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Shadow Prices and Sensitivity Analysis in Linear Programming under Degeneracy

State-of-the-Art-Survey

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Summary. In linear programming applications the economic meaning of shadow prices is important. In the case primal degeneracy occurs in the optimal solution, the values of the dual real variables are not, in general, identical with the corresponding shadow prices, or, in other words, these values have not the usual meaning in comparison with LP optimal solutions without primal degeneracy. Several proposals on how to interpret such values or how to find the "true" shadow prices have been made and terms like "many-sided-" or "two-sided-shadow prices" have been coined. Also, when performing sensitivity analysis in the case primal degeneracy occurs, the so called critical ranges of the right hand side or of the objective function coefficients cannot be determined in the usual way. In this paper, a state-of-the-art-survey on these questions is given.

Zusammenfassung. Bei den Anwendungen der linearen Optimierung ist der ökonomische Inhalt der Schattenpreise von Bedeutung. Falls eine optimale Lösung primal entartet ist, sind die Werte der dualen Strukturvariablen im allgemeinen nicht identisch mit den entsprechenden Schattenpreisen, oder – anders ausgedrückt – diese Werte kann man nicht so interpretieren wie bei nichtentarteten optimalen Lösungen eines linearen Optimierungsproblems. Es gibt in der Literatur verschiedene Vorschläge, wie diese Werte interpretiert werden sollen oder wie der „richtige" Schattenpreis bestimmt werden soll. Dabei werden Bezeichnungen wie „vielseitige" bzw. „zweiseitige Schattenpreise" eingeführt. Auch bei der Durchführung einer Sensitivitätsanalyse können im Falle einer primalen Entartung die kritischen Bereiche für Parameter in der rechten Seite oder in den Zielkoeffizienten nicht auf die übliche Weise bestimmt werden. In diesem Artikel ist eine Übersicht des gegenwärtigen Standes zu den obigen Problemen gegeben.

0. Introduction

Consider

(LP) $\max\limits_{x \in X} z = c^T x$

with $X = \{x \in \mathbb{R}^n \mid Ax \leqslant b, x \geqslant 0\}$

(LP_D) $\min\limits_{y \in Y} f = y^T b$

with $Y = \{y \in \mathbb{R}^m \mid A^T y \geqslant c, y \geqslant 0\}$

where $c = (c_1, \ldots, c_j, \ldots, c_n)^T \in \mathbb{R}^n$, $b = (b_1, \ldots, b_i, \ldots, b_m)^T \in \mathbb{R}^m$, $x = (x_1, \ldots, x_j, \ldots, x_n)^T \in \mathbb{R}^n$, $y = (y_1, \ldots, y_i, \ldots, y_m)^T \in \mathbb{R}^m$, A an $(m \times n)$-matrix, $A = (a^1, \ldots, a^j, \ldots, a^n)$, $a^j = (a_{1j}, \ldots, a_{ij}, \ldots, a_{mj})^T$, $j = 1, \ldots, n$. After introducing slacks x_{n+i}, $i = 1, \ldots, m$, and computing an optimal solution, suppose that $B = (a^{j_1}, \ldots, a^{j_m})$, $a^{j_i} \in \mathbb{R}^m$, $i = 1, \ldots, m$, is the corresponding optimal basis with the characteristic basis-index $\rho = \{j_1, \ldots, j_m\}$ such that $x_{j_1}, \ldots, x_{j_m}$ are basic variables.

Let us summarize the needed notation in the following optimal simplex tableau, whereas – without loss of generality – suppose that after some rearranging $j_1 = 1, \ldots, j_m = m$.

