Articles 1

CONDORCET'S PARADOX AND ANONYMOUS PREFERENCE PROFILES

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Condorcet's paradox [6] of simple majority voting occurs in a voting situation with n voters and m candidates or alternatives if for every alternative there is a second alternative which more voters prefer to the first alternative than conversely. The paradox can arise only if the strict simple majority relation on the alternatives is cyclic, provided that m is finite.

Studies of the paradox are usually based either on profiles or A-profiles (anonymous preference profiles). A *profile* is a function that assigns a preference order on the alternatives to each voter. An *A-profile*, which has also been called a return [28], profile [31] and pattern [20], is a function that assigns a nonnegative number of voters to each potential preference order on the alternatives such that the sum of the assigned integers equals n. In general, many different profiles – which retain voter identities – map into the same A-profile, and any two profiles that map into the same A-profile bear the same simple majority relation on the alternatives. Hence, it may appear that it is purely a matter of personal taste or analytical convenience whether one works with profiles or with A-profiles in studying Condorcet's paradox. Although this is true in one sense, there are important differences between the two bases that will be explored in the present paper. Of special concern will be the fact that some A-profiles correspond to very few profiles (consider an A-profile that assigns all n voters to the same preference

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order) whereas other A-profiles correspond to a multitude of profiles.

An examination of the spaces of profiles and of A-profiles as they relate to Condorcet's paradox can provide new insights into the paradox. Working with three alternatives and assuming that n is odd with each voter having a linear preference order, *exact* expressions are given for the proportion of A-profiles that avoid the paradox and for the expected proportions of voters who have the simple majority alternative ranked first, second and third in their preference orders, given that there is a simple majority alternative. The numbers derived from these formulas will be compared to their counterparts obtained from the space of profiles. In addition, we shall show how the likelihood of the paradox can be computed on the basis of a probability distribution over A-profiles and note that this approach to likelihood computation cannot in general be duplicated by the more traditional method in which each voter independently selects a preference order according to some probability distribution over orders.

Although many studies of social choice [e.g. 2, 10, 27] are based on profiles since this beginning allows greater flexibility in discussing various aspects of the theory, the A-profile base has proved most useful in examinations of anonymous social choice rules. Moreover, the A-profile viewpoint is most natural when one considers voting blocs or coalitions whose members share common interests. In many real situations with small numbers of candidates or alternatives the parties at interest can be identified with specific preference orders, thus providing an appealing interpretive base for further analysis. If one party or bloc dominates the situation then the outcome may be obvious, but if power is dispersed among a number of blocs (whose actual sizes may be subject to considerable uncertainty) then the picture can be very complex. The latter type of situation seems intuitively more likely to give rise to Condorcet's paradox, and this will be borne out by our comparison of the profile and A-profile spaces. Further evidence on this point will be provided by relating the occurrence of the paradox to the sum of squares of the numbers of voters who have the same preference orders. This sum is maximized when the electorate is unanimous and is minimized when voters are evenly distributed over the potential preference orders.

Within the sphere of anonymous social choice rules, A-profiles have been used extensively in variable-electorate social choice theory for axiomatizations of Borda's rule [4, 30] and other positional voting rules [8, 9, 26, 29]. Within the simple majority domain, the A-profile viewpoint is used to identify restrictions on sets of individuals' preferences which prohibit the occurrence of Condorcet's paradox [2, 3, 10, 16, 24, 27, 29]. With n_i the number of voters who have the i^{th} preference order in an A-profile, these restrictions (the best known of which is single-peaked preferences) can be characterized by subsets of orders whose n_i values are required to equal zero. In terms of the A-profile space as defined in the next section, the restrictions identify forbidden regions of the space.

