau}(f_k \circ \omega_k) \right\|$$

(both sides are independent of ε), and the main lemma is proved.

4. Proof of the Theorems

Before proving the theorems, we need the following:

Lemma 4.1 (R. J. AUMANN)[5]):

Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be measures on $(I, \mathscr{C})$. Let $X_1, X_2, \ldots, X_m$ be real-valued random variables on the space $(J, \mathscr{D})$.

Then $\left\| \sum_{k=1}^{m} X_k \cdot \lambda_k \right\|$ is a random variable (on $(J, \mathscr{D})$), and

$$\left\| \sum_{k=1}^{m} E(X_k) \cdot \lambda_k \right\| \leq E \left\| \sum_{k=1}^{m} X_k \cdot \lambda_k \right\|.$$

Proof:

Without loss of generality, let $(I, \mathscr{C})$ be $([0,1], \mathscr{B})$ (assumption 1.1). Let $\mathscr{S}$ be the family of all unions of dyadic intervals (i.e., intervals of the form $\left[\frac{m}{2^k}, \frac{m+1}{2^k}\right]$ for positive integers m and k). $\mathscr{S}$ is countable, and it is well known that for any measure θ on $(I, \mathscr{C})$, every measurable set in I can be approximated in θ-measure by a member of $\mathscr{S}$.

Let $\lambda = \sum_{k=1}^{m} X_k \cdot \lambda_k$. For each $t \in J$, $\lambda(t) = \sum_{k=1}^{m} X_k(t) \cdot \lambda_k$ is a measure on $(I, \mathscr{C})$. By the Hahn decomposition, there is a set $S = S(t)$ in $\mathscr{C}$ such that

$$\|\lambda(t)\| = |\lambda(t)(S)| + |\lambda(t)(I \backslash S)|$$

(cf. AUMANN and SHAPLEY [1968], proof of Proposition 7.6). Since we can approximate $S(t)$ in $\lambda(t)$ – measure by members of $\mathscr{S}$, it follows that:

$$\|\lambda(t)\| = \sup_{S \in \mathscr{S}} (|\lambda(t)(S)| + |\lambda(t)(I \backslash S)|) \tag{*}$$

(the opposite inequality is always true: for each $S \in \mathscr{S}$ let Λ be the chain $\emptyset \subset S \subset I$, then:

$$\|\lambda(t)\| \geq \|\lambda(t)\|_\Lambda = |\lambda(t)(S) - \lambda(t)(\emptyset)| + |\lambda(t)(I) - \lambda(t)(S)| = |\lambda(t)(S)| + |\lambda(t)(I \backslash S)|$$

because $\lambda(t)$ is a measure, hence additive). For each fixed S,

$$\lambda(S) = \sum_{k=1}^{m} X_k \cdot \lambda_k(S)$$

is a linear combination of random variables, hence measurable. Thus $|\lambda(S)| + |\lambda(I \backslash S)|$ is measurable for each S in $\mathscr{S}$. Since the supremum of a denumerable number of measurable functions is measurable, if follows from (*) that $\|\lambda\|$ is measurable, i.e. a random variable (on $(J, \mathscr{D})$).

Next, let $\xi = \sum_{k=1}^{m} E(X_k) \cdot \lambda_k$. Then ξ is a measure on $(I, \mathscr{C})$, and let $(S, I \backslash S)$ be the corresponding Hahn decomposition. We get:

[5]) Private communication.

$$\|\xi\| = |\xi(S)| + |\xi(I\backslash S)| = |E(\lambda(S))| + |E(\lambda(I\backslash S))| \leq E(|\lambda(S)| + |\lambda(I\backslash S)|) \leq E\|\lambda\|$$

$$\text{i.e.:} \quad \left\|\sum_{k=1}^{m} E(X_k)\cdot\lambda_k\right\| \leq E\left\|\sum_{k=1}^{m} X_k\cdot\lambda_k\right\|$$

as was to be proved.

Proof of Theorem A:

Let $v = \sum_{k=1}^{m} f_k \circ \omega_k$ be as in the main lemma. Then

$$\left\|\sum_{k=1}^{m} \Phi_p(f_k\circ\omega_k)\right\| = \sum_{s\in N}\left|\sum_{k=1}^{m} \Phi_p(f_k\circ\omega_k)(\{s\})\right| + \left\|\sum_{k=1}^{m} \alpha_p(f_k\circ\omega_k)\cdot\mu_k\right\|.$$

The first term is

$$\sum_{s\in N}\left|\sum_{k=1}^{n} E\left(\frac{1}{n!}\sum_{\pi\in\Pi}\psi_{\pi,T}(f_k\circ\omega_k)(\{s\})\right)\right| \leq E\left(\frac{1}{n!}\sum_{\pi\in\Pi}\left(\sum_{s\in N}\left|\sum_{k=1}^{m}\psi_{\pi,T}(f\circ\omega_k)(\{s\})\right|\right)\right).$$