Optimal solution

ρ	$x_1 \ldots x_m$	$x_{m+1} \ldots x_n$	$x_{n+1} \ldots x_{n+m}$	x_B
1	$1 \ldots 0$	$\bar{a}_{1,m+1} \ldots \bar{a}_{1n}$	$\bar{a}_{1,n+1} \ldots \bar{a}_{1,n+m}$	$\bar{b}_1$
$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$
m	$0 \ldots 1$	$\bar{a}_{m,m+1} \ldots \bar{a}_{mn}$	$\bar{a}_{m,n+1} \ldots \bar{a}_{m,n+m}$	$\bar{b}_m$
Δz_j	$0 \ldots 0$	$\bar{c}_{m+1} \ldots \bar{c}_n$	$\bar{y}_1 \ldots \bar{y}_m$	$z_{\max}$

The corresponding optimal basic solution is $x_B = (x_{B_1}, \ldots, x_{B_m})^T$ with $x_{B_i} = \bar{b}_i$, $i = 1, \ldots, m$, and the optimal objective function value $z_{\max} = c_B^T x_B$.

The elements of the last row labeled by Δz_j are

$$\Delta z_j = \bar{c}_j \quad \text{for } j = 1, \ldots, n,$$

$$\Delta z_j = \bar{y}_i \quad \text{for } j = n + 1, \ldots, n + m.$$

These elements are sometimes called (optimality) criterion elements (hence, the row is called criterion row) or reduced costs. In the literature, one does not find a unique distinction between $\bar{c}_j$ and $\bar{y}_i$ in the sense of their economic interpretation. For example, in [24]: "Scarcity prices or opportunity costs are defined as the amount by which the objective function value increases when a scarce resource is increased by one" (p. 92) and "Opportunity costs are called in the terminology of linear programming shadow prices or dual values" (p. 104).

In [25]: "Value of capacity of machine 1 plays the role of costs; in economics the term opportunity costs would be used ... other terms are imputed costs or shadow prices" (p. 91).

In [32]: "The final row coefficients (the $\bar{c}_j$'s – T.G.) are sometimes referred to as relative or shadow costs ... the final row coefficients of the slack variables (the $\bar{y}_i$'s – T.G.) are sometimes called shadow prices" (p. 109).

We shall distinguish two categories of the reduced costs Δz_j according, e.g., to Cook and Russel [9], pp. 151 f and 183f.

(1) the $\bar{c}_j$'s for $j \notin \rho$ as the opportunity costs

(2) the $\bar{y}_i$'s as the optimal dual values (i.e. the optimal values of the dual real variables), or marginal values or shadow prices.

Without going into technical details of construction of an initial simplex tableau, let us note, that in the case the original LP has inequalities of the type "$\geqslant$" and/or equalities, they $\bar{y}_i$'s are found in the columns of the inverse B^{-1} which are not all identical with the columns of slacks. In our case with only "$\leqslant$" the $\bar{y}_i$'s are in the columns of the slacks.

The usual interpretation of the opportunity costs $\bar{c}_j$ is either

(1) the amount by which the objective function value decreses setting $x_j = 1, j \notin \rho$, provided that $x_j = 1$ is feasible, i.e., the cost $\bar{c}_j$ for introducing a product P_j into production on the level $x_j = 1$, when – in accordance with the optimal solution – P_j is not to be produced; or

(2) $\bar{c}_j$ is the amount by which c_j must increase in order to enter $x_j, j \notin \rho$, into the basis; or

(3) it is the right partial derivative

$$\bar{c}_j = \frac{\partial z}{\partial x_j^+}, \quad j \notin \rho.$$

The usual interpretation of the marginal value or shadow price $\bar{y}_i$ is either (1)

$$\bar{y}_i = \frac{\partial z}{\partial b_i}$$

or (2) the "price" for selling or buying one unit of the i-th resource, i.e. $\bar{y}_i$ is the amount by which $z_{\max}$ changes changing b_i by one unit.

Let us present a simple illustration.