Several studies [12, 17, 20, 22] have examined relationships between the likelihood of Condorcet's paradox and various measures of 'social homogeneity' that reflect the degree to which voters have 'similar' preference orders. Niemi [22], who assesses social homogeneity by the maximum number of voters whose preference orders are single peaked for some linear order on the alternatives, and Fishburn [12], who measures homogeneity by the Kendall-Smith coefficient of concordance [19], consider the aforementioned relationship under the assumption that each profile is equally likely to obtain. On the other hand, Kuga and Nagatani [20], whose social homogeneity measure is inversely related to their 'antagonism intensity' within an A-profile, work with the assumption that each A-profile is equally likely to obtain. Despite the differences in these studies they share the common conclusion that an increase in social homogeneity tends to decrease the likelihood of the paradox.

Later in the paper we use the sum of squares measure Σn_i^2 as an approximate measure of the imbalance of power among blocs or coalitions. Unlike the social homogeneity measures mentioned in the preceding paragraph, Σn_i^2 is concerned only with the relative magnitudes of the n_i and pays no attention to similarities among the orders attached to the n_i. Our point in using Σn_i^2 is to show that, even when similarities among nonidentical orders are ignored, the incidence of Condorcet's paradox tends to decrease as Σn_i^2 increases. This conclusion reflects our finding that A-profiles near the center of the A-profile space have a greater propensity for the paradox than do A-profiles that are near the boundary of the space.

A final feature of prior studies that relates to the present work is the efforts to compute the likelihood of the paradox when each voter is assumed to independently select a preference order according to a probability distribution over orders [5, 7, 14, 15, 21, 23, 25]. Here we shall examine the alternative of basing likelihood computations on probability distributions over A-profiles. This is tantamount to using probability distributions over profiles since the corresponding A-profile distribution is simply obtained from the profile distribution by summing probabilities over the profiles that map into the same A-profile. However, we prefer to work with A-profiles since, as argued above, they form a natural unit for analysis in many real situations and, in addition, allow one to avoid the sometimes elusive details of voter identities in profiles. It is also important to note that some probability distributions over A-profiles that may capture aspects of voter interdependence correspond to no distribution on A-profiles that is derived under the assumption of independent voter selection in the more traditional method, even when each voter can make his selection from a different distribution over the preference orders. A simple example of this is provided by a two-voter case in which each voter is confined with probability 1 to select a preference order from a three-order set. In this case there are five degrees of freedom in specifying a probability distribution on A-profiles (since there are six possible A-profiles) but only four degrees of freedom (two for each voter) in specifying the probability distributions used by the voters for selecting their orders from the three-order set. Hence, as noted above, likelihood computations based on probability distributions over A-profiles (or over profiles) cannot in general be duplicated by the more traditional independent-selection approach.

The derivations of many of the results given later are presented in the appendix.

I. Anonymous Preference Profiles

This section shows how to compute the likelihood of Condorcet's paradox for any given probability distribution on the set of A-profiles when $m = 3$ and the number n of voters is odd with each voter having a linear preference order on the three alternatives. A simple, exact expression is then given for the proportion of A-profiles that have a simple majority winner. A similar formula is presented for the proportion of A-profiles that have a simple majority winner when $m = 4$. Finally, we note the average proportions of voters who have the simple majority winner in first, second and third places in their preference orders when $m = 3$, where the average is taken over the A-profiles that have a simple majority winner.

With $m = 3$, the numbers of individuals who have the six linear orders on the set $\{1,2,3\}$ of alternatives are as follows:

n_1 prefer 2 to 3 to 1
n_2 prefer 3 to 2 to 1
n_3 prefer 2 to 1 to 3
n_4 prefer 3 to 1 to 2
n_5 prefer 1 to 2 to 3
n_6 prefer 1 to 3 to 2.

The space of all A-profiles for three alternatives and n voters with linear orders is

$$A_n = \left\{(n_1, \ldots, n_6): n_i \in \left\{0,1, \ldots, n\right\} \text{ for each i, and } \Sigma n_i = n\right\}.$$

A-profile $(n_1, \ldots, n_6)$ corresponds to $(n!) / (n_1!n_2! \ldots n_6!)$ distinct profiles.