The second term is

$$\left\|\sum_{k=1}^{m} E\left(\frac{1}{n!}\sum_{\pi\in\Pi}\alpha_{\pi,T}(f_k\circ\omega_k)\right)\cdot\mu_k\right\| \leq E\left(\frac{1}{n!}\sum_{\pi\in\Pi}\left\|\sum_{k=1}^{m}\alpha_{\pi,T}(f_k\circ\omega_k)\cdot\mu_k\right\|\right)$$

(by lemma 4.1). Adding the above two inequalities, and recalling (3.3), we obtain

$$\left\|\sum_{k=1}^{m} \Phi_p(f_k\circ\omega_k)\right\| \leq E\left(\frac{1}{n!}\sum_{\pi\in\Pi}\left\|\sum_{k=1}^{m}\psi_{\pi,T}(f_k\circ\omega_k)\right\|\right). \tag{4.2}$$

For each $k(1 \leq k \leq m)$, f_k is the difference of two monotonic functions, hence the number of its discontinuities is countable. Since p was assumed to be a continuous probability measure, the probability that at least one of the $2n$ points $T^{(i)}\mu_k(I) + v_k \circ \pi^{-1}(J_i)$ and $T^{(i)}\mu_k(I) + v_k\circ\pi^{-1}(J_{i-1})$ $(1 \leq i \leq n)$ is a discontinuity point of f_k (i.e., (3.2) is not satisfied) is zero. Furthermore, let M_k be such that $|f_k(t)| \leq M_k$ for all $t \in [0, 1]$. Then, by (3.5)

$$\left\|\sum_{k=1}^{m}\psi_{\pi,\tau}(f_k\circ\omega_k)\right\| \leq \sum_{i=1}^{n}\sum_{k=1}^{m} 2\cdot M_k + \sum_{k=1}^{n}(2n+1)\cdot M_k = (4n+1)\sum_{k=1}^{n} M_k.$$

Hence, the expectation in (4.2) over all values τ of T such that condition (3.2) is not satisfied is zero (a bounded variable over a zero-probability set). Using the main lemma for all other values τ of T, we get finally

$$\left\|\sum_{k=1}^{m} \Phi_p(f_k\circ\omega_k)\right\| \leq \|v\|. \tag{4.3}$$

Let A be the subspace of $bv'FL$ consisting of all set functions v of the form $v = \sum_{k=1}^{m} f_k\circ\omega_k$, where $f_1, f_2, \ldots, f_m \in bv'$ and $w_1, w_2, \ldots, w_k \in FL^+$ are probability measures. We define Φ on A by:

$$\Phi v = \Phi\left(\sum_{k=1}^{m} f_k \circ \omega_k\right) = \sum_{k=1}^{m} \Phi_p(f_k \circ \omega_k).$$

It follows from (4.3) that Φ is well defined on A (i.e., for any representation of v, Φv is the same), and $||\Phi|| \leqslant 1$. By definition, A is a dense linear subspace of $bv'FL$; Φ is a linear continuous operator from A into FA, which is complete (cf. AUMANN and SHAPLEY [1968], Propositions 4.3 and 4.4). Hence Φ can be uniquely extended to a continuous linear operator from $bv'FL$ into FA, (we denote it also Φ), such that $||\Phi|| \leq 1$ (in fact, $||\Phi|| = 1$ because the restriction of Φ over pNA is the unique value there).

We will show that Φ is indeed a value on $bv'FL$. Φ is linear by definition, hence (1.2) holds. For all v of the form $v = f \circ \omega (f \in bv', \omega \in FL^+$ a probability measure), $(\Phi v)(I) = (\Phi_p v)(I) = f(1) = v(I)$. Hence, $(\Phi v)(I) - v(I)$ is a continuous linear functional that vanishes on a spanning set of $bv'FL$, and is therefore identically zero; this proves (1.5). Let θ be an automorphism in $\mathscr{J}$, then it is easy to verify that $\Phi\theta_* v = \theta_* \Phi v$ for all $v = f \circ \omega$, hence for all $v \in bv'FL (\Phi\theta_* - \theta_*\Phi$ is continuous and linear, and vanishes on a spanning subset); thus (1.4) holds. From (1.5) and $||\Phi|| \leq 1$ follows the positivity (1.3) (cf. AUMANN and SHAPLEY [1968], Proposition 4.6). Φ satisfies (1.2)–(1.5), hence is a value on $bv'FL$, as was to be proved.

Proof of Theorem B:

The proof of Theorem B is exactly the same as the proof of Theorem A, taking lc' (rc', respectively) instead of bv'. The only thing to prove here is inequality (4.3) for all p (not necessarily continuous).

Let $v = \sum_{k=1}^{m} f_k \circ \omega_k$, where $f_1, f_2, \ldots, f_m \in lc'$, and $\omega_1, \omega_2, \ldots, \omega_m \in FL^+$ probability measures. For each τ satisfying (3.2), the inequality

$$\left\| \sum_{k=1}^{m} \psi_{\pi,\tau}(f_k \circ \omega_k) \right\| \leq ||v|| \tag{5.4}$$

holds by the main lemma. We will prove that it holds for all τ.