Example 0.1:

$$\max z = 4x_1 + 2x_2 + 9x_3 + 6x_4$$

s.t. (with slacks x_{n+i}, $i = 1, 2, n = 4$, included)

$$x_1 + 2x_2 + 3x_3 + x_4 + x_5 = 50$$

$$2x_1 + x_2 + 4x_3 + 3x_4 + x_6 = 80$$

$$x_j \geqslant 0, \; j = 1, \ldots, 6.$$

The interpretation could be: maximize the profit producing some of the four products P_j, $j = 1, \ldots, 4$, restricted to two machine-capacity constraints.

The optimal solution is in Table 1.

Table 1

ρ	1	2	5	6	x_B
3	0.2	1	0.6	−0.2	14
4	0.4	−1	−0.8	0.6	8
Δz_j	0.2	1	0.6	1.8	174

Here $\bar{c}_1 = 0.2$, $\bar{c}_2 = 1$, $\bar{y}_1 = 0.6$, $\bar{y}_2 = 1.8$, $z_{\max} = 174$. Setting, e.g. $x_1^* = 1$ ($0 \leqslant x_1 \leqslant 20$, hence x_1^* is feasible) we obtain $x_3(x_1^*) = 14 - 0.2x_1^* = 13.8$, $x_4(x_1^*) = 8 - 0.4x_1^* = 7.6$, $z_{\max}(x_1^*) = 174 - 0.2x_1^* = 173.8$. The value $\bar{c}_1 = 0.2$ represents the opportunity costs for the product P_1.

Changing $b_1 = 50$ by +1 or −1, the price for selling or buying respectively, one unit of this resource is 0.6 or −0.6, respectively, hence the objective function value changes to 174.6 or to 173.4, respectively. In other words: The shadow price for capacity resource 1 is 0.6. Analogously the shadow price for resource 2 is 1.8.

Let us recall that a solution is called primal degenerate if $\bar{b}_i = 0$ for at least one $i \in \{1, \ldots, m\}$, it is called dual generate if $\Delta z_j = 0$ for at least one $j \notin \rho$.

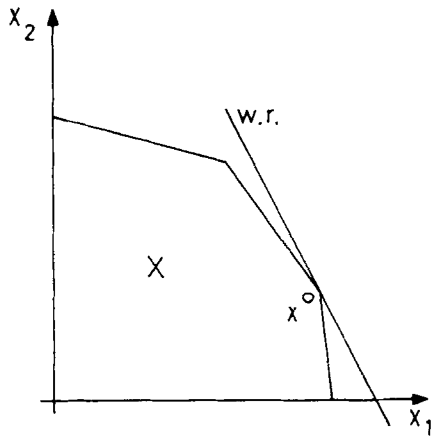

Fig. 1

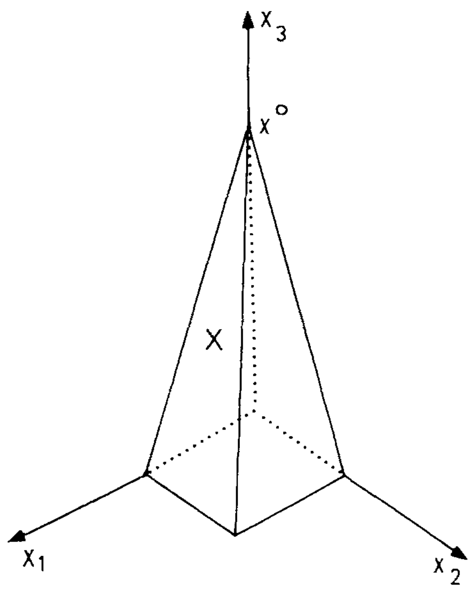

Fig. 2

Let x^s be a vertex or extreme point (EP) of the convex polyhedron $X = \{x \in \mathbb{R}^n \mid Ax \leqslant b, x \geqslant 0\}$. If x^s is degenerate, one speaks in general about degenerate polytopes [15]. Considering X as the constraint set of (LP), the connection between (LP) and degenerate polytopes is obvious. As is well known, an EP of X is defined by n hyperplanes $\sum_{j=1}^{n} a_{ij}x_j = b_i$, $i = 1, \ldots, m$ (see, e.g., [17]). If x^s is overdetermined (i.e. x^s is determined by at least $n + 1$ hyperplanes) then x^s is degenerate. Degeneracy can be caused by weak redundancy ([18]) (cf. Figs. 1 and 2).