Let P be a probability distribution on A_n with $P(n_1, \ldots, n_6) \geqslant 0$ and $P(A_n) = 1$. To compute the likelihood that alternative 1 is the simple majority winner, given P, the $P(n_1, \ldots, n_6)$ values are summed over the subspace S_1 of A_n in which 1 beats each of 2 and 3 by simple majority. With n odd,

$$S_1 = \{(n_1, \ldots, n_6) \in A_n: \; n_1 + n_2 + n_3 \leq \frac{n-1}{2}$$
$$\text{and } n_1 + n_2 + n_4 \leq \frac{n-1}{2}\}.$$

The probability that 1 is the simple majority winner is then $P(S_1)$, where, with $n_{12} = n_1 + n_2$,

$$P(S_1) = \sum_{n_1=0}^{\frac{n-1}{2}} \sum_{n_2=0}^{\frac{n-1}{2}-n_1} \sum_{n_3=0}^{\frac{n-1}{2}-n_{12}} \sum_{n_4=0}^{\frac{n-1}{2}-n_{12}} \sum_{n_5=0}^{n-n_{12}-n_3-n_4} P(n_1, \ldots, n_6).$$

In a manner that is entirely analogous to the development of $P(S_1)$, we can specify summation formulas for $P(S_2)$ or $P(S_3)$, for the probability that 2 or 3 is the simple majority winner, given P. The total probability that P will give rise to a simple majority winner is therefore $P(S_1) + P(S_2) + P(S_3)$, and hence the likelihood of Condorcet's paradox is $1-P(S_1)-P(S_2)-P(S_3)$.

When P is taken to be the distribution that assigns an equal probability to all points in A_n, the preceding sum gives rise to a simple algebraic expression. Since this case is tantamount to an examination of the proportion of points in A_n for which 1 is the simple majority winner, we shall consider it from the latter viewpoint.

Consider first the number of points in A_n for which 1 is the simple majority winner. Letting $\#B$ denote the number of points in set B, this number is $\#S_1$ and

$$\#S_1 = \sum_{n_{12}=0}^{\frac{n-1}{2}} (n_{12}+1) \sum_{n_3=0}^{\frac{n-1}{2}-n_{12}} \sum_{n_4=0}^{\frac{n-1}{2}-n_{12}} \sum_{n_5=0}^{n-n_{12}-n_3-n_4} f(n_1,\ldots,n_6) \tag{1}$$

where $f(n_1 \ldots n\ ,6) = 1$ for all points in S_1 and $(n_{12}+1)$ is the number of ways and n_2 can sum to n_{12} in collapsing the double sum over n_1 and n_2 to the single sum of n_{12}. This equation can be computed by sequential summation using the formulas for sums of powers of integers [26]. As shown by (8) in the appendix, the result is

$$\#S_1 = \frac{(n+1)(n+3)^3(n+5)}{384}.$$

Since the number of points in A_n for which one of the three alternatives is the simple majority winner is $3(\# S_1)$, A_n contains $(n+1)(n+3)^3(n+5)/128$ points that do not exhibit Condorcet's paradox. In addition, [see (7) in the appendix], the number of points in A_n is

$$\#A_n = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n-n_1} \cdots \sum_{n_5=0}^{n-n_1-n_2-n_3-n_4} 1 = \frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{120},$$

so that the proportion $p(n,3)$ of points in A_n that have a simple majority winner is $3(\# S)/\# A_n$, or

$$p(n,3) = \left(\frac{15}{16}\right)\frac{(n+3)^2}{(n+2)(n+4)}, \qquad n \in \{1,3,5,\ldots\}. \tag{2}$$

The space of A-profiles with $m = 4$ and n voters with linear preference orders consists of 24-dimensional vectors with nonnegative integer components that sum to n. Although this space is much more complex than A_n for $m = 3$, it is a simple matter to compute the proportion $p(n,4)$ of points in the $m = 4$ space that have a simple majority winner. In particular, the proof given in [11] for May's formula [19], which says that the likelihood of Condorcet's paradox for $m = 4$ is twice its probability for $m = 3$ when all voters are assumed to have linear preference orders on the alternatives (n odd) and all possible profiles are equally likely, applies also in the A-profile setting, and therefore $1 - p(n,4) = 2[1 - p(n,3)]$, or

$$p(n,4) = \frac{7n^2 + 42n + 71}{8(n + 2)(n + 4)}, \quad n \in \{1,3,5,\ldots\}. \tag{3}$$

