Z. Phys. B – Condensed Matter 63, 517-520 (1986)

Condensed Matter
Zeitschrift für Physik B
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Field Theory of Long Time Behaviour in Driven Diffusive Systems

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Received February 25, 1986

We present a renormalisation group study for the long time behaviour of a diffusive system with a single conserved density which is subjected to an external driving force. In the asymptotic long wavelength limit the system approaches an infrared stable fixed point where detailed balance is satisfied. We obtain the exact scaling form of the density correlation function. In one dimension, the corresponding universal amplitude agrees excellently with a recent Monte Carlo simulation.

The long time (low frequency, long wavelength) behaviour of hydrodynamic systems has attracted considerable interest over the past two decades. The typical scenario which is also applicable to the work reported here is as follows: in high spatial dimension, standard perturbational methods suffice to capture the long time behaviour of hydrodynamic systems, characterised by the usual long time tails. In low dimensions, however, anomalous properties arise which require more elaborate methods.

Recently, van Beijeren et al. [1] (henceforth abbreviated BKS) studied a continuum model for the diffusion of particles subjected to a driving force. The equation of motion for the particle density in the BKS model is given by

$$\frac{\partial}{\partial t} c(\mathbf{r}, t)+\nabla\mathbf{j}(\mathbf{r}, t)=0, \tag{1a}$$

$$\mathbf{j}(\mathbf{r}, t)=-D\nabla c(\mathbf{r}, t)+c(\mathbf{r}, t)\,\mathbf{u}(c(\mathbf{r}, t))+\mathbf{j}_L(\mathbf{r}, t). \tag{1b}$$

The first term on the right hand side of (1b) is simply the unperturbed diffusion current. The second term models the current contribution that is generated by the driving force. The direction of the velocity field $\mathbf{u}$ is given by the direction of the driving force and will be labelled "parallel" from now on. In addition, there is a Gaussian white noise contribution $\mathbf{j}_L$ which is assumed to summarise the effects of the fast microscopic degrees of freedom.

BKS use the mode coupling approximation [2–4] to analyse their model. For spatial dimensions $d>2$, they find that the current-current correlation function exhibits the standard hydrodynamic long time tail $\sim t^{-d/2}$. For $d\leqq 2$, however, the long time behaviour turns out to be anomalous, implying that density fluctuations spread faster than diffusively.

In this paper, we analyse a generalisation of the BKS model by renormalisation group methods [5, 6], below the critical dimension $d_c=2$. The reason for our work is twofold:

(i) The mode coupling approximation, in its standard form, involves calculating the self energy on the basis of the full propagators but neglecting vertex corrections. It is therefore essentially an uncontrolled expansion. Problems typically arise if the vertices require a non-trivial renormalisation and hence change the general scaling behaviour (see also Hohenberg and Halperin [7] for a discussion). In general, it is not a priori obvious whether a certain vertex need not be renormalised. Thus, in the presence of strong infrared singularities renormalisation group methods in conjunction with systematic loopwise perturbation expansions are more appropriate.

(ii) The equation of motion as written down by BKS implicitly assumes that detailed balance is val-

id. However, in general this need not be justified for strong non-equilibrium situations. One should therefore start with a suitably modified equation of motion.

Following this programme, we shall show explicitly that detailed balance is dynamically generated in the asymptotic low frequency and long wavelength limit. It will then become clear that no vertex renormalisation is required, due to the additional symmetry and the structure of the nonlinearity. This is the reason why the mode coupling approximation predicts the correct scaling behaviour for this model. Further, we compute the (universal) amplitude of the density-density correlation function. In one dimension, its value is compared with the corresponding Monte Carlo result given by BKS. The agreement is excellent.

In one dimension, our model is equivalent to the Burgers' equation with external noise. This case was studied by Forster et al. [8] who performed a Wilson-Fisher recursion relation analysis. Note, however, that the continuation of Burgers' equation to dimensions greater than one which was suggested by these authors genuinely differs from our model: their model is isotropic and involves a vector order parameter.