In Fig. 1 the degeneracy of x^0 is caused by the weakly redundant constraint "w.r." (x^0 is overdetermined because three "hyperplanes" go through x^0), in Fig. 2 x^0 is overdetermined (4 "hyperplanes") but there is no weak redundancy.

Turning back to (LP), having a primal degenerate solution it is well known (see, e.g., [10, 16]) that to such an EP more than 1 basis can be assigned. This lead to the discovery of so called degeneracy graphs ([15, 22]) by the aid of which various properties of the set bases associated with a degenerate EP can be studied.

Primal degeneracy can be viewed at from various viewpoints:

1. Degeneracy in the LP-model

2. Degeneracy occuring in the course of computing an optimal solution

3. Degeneracy in an optimal solution.

We regard cases 1. and 2. as rather technical problems because they are closely related to the computing techniques used to find an optimal solution. In this connection known anticycling methods ([6, 8, 11, 35] – see also [7]) have been elaborated.

The problem we are interested in is the third case, especially with respect to the interpretation of dual values and to sensitivity analysis when (primal) degeneracy occurs in the optimal solution.

In Sect. 1 we give an overview on the determination and interpretation of shadow prices under degeneracy; in Sect. 2 we give an overview on performing sensitivity analysis under primal degeneracy.

1. Two- or Many-Sided Shadow Prices

Let us first introduce some more notation:

$$T = \{i \mid \bar{b}_i = 0\}$$

$$x^{(s)} = (x_{n+1}, \ldots, x_{n+i}, \ldots, x_{n+m}) \quad \text{slacks}$$

$$\bar{X} = \left\{ \begin{pmatrix} x \\ x^{(s)} \end{pmatrix} \in \mathbb{R}^{m+n} \mid (A/I_m) \begin{pmatrix} x \\ x^{(s)} \end{pmatrix} = b, x \geqslant 0, x^{(s)} \geqslant 0 \right\}$$

p_i – shadow price of i-th resource b_i,

p_i^+ – shadow price of the i-th resource for the case b_i increases by 1

p_i^- – shadow price of the i-th resource for the case b_i decreases by 1.

Let us state that in case there is no primal degeneracy the following holds:

$$\bar{y}_i = p_i^+ = p_i^- = p_i \quad \text{for all } i = 1, \ldots, m.$$

The (probably) first author who dealt in a publication with the interpretation of dual values in an optimal solution to (LP) under degeneracy in the sense of shadow prices was Strum [30]. He mentions, among other things, the paper by Wright [36] who dealt with asset services and who based his work on the theory of shadow pricing developed within LP, however, without taking account of degeneracy. Strum says that "... uses of shadow prices are to be found also in other papers on accounting research ...". The purpose of this note is to show, by means of an example, the need to make such distinctions" (i.e. "gain in having one more unit of a resource" and "loss in having one less unit"). Strum uses a 3 x 2-example to illustrate the necessity to introduce "two-sided" shadow prices. His example reads:

Example 1.1. (Strum [30]):

$$\max z = 2x_1 + 3x_2$$

$$\text{s.t.} \quad 3x_1 + x_2 \leqslant 48$$
$$3x_1 + 4x_2 \leqslant 120$$
$$x_1 + 2x_2 \leqslant 56$$
$$x_1 \geqslant 0, \; x_2 \geqslant 0.$$

In Table 2 is the "first" primal degenerate optimal solution, in Table 3 the second one and in Table 4 is another primal degenerate solution associated with the same EP, which we shall need in connection with the paper by Eilon and Flavell [12].