We consider next the proportions of voters who have the simple majority winner in the kth position of their preference orders for $k = 1, \ldots, m$, given that there is a simple majority winner. Although this has not received much attention, it is an interesting dimension of voting models. For example, a recent study [13] shows that these proportions are an important factor in determining which single-stage voting rule is most likely to elect the simple majority alternative when it exists. The generic single-stage rule examined in [13] requests each voter to vote for K of the m alternatives (without ranking) for a given $K \in \{1, \ldots, m - 1\}$. The winner under rule K is the alternative with the most votes. As the expected proportions for $k = 1 \ldots, m$ become more different, the value of K that maximizes agreement between rule K and the simple majority rule (when a simple majority alternative exists) decreases from about $K = m/2$ to $K = 1$. The $K = 1$ rule is of course the common simple plurality rule under which each voter votes for exactly one candidate.

Within the context of A_n with n odd, let P_n^k be the average proportion of voters that have the simple majority winner in kth place in their preference orders, where the average is taken over all A-profiles that have a simple majority winner. Since this average will not change if we consider only the profiles in which alternative 1 is the simple majority winner, P_n^3 is the average value of $(n_1 + n_2)/n$ over the A-profiles in S_1. This average is obtained from (1) with $f(n_1, \ldots, n_6) = (n_1 + n_2) / (n\# S_1)$. The result is equation (9) of the appendix divided by n times (8):

$$P_n^3 = \frac{3n^2 + 3n - 6}{15n(n + 3)}, \quad n \in \{1,3,5,\ldots\}. \tag{4}$$

Similarly, P_n^2 is obtained from (1) with $f(n_1, \ldots, n_6) = (n_3 + n_4) / (n\# S_1)$, which is (10) of the appendix divided by n times (8):

$$P_n^2 = \frac{(n - 1)(4n + 13)}{15n(n + 3)}, \quad n \in \{1,3,5,\ldots\}. \tag{5}$$

Finally, $P_n^1 = 1 - P_n^3 - P_n^2$, so that (4) and (5) yield

$$P_n^1 = \frac{8n^2 + 33n + 19}{15n(n + 3)}, \quad n \in \{1,3,5,\ldots\}.$$

These proportions will be used further in the next section.

II. Comparisons with Profile Results

The proportions in equations (2) through (5) can be viewed as probabilities for the indicated events when each A-profile is assumed to be equally likely. We shall refer to the equally likely A-profile assumption as ELAP. The purpose of this section is to compare the results obtained under ELAP with similar results derived under the assumption that each profile is equally likely. The latter assumption, which is often termed "impartial culture", will be referred to here as ELP. We shall consider first the paradox probabilities and then look at voter proportions. In both instances the differences between ELAP and ELP will be related to the fact that ELP places considerably more weight on A-profiles that are near to the center of the A-profile space (with approximately equal n_i) than on A-profiles that are near the edges of the A-profile space (with one or two n_i much larger than the others). Recall that, with the linear orders on m alternatives indexed as $1,2,\ldots,m!$, ELP assigns probability $(m!)^{-n}\, n!/[n_1!n_2!\ldots(n_{m!})!]$ to the A-profile $(n_1,\ldots,n_{m!})$, whereas ELAP assigns equal probability to each A-profile.

Beginning with ELAP, (2) and (3) specify the ELAP probabilities for $m = 3$ and $m = 4$ of *avoiding* Condorcet's paradox. Since the derivatives of $p(n,3)$ and $p(n,4)$ with respect to n are negative, $p(n,3)$ decreases in n to the limit 15/16, and $p(n,4)$ decreases in n to the limit 7/8. Moreover, $p(n,3) > p(n,4)$ for each n. Therefore, within the confines of $m \in \{3,4\}$ and odd n, the likelihood of Condorcet's paradox under ELAP increases in both n and m.