Expanding Eqs. (1a, b) in $s(\mathbf{r},t):=c(\mathbf{r},t)-\bar{c}$, i.e. in the deviation of the density $c(\mathbf{r},t)$ from its uniform average $\bar{c}$ and keeping remark (ii) in mind one obtains

$$\dot{s}(\mathbf{r},t)=\lambda(\Delta_\perp+\rho\Delta_\parallel)s(\mathbf{r},t)+\tfrac{1}{2}\lambda g\nabla_\parallel s^2(\mathbf{r},t)+\zeta(\mathbf{r},t), \quad (2\,\mathrm{a})$$

$$\langle\zeta(\mathbf{r},t)\zeta(\mathbf{r}',t')\rangle=-\lambda(\Delta_\perp+\sigma\Delta_\parallel)\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') \quad (2\,\mathrm{b})$$

after a suitable Galilean transformation which takes care of linear terms in the expansion of $c(\mathbf{r},t)\mathbf{u}(c(\mathbf{r},t))$, cf. also BKS. The equations of motion (2a, b) are the most general ones for a conserved density which is spatially isotropic with respect to the transverse directions. The parameters ρ and σ take care of anisotropies which the driving force induces in the diffusion constant and the Gaussian noise term. Dimensional analysis near the critical dimension $d_c=2$ shows that terms which might be expected to modify (2) are irrelevant in the renormalisation group sense.

In contrast to BKS who assume $\rho=\sigma=1$, we allow for any value of the parameters ρ and σ in order to obtain a renormalisable model. Detailed balance is satisfied if $\rho=\sigma$.

To set up a renormalised field theory it is convenient to recast the model in terms of a dynamic functional [9–11]

$$J[s,\tilde{s}]=\int \mathrm{d}t\,\mathrm{d}^dx\{\tilde{s}[\dot{s}-\lambda(\Delta_\perp+\rho\Delta_\parallel)s]+\tfrac{1}{2}\lambda g(\nabla_\parallel\tilde{s})s^2 +\lambda\tilde{s}(\Delta_\perp+\sigma\Delta_\parallel)\tilde{s}\} \quad (3)$$

where $\tilde{s}(\mathbf{r},t)$ is a Martin-Siggia-Rose [12] response field. Correlation and response functions can now be expressed as functional averages with weight $\exp(-J)$. The functional J is invariant under the scale transformation

$$\begin{aligned}&x_\parallel\to\alpha^2x_\parallel, \quad x_\perp\to x_\perp\\ &s\to\alpha^{-1}s, \quad \tilde{s}\to\alpha^{-1}\tilde{s}\\ &\rho\to\alpha^4\rho, \quad \sigma\to\alpha^4\sigma, \quad g\to\alpha^3g.\end{aligned} \quad (4)$$

Thus the appropriate invariant parameters of the model are

$$w:=\sigma/\rho, \quad f:=g^2/\rho^{3/2}. \quad (5)$$

Once J is given, the study of infrared properties follows standard renormalisation group methods [11, 13, 14]. We use dimensional regularisation in $d=2-\varepsilon$ followed by minimal subtraction. Denoting by $\Gamma_{\tilde{n}n}$ the one particle irreducible vertex functions with $\tilde{n}$ $\tilde{s}$-legs and n s-legs we find that only Γ_{11}, Γ_{20} and Γ_{12} are primitively divergent. The divergences are absorbed multiplicatively in a redefinition of the parameters (from now on, bare quantities are given an extra index zero):

$$\rho_0=Z_\rho\rho, \quad \sigma_0=Z_\sigma\sigma, \quad g_0^2=Z_u u\mu^\varepsilon \quad (6)$$

where μ^{-1} is an arbitrary external length scale. To the order of one loop we find

$$\begin{aligned}&Z_\rho=1-\frac{1}{\varepsilon}(3+w)\frac{v}{8}+O(v^2)\\ &Z_\sigma=1-\frac{1}{\varepsilon}(3w+2+3w^{-1})\frac{v}{16}+O(v^2)\\ &Z_u=1+O(v^2)\end{aligned} \quad (7)$$

where

$$v=\frac{\Gamma\left(1+\frac{\varepsilon}{2}\right)}{(4\pi)^{d/2}}u\rho^{-3/2}$$

and $\Gamma(z)$ is Euler's Γ-function.