The number of discontinuities of each f_k being countable, for each τ and $\varepsilon > 0$ there is $\tau' = (\tau'_1, \tau'_2, \ldots, \tau'_n)$ $(0 < \tau'_1 \leq \tau'_2 \leq \cdots \leq \tau'_n < 1)$ such that:

(i) τ' satisfies (3.2)

(ii) $\tau'_i \leq \tau_i$ for all $i (1 \leq i \leq n)$

(iii) $|f_k[\tau_i \mu_k(I) + \nu_k \circ \pi^{-1}(J_i)] - f_k[\tau'_i \mu_k(I) + \nu_k \circ \pi^{-1}(J_i)]| < \frac{\varepsilon}{4nm}$

and

$$|f_k[\tau_i \mu_k(I) + \nu_k \circ \pi^{-1}(J_{i-1})] - f_k[\tau'_i \mu_k(I) + \nu_k \circ \pi^{-1}(J_{i-1})]| < \frac{\varepsilon}{4nm}$$

for all $i (1 \leq i \leq n)$ and $k (1 \leq k \leq m)$ ((iii) holds because each f_k is left continuous). From (3.5), we get:

$$\left| \left\| \sum_{k=1}^{m} \psi_{\pi,\tau}(f_k \circ \omega_k) \right\| - \left\| \sum_{k=1}^{m} \psi_{\pi,\tau'}(f_k \circ \omega_k) \right\| \right| < \sum_{i=1}^{n} \sum_{k=1}^{m} 2 \cdot \frac{\varepsilon}{4nm} + \sum_{k=1}^{m} 2n \cdot \frac{\varepsilon}{4nm} = \varepsilon.$$

Applying the main lemma to τ' (by (i)), it follows:

$$\left\| \sum_{k=1}^{m} \psi_{\pi,\tau}(f_k \circ \omega_k) \right\| < \|v\| + \varepsilon .$$

The inequality holds for all $\varepsilon > 0$, hence (4.4) also holds – for *all* values τ of T. Summing over $\pi \in \Pi$ and taking expectation, it follows from (4.2) that

$$\left\| \sum_{k=1}^{m} \Phi_p(f_k \circ \omega_k) \right\| \leq \|v\|$$

i.e. (4.3) is proved, and we continue as in the proof on Theorem A. In case of $rc'FL$, only condition (ii) needs changing to: $\tau_i' \geq \tau_i$ for all $i (1 \leq i \leq n)$.

5. Discussion

It is obvious that the values we have defined are different. Let p_1 and p_2 be different probability measures (we will assume them also continuous such that Φ_{p_1} and Φ_{p_2} will be values on $bv'FL$), then by definition of the Borel σ-field of $(0,1)$ (on which p_1 and p_2 are defined), there are $0 \leq \alpha < \beta < 1$ with $p_1((\alpha,\beta]) \neq p_2((\alpha,\beta])$. Let $f \in bv'$ be:

$$f(t) = \begin{cases} 0 & 0 \leq t \leq \dfrac{\beta}{1 + (\beta - \alpha)} \\ 1 & \dfrac{\beta}{1 + (\beta - \alpha)} < t \leq 1 ; \end{cases}$$

let $\mu \in NA^+$ with $\mu(I) = \dfrac{1}{1 + (\beta - \alpha)}$, and let $v \in FC^+$ with carrier $N = \{1\}$ and $v(N) = \dfrac{\beta - \alpha}{1 + (\beta - \alpha)}$. Then $v = f \circ (\mu + v) \in bv'FL$. It is easy to verify that $\Delta(i, \tau, v, f)$ is 1 for $\tau \in (\alpha, \beta]$ and 0 otherwise, hence:

$$(\Phi_{p_1} v)(\{1\}) = p_1((\alpha,\beta]) \neq p_2((\alpha,\beta]) = (\Phi_{p_2} v)(\{1\}) .$$

Thus $\Phi_{p_1} v \neq \Phi_{p_2} v$ and $\Phi_{p_1} \neq \Phi_{p2}$.

Therefore, there is no unique value on the spaces of mixed games, whereas this is the case on the spaces of non-atomic games (cf. AUMANN and SHAPLEY [1968]) and games with finite carriers (i.e., "finite games" – cf. SHAPLEY [1953]). Furthermore, the number of values is uncountable.

This suggests that an improvement in the definition of the axiomatic value is needed, such that only the value corresponding to the uniform distribution will be a value (cf. SHAPLEY and SHAPIRO [1960] and MILNOR and SHAPLEY [1961]; this is the limit of the values of games where the "ocean" was divided into a large number of "minor" players with measures tending to zero).

We will remark that each value Φ defined here restricted to $bv'NA$ is identical with the unique value there, and the same is true for the space FIN of all finite games (obviously $FIN \subset bv'FL$).

A problem that is still open is whether Theorem A remains true for non-continuous probability measures p (i.e., whether Φ_p defines a value on $bv'FL$ or not).

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