The situation is similar with ELP. Studies on the probability of the paradox [7, 14, 21, 23] under ELP reveal that it tends to increase in both n and m. A more precise study of these trends is made by Kelly [18]. Given $m \in \{3,4\}$ and odd n, the paradox's probability increases in n and m under ELP. Kelly notes, however, that the status of these trends remains conjectural for larger values of m. Likewise, the behavior of the paradox probability under ELAP is an open question for larger values of m.

Table 1 offers a detailed comparison between ELAP and ELP. The ELAP values come from (2) and (3). The ELP values are computed from a recently-derived formula [15]

$$p^*(n,3) = 3^{-n+1} \sum_{n_{12}=0}^{\frac{n-1}{2}} \sum_{n_3=0}^{\frac{n-1}{2}-n_{12}} \sum_{n_4=0}^{\frac{n-1}{2}-n_{12}} \left(\frac{n!}{n_{12}!\,n_3!\,n_4!\,n_{56}!}\right) 2^{-(n_3+n_4)} \qquad (6)$$

TABLE 1

Probabilities of a Simple Majority Winner
Under ELAP and ELP

	m = 3		m = 4	
n	ELAP	ELP	ELAP	ELP
3	.96429	.94444	.92857	.88889
5	.95238	.93056	.90476	.86111
7	.94697	.92498	.89394	.84997
9	.94406	.92202	.88811	.84405
11	.94231	.92019	.88462	.84037
13	.94118	.91893	.88235	.83786
15	.94040	.91802	.88081	.83604
17	.93985	.91733	.87970	.83466
∞	.93750	.91226	.87500	.82452

with $n_{12} = n_1 + n_2$ and $n_{5\,6} = n_5 + n_6$, and from May's formula $1 - p^*(n,4) = 2[1 - p^*(n,3)]$, where $p^*(n,m)$ is the probability of a simple majority winner under ELP. The p^* values in the table are also directly obtainable from Table I in Garman and Kamien [14].

Table 1 shows that the likelihood of getting a simple majority winner is greater under ELAP than under ELP. In fact, except for $n = 3$, $p(\infty,m) > p^*(n,m)$ for all n. Because ELP places more weight on A-profiles that are near the center of the A-profile space, this indicates that centrally located A-profiles are, on average, more susceptible to Condorcet's paradox than are A-profiles near the edges of the space. In other words, A-profiles whose n_i values are similar are, on balance, more likely to exhibit the paradox than are A-profiles whose n_i values are quite dissimilar. The distributions of n_i values under the two assumptions are examined further in the next section.

Another comparison between ELAP and ELP is provided for $m = 3$ by the expected proportions of voters who have the simple majority winner in first, second and third places in their preference orders, given that there is a simple majority winner. These proportions are shown in Table 2 for $n \in \{3, 5, 7, 9, 11\}$ and for the limit as $n \to \infty$. The values for ELAP come from (4) and (5). The values for ELP were obtained by computer enumeration over the A-profiles. The limiting values of 1/3 for ELP are obtained by the following argument. Since the probability of A-profiles under ELP is given by a multinomial distribution with probability 1/6 for each of the six linear orders, the proportion of profiles that have $1/6 - \delta \leqslant n_i/n \leqslant 1/6 + \delta$ for $i = 1,\ldots,6$ approaches 1 as $n \to \infty$ for each positive δ.