The fact that the unrenormalised theory is independent from μ leads to the renormalisation group equation

$$\left\{\mu\frac{\partial}{\partial\mu}+\beta_v\frac{\partial}{\partial v}+\beta_w\frac{\partial}{\partial w}+\rho\cdot\zeta_\rho\cdot\frac{\partial}{\partial\rho}\right\} \cdot\Gamma_{\tilde{n}n}(\{q\omega\},\rho,v,w,\mu,\lambda)=0. \quad (8)$$

The parameter functions are given by

$$\beta_v = v(-\varepsilon + \tfrac{3}{16}(3+w)\,v + O(v^2))$$

$$\beta_w = -(w^2 - 4w + 3)\frac{v}{16} + O(v^2)$$

$$\zeta_\rho = -(3+w)\frac{v}{8} + O(v^2). \qquad (9)$$

We find an infrared stable fixed point

$$v^* = \tfrac{4}{3}\varepsilon + O(\varepsilon^2), \qquad w^* = 1 + O(\varepsilon) \qquad (10)$$

with correction exponents

$$\bar{\omega}_w = \frac{\partial}{\partial w}\beta_w|_* = \frac{\varepsilon}{6} + O(\varepsilon^2)$$

$$\bar{\omega}_v = \frac{\partial}{\partial v}\beta_v|_* = \varepsilon + O(\varepsilon^2) \qquad (11)$$

and a region of attraction $v > 0$, $w < 3 + O(\varepsilon)$.

If $w > 3 + O(\varepsilon)$, the infrared behaviour is dominated by a degenerate fixed point $v = 0$, $w = \infty$. We believe that this state may be reached if the driving force is very large. In the following, however, we shall concentrate on the case $w < 3 + O(\varepsilon)$.

If $w = 1$, the model satisfies detailed balance. This is easily seen by rewriting the dynamic functional J, Eq. (3) for $\rho = \sigma$, in the form

$$J = \int \mathrm{d}t\, \mathrm{d}^d x \left\{ \tilde{s}\left(\dot{s} + R\frac{\delta H}{\delta s}\right) - \tilde{s}R\tilde{s} \right\}$$

$$-R = \lambda[(\Delta_\perp + \rho\Delta_\parallel) + \frac{g}{3}(s\nabla_\parallel + \nabla_\parallel s)$$

$$H = \int \mathrm{d}^d x\, \tfrac{1}{2}s^2. \qquad (12)$$

Obviously, J now obeys the detailed balance symmetry [11]

$$s(t) \to -s(-t), \qquad (13\,\mathrm{a})$$

$$\tilde{s}(t) \to s(-t) - \left.\frac{\delta H}{\delta s}\right|_{-t} = \tilde{s}(-t) - s(-t) \qquad (13\,\mathrm{b})$$

where $\exp(-H[s])$ is the (purely Gaussian) stationary state distribution. Equation (13 b) implies in particular that

$$\langle s(t)\,\tilde{s}(0)\rangle = \Theta(t)\langle s(t)\,s(0)\rangle. \qquad (14)$$

A few remarks are in order:

(i) The mode coupling equation of BKS may be obtained by writing down a Dyson equation for the full propagator $\langle s(t)\,\tilde{s}(0)\rangle$, using Eq. (14), but neglecting all vertex corrections. This confirms explicitly that BKS have used detailed balance.

(ii) Above, we have seen that the infrared behaviour of the system is dominated by the detailed balance fixed point $w^* = 1$, up to first order in ε. Since this fixed point is an exact fixed point, due to its additional symmetry, it dominates the infrared behaviour to all orders in ε. Consequently, higher order calculations may now be simplified by working directly at $w = w^* = 1$.

(iii) The perturbation vertices $\lambda g(\nabla_\parallel \tilde{s})s^2$ are linearly dependent on momentum. In conjunction with momentum conservation and Eq. (14) this leads to the result that the ultraviolet divergent graphs associated with Γ_{12} sum to zero order by order in perturbation theory.

Thus the coupling g need not be renormalised:

$$g_0^2 = g^2 = u\mu^\varepsilon \qquad (15)$$

whence, to all orders in ε,

$$\beta_v = -(\varepsilon + \tfrac{3}{2}\zeta_\rho)\,v \qquad (16)$$

implying that

$$\zeta_\rho(v^*) = -\tfrac{2}{3}\varepsilon \qquad (17)$$

at the fixed point $v^* \neq 0$, to all orders.