TABLE 2

Expected Proportions of Voters who have the Simple Majority Winner in First Place, Second Place, and Third Place Under ELAP and ELP with m = 3, Given that there is a Simple Majority Winner

	First Place		Second Place		Third Place	
n	ELAP	ELP	ELAP	ELP	ELAP	ELP
3	.70370	.64706	.18519	.23529	.11111	.11765
5	.64000	.56219	.22000	.27861	.14000	.15920
7	.61143	.52118	.23429	.29549	.15428	.18333
9	.59506	.49609	.24198	.30439	.16296	.19952
11	.58442	.47877	.24675	.30990	.16883	.21133
∞	.53333	.33333	.26667	.33333	.20000	.33333

By Table 1, more than 90 percent of the profiles have a simple majority winner for each n, with m = 3. Therefore, under ELP, the probability that a profile has $1/6 - \delta \leqslant n_i/n \leqslant 1/6 + \delta$ for all i, given that it has a simple majority winner, approaches 1 as $n \to \infty$. Then, letting δ approach zero, the expected proportion of voters who have the simple majority winner in a given place under ELP, given that there is a simple majority winner, must approach $1/6 + 1/6 = 1/3$ as $n \to \infty$.

Table 2 shows the expected result that, under either ELAP or ELP, the expected number of voters who have the simple majority winner in jth place exceeds the expected number of voters who have the simple majority winner in kth place whenever $j < k$. In addition, the first-place proportions decrease in n, and the second-place and third-place proportions increase in n, but while the limit values under ELP are all 1/3, the limit values under ELAP are quite different for the three places. Under ELAP, at least half the voters are expected to have the simple majority winner as their first choice, regardless of the size of n, given that there is a simple majority winner. This implies, for example, that if there is a simple majority alternative and if n is very large (to make the variance about the expected proportion near to zero) and ELAP holds, then the simple majority winner will almost surely be elected by the common simple plurality rule (i.e., K = 1, as discussed after (3)). The differences between ELAP and ELP in Table 2 show that the average proportion of voters who have the simple majority winner in first place, given that there is a simple majority winner, is considerably higher for A-profiles near the edges of the space than for profiles near the center. Since Σn_i^2 increases as one moves away from the center of the A-profile space, larger values of Σn_i^2 for fixed n tend to be associated with higher proportions of voters who have the simple majority winner in first place.

III. An Approximate Measure of Power Imbalance

Tables 1 and 2 show that the likelihood of Condorcet's paradox and the expected proportions of voters who have the simple majority winner in various positions in their preference orders are quite sensitive to the dispersion of the n_i values, or to the distance of A-profiles from the center of the A-profile space. Perhaps the simplest measure of dispersion or difference with m alternatives is $\Sigma_{i=1}^{m!} n_i^2$, which is monotonically related to the sum of squared mean differences $\Sigma[n_i - n/m!]^2$. The measure Σn_i^2 on A-profiles can of course be viewed as an approximate measure of the imbalance of power among coalitions or voting blocs whose members have common preferences. For a given n, this measure is smallest when voters are evenly distributed over the m! linear orders. This reflects a balance of power situation since no one coalition can enforce its viewpoint without 'cooperation' from other coalitions. At the other extreme, Σn_i^2 is largest when almost all voters have the same preference order, or when one coalition is very large compared to the others.

An indication of the differences between ELAP and ELP for Σn_i^2 is shown in Table 3 for (m,n) = (3,7). Similar computations were made for other values of n combined with m=3. They show that the seven-voter results of Table 3 are typical of the general situation. The entries in the table give the probability of $\Sigma n_i^2 \leqslant D$ under ELAP and ELP, and they clearly show the tendency of ELP to generate smaller Σn_i^2 values than does ELAP. This of course reflects the greater emphasis that ELP places on A-profiles near the center of the space, and, in conjunction with Tables 1 and 2, it supports the findings of the preceding section that smaller values of Σr_i^2 tend to be associated with a higher incidence of Condorcet's paradox and with more even distributions of voters over the three places in the preference orders for the simple majority winner when it exists.

The differences illustrated by the tables can also be thought of in terms of social homogeneity even though there is no general agreement on the most appropriate measure of this concept [1,12, 17, 20, 22]. Nevertheless, it is generally agreed that social homogeneity is maximized when all voters have the same preference order, in which case Σn_i^2 is maximized and Condorcet's paradox cannot arise. And, at least by several measures [12, 20, 22], homogeneity is minimized when voters are evenly dispersed over the preference orders, or when Σn_i^2 is minimized. Therefore it seems reasonable to view Σn_i^2 as at least a crude measure of social homogeneity. We say "crude" because, unlike several other measures [12, 20, 22], Σn_i^2 takes no account of similarities among the different orders chosen by the voters.

As mentioned earlier, other studies of social homogeneity and Condorcet's paradox show that an increase in social homogeneity tends to decrease, either as a general trend or in a precise sense[20], the likelihood of the paradox. The present study shows that this is true also for Σn_i^2 : as Σn_i^2 increases, the likelihood of the paradox tends to decrease. Hence, even when similarities among different preference orders are ignored, there is a measure of homogeneity that relates to the likelihood of Condorcet's paradox in the same way as do more refined measures.

TABLE 3

Probabilities that ELAP and ELP will Generate Σn_i^2 Values Which are Less Than or Equal to D, for Three Alternatives and Seven Voters

D	ELAP	ELP
9	.00758	.05401
11	.08333	.32407
13	.19697	.54913
15	.42424	.81919
17	.50000	.86420
19	.65152	.93921
21	.80303	.98422
25	.84091	.98798
27	.91667	.99698
29	.95455	.99923
37	.99243	.99998
49	1.00000	1.00000

Appendix

This appendix outlines four derivations whose results are used in the paper. The four relations that are required are:

$$\sum_{n_{12}=0}^{n} (n_{12}+1) \sum_{n_3=0}^{n-n_{12}} \; \sum_{n_4=0}^{n-n_{12}-n_3} \; \sum_{n_5=0}^{n-n_{12}-n_3-n_4} = \frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{120} \tag{7}$$

$$\sum_{n_{12}=0}^{\frac{n-1}{2}} (n_{12}+1) \sum_{n_3=0}^{\frac{n-1}{2}-n_{12}} \; \sum_{n_4=0}^{\frac{n-1}{2}-n_{12}} \; \sum_{n_5=0}^{n-n_{12}-n_3-n_4} = \frac{(n+1)(n+3)^3(n+5)}{384} \tag{8}$$

$$\sum_{n_{12}=0}^{\frac{n-1}{2}} n_{12}(n_{12}+1) \sum_{n_3=0}^{\frac{n-1}{2}-n_{12}} \; \sum_{n_4=0}^{\frac{n-1}{2}-n_{12}} \; \sum_{n_5=0}^{n-n_{12}-n_3-n_4} = \frac{(n-1)(n+1)(n+2)(n+3)^2(n+5)}{1920} \tag{9}$$

$$\sum_{n_{12}=0}^{\frac{n-1}{2}} (n_{12}+1) \sum_{n_3=0}^{\frac{n-1}{2}-n_{12}} \; \sum_{n_4=0}^{\frac{n-1}{2}-n_{12}} (n_3+n_4) \sum_{n_5=0}^{\frac{n-1}{2}-n_{12}} = \frac{(n-1)(n+1)(n+3)^2(n+5)(4n+13)}{5760}. \tag{10}$$

The derivations employ sequential application of the relations for sums of powers of integers. The relations for sums of powers of integers that are used are taken from Selby [26], with

$$\sum_{i=1}^{m} = m \tag{11}$$

$$\sum_{i=1}^{m} i = \frac{m(m+1)}{2} \tag{12}$$

$$\sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6} \tag{13}$$

$$\sum_{i=1}^{m} i^3 = \frac{m^2(m+1)^2}{4} \tag{14}$$

$$\sum_{i=1}^{m} i^4 = \frac{m}{30}(m+1)(2m+1)(3m^2+3m-1) \tag{15}$$

$$\sum_{i=1}^{m} i^5 = \frac{m^2}{12}(m+1)^2(2m^2+2m-1). \tag{16}$$

Applying (11) to the left hand side of (7), we obtain

$$\sum_{n_{12}} (n_{12}+1) \sum_{n_3} \sum_{n_4=0}^{n-n_{12}-n_3} (n+1-n_{12}-n_3-n_4).$$