A two loop calculation yields

$$\zeta_\rho = -\tfrac{1}{2}[1 + \tfrac{1}{8}(1 + 3\ln\tfrac{3}{4})\,v + O(v^2)]\,v \qquad (18)$$

giving the fixed point to order ε^2

$$v^* = \tfrac{4}{3}\varepsilon[1 - \tfrac{1}{6}(1 + 3\ln\tfrac{4}{3})\,\varepsilon + O(\varepsilon^2)]. \qquad (19)$$

The scale transformation Eq. (4), dimensional analysis, and the renormalisation group equation (8) may now be used to derive the exact scaling form of the density correlation function:

$$S(q,\omega) = \omega^{-1} f(\omega^{-\frac{1}{6}(5-d)} q_\parallel, \omega^{-\frac{1}{2}} q_\perp) \qquad (20)$$

which reads, in $d = 1$,

$$S(q,\omega) = \omega^{-1} f(\omega^{-2/3} q), \qquad (21\,\mathrm{a})$$

$$\hat{S}(q,t) = \hat{f}(q t^{2/3}) \qquad (21\,\mathrm{b})$$

where $\hat{S}(q,t)$ denotes the Fourier transform of $S(q,\omega)$. This result agrees with the (approximate) form given by BKS and also with the form derived by Forster et al. [8].

Explicitly, we find, for $q^2/\mu^2 \ll 1$, neglecting terms of order ε^2

$$S(q,\omega) = -i\omega^{-1}\left\{1 - \frac{1}{2}\left[\frac{2\lambda\mu^2}{i\omega}\cdot\frac{q_\perp^2}{\mu^2} + \left(\frac{2\lambda\mu^2}{i\omega}\right)^{\frac{5-d}{3}} \frac{\rho q_\parallel^2}{\mu^2}\right] + O(q^4/\mu^4)\right\}. \qquad (22)$$

Expressing $\hat{S}(q,t)$ in the dimensionless variables

$$q^2 = (g^4/2^8)^{1/\varepsilon}\,\hat{q}^2$$
$$\lambda t = (2^7/g^4)^{1/\varepsilon}\,\hat{t} \qquad (23)$$

which correspond to the scales defined by BKS, and setting $\varepsilon = 1$, we find

$$\hat{S}(q,t) = 1 - \tfrac{1}{2} C \hat{q}^2 \cdot \hat{t}^{4/3} + O(\hat{q}^4) \qquad (24\,\mathrm{a})$$

where

$$C = \frac{3^{8/3}}{4\Gamma(1/3)} (\tfrac{3}{4} v^*)^{-2/3}. \qquad (24\,\mathrm{b})$$

The prefactor in the previous equation is of purely geometric origin, i.e. independent of the fixed point, and has therefore not been expanded in ε. This procedure is empirically known to give the best numerical results [15]. Setting now $\varepsilon = 1$ in (24 b) yields $C = 2.15 \pm 0.05$ which agrees excellently with the Monte Carlo value $C_{\mathrm{MC}} \approx 2.1$ that was calculated by BKS. The error bars on our result reflect the fact that the value of C depends slightly on how the ε-expansion is extrapolated.

In order to analyse the asymptotic behaviour of the model in two dimensions, one sets ε equal to zero in (9) and solves the renormalisation group equation (8) with the modified parameter functions. The results for the logarithmic corrections in the density correlation function agree with BKS, up to a non-universal scale factor.

In conclusion, we wish to remark that the reason why BKS have found the correct scaling form can now be fully understood: Our renormalisation group analysis vindicates the assumption made by BKS that detailed balance is asymptotically valid. As a consequence, no renormalisation of the vertex g_0 is needed, and hence the mode coupling approximation predicts the correct scaling behaviour. An analogous situation is found in the critical dynamics of models with reversible non-linear couplings [16] (e.g. isotropic ferromagnets and antiferromagnets) close to equilibrium. Yet in contrast to the mode coupling approximation, the renormalisation group provides a systematic account of corrections, due to the non-trivial renormalisation of the coupling v. They are specifically required for the computation of universal amplitude ratios like the constant C. Thus the RG study of our model provides both an improvement and a deeper understanding of the mode coupling approximation. The extension of our calculations to higher orders is straightforward.

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