Using (11) and (12), this reduces to

$$\frac{1}{2}\sum_{n_{12}}(n_{12}+1)\sum_{n_3=0}^{n-n_{12}}\left[\begin{array}{l}(n+1-n_{12})(n+2-n_{12})\\ -(2n+3-2n_{12})n_3+n_3^2\end{array}\right].$$

Using (11), (12) and (13) this reduces to

$$\frac{1}{6}\sum_{n_{12}=0}^{n}\left[\begin{array}{l}(n+1)(n+2)(n+3)\\ +(n^3+3n^2-n-5)n_{12}\\ -(3n^2+9n+5)n_{12}^2\\ +(3n+5)n_{12}^3\\ -n_{12}^4\end{array}\right].$$

Formulas (11), (12), (13), (14) and (15) then give

$$\frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{120},$$

which verifies (7).

Applying (11) to the left hand side of (8) we obtain

$$\sum_{n_{12}}(n_{12}+1)\sum_{n_3}\sum_{n_4=0}^{\frac{n-1}{2}-n_{12}}(n+1-n_{12}-n_3-n_4).$$

Using (11) and (12) this reduces to

$$\left(\frac{n+3}{2}\right)\sum_{n_{12}=0}^{\frac{n-1}{2}}\left[\begin{array}{l}\left(\frac{n+1}{2}\right)^2\\ +\frac{(n^2-2n-3)}{4}n_{12}\\ -n\,n_{12}^2\\ +n_{12}^3\end{array}\right].$$

Then (11), (12), (13) and (14) give the desired result

$$\frac{(n+1)(n+3)^3(n+5)}{384}.$$

The reduction for (9) follows the same format as the reduction of the left hand side of (8) to (17). For the left hand side of (9) we have

$$\left(\frac{n+3}{2}\right)\sum_{n_{12}=0}^{\frac{n-1}{2}}\left[\begin{array}{l}\left(\frac{n+1}{2}\right)^2 n_{12}\\ +\frac{(n^2-2n-3)}{4}n_{12}^2\\ -n\,n_{12}^3\\ +n_{12}^4\end{array}\right].$$

Using (12), (13), (14) and (15) we obtain the desired result

$$\frac{(n-1)(n+1)(n+2)(n+3)^2(n+5)}{1920}.$$

Finally, using (11) on the left hand side of (10), we get

$$\sum_{n_{12}}(n_{12}+1)\sum_{n_3}\sum_{n_4=0}^{\frac{n-1}{2}-n_{12}}\left[\begin{array}{l}n_3(n+1-n_{12}-n_3)\\ +(n+1-n_{12}-2n_3)n_4\\ -n_4^2\end{array}\right].$$

Then (11), (12) and (13) reduce this to

$$\sum_{n_{12}}(n_{12}+1)\left(\frac{n+1}{2}-n_{12}\right)\sum_{n_3=0}^{\frac{n-1}{2}-n_{12}}\left[\begin{array}{l}\left(\frac{n-1}{2}-n_{12}\right)\left(\frac{2n+3-n_{12}}{6}\right)\\ +\left(\frac{n+3}{2}\right)n_3\\ -n_3^2\end{array}\right].$$

Further use of (11), (12) and (13) gives

$$\frac{1}{12}\sum_{n_{12}=0}^{\frac{n-1}{2}}\left[\begin{array}{l} \frac{5(n+1)^2(n-1)(n+3)}{8} \\ +\left(\frac{n+1}{2}\right)\left(\frac{5n^3-13n^2-85n+13}{4}\right)n_{12} \\ -\left(\frac{14n^3+30n^2-70n-46}{4}\right)n_{12}^2 \\ +\left(\frac{12n^2+44n-12}{2}\right)n_{12}^3 \\ -2(n+8)n_{12}^4 \\ -2n_{12}^5 \end{array}\right],$$

and (11), (12), (13), (14), (15) and (16) reduce this to

$$\frac{(n-1)(n+1)(n+3)^2(n+5)(4n+13)}{5760}.$$

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