Table 2 (Strum [30])

	4	5	x_B
3	−2.5	4.5	0
1	1	−2	8
2	−0.5	1.5	24
Δz_j	0.5	0.5	88

Table 3

	3	5	x_B
4	−0.4	−1.8	0
1	0.4	−0.2	8
2	−0.2	0.6	24
Δz_j	0.2	1.4	88

Table 4

	3	4	x_B
5	2/9	−5/9	0
1	4/9	−1/9	8
2	−1/3	1/3	24
Δz_j	−1/9	7/9	88

Note that, from Table 3 it follows that constraint No. 2 is weakly redundant.

Strum [30] shows that (cf. Table 2, column "5") with $b_3 + 1$ the shadow price is $p_3^+ = 0.5$, with $b_3 - 1$ (cf. Table 3, column "5") the shadow price is $p_3^- = 1.4$. Hence $\bar{y}_3 \neq p_3$, $p_3^+ \neq p_3^-$, thus p_3 does not exist. He speaks therefore of "two-sided shadow prices" and concludes that "from the viewpoint of foundations of accounting measurement, care must be taken to distinguish between the concepts 'value of one more unit of resource' and 'value of one less unit of the same resource' ".

Eilon and Flavell try to show in their paper [12] that other than two marginal values exist and call the shadow prices in the case of primal degeneracy "many-sided". They generate for Strum's example a third tableau (Table 4) assigned to the optimal EP; this tableau is however formally not optimal ($\Delta z_3 = \bar{y}_1 = -1/9 < 0$). They establish an interesting correspondence between dual infeasibility and the effects of omission of a constraint. However, as is shown in several other papers [1, 13, 19] and, as shown below, "many-sided" is not reasonable either from the economic or from the formal points of view.

In case primal degeneracy occurs in an optimal solution to (LP), Aucamp and Steinberg [2] introduce left- $\left(\frac{\partial z}{\partial b_i^-}\right)$ and right-partial derivatives $\left(\frac{\partial z}{\partial b_i^-}\right)$ for characterizing shadow prices when the i-th resource increases or decreases, respectively. They also state that "although a constraint is redundant (causing degeneracy in the optimal solution – T.G.), its presence does affect the shadow prices of the remaining constraints defining the optimal EP". Hence an omission of the redundant constraint is not possible. Aucamp and Steinberg [2] also clarify the connection between the primal and dual problems showing that, if in the dual there exist multiple optimal solutions, then the optimal values of the dual variables are not equal to shadow prices. They confine their investigations to increasing resource b_i, and say that "a similar analysis applies to marginal decreases in a resource ..." (p. 558). They prove then the following Theorem 1 (the detailed proof is in [3]).

Theorem 1 (Aucamp and Steinberg [2, 3]). *If there are K optimal* EP's *for* (LP_D), *then*

$$p_i^+ = \min_k \{\bar{y}_i^{(k)}\}$$

and if $K > 1$ *and* $b > 0$ *then further*

$$p \neq \bar{y}^{(k)}, \quad 1 < k \leqslant K,$$

where $\bar{y}^{(k)}$ *is the* k-th *dual solution vector and* p *is the shadow price vector.*

They quote Williams [34] saying that "a more general version of this theorem is stated and proved in Williams" (p. 559). We will discuss the results of Williams in Sect. 2.

Aucamp and Steinberg [2] introduce the so called VDPR (valid dual pivot row) defining the r-th row being a VDPR iff

(a) $\bar{b}_r = 0$

(b) $\bar{a}_{rj} \neq 0 \Rightarrow \bar{c}_j > 0 \quad$ for all $j \notin \rho$.

To obtain an alternate optimal dual solution in one simplex iteration such that the "degenerate variable" x_{Br}, $r \in T$, is replaced by the nonbasic variable x_k, the following should hold:

$$k = \arg\min \left\{ \left| \frac{\bar{c}_j}{\bar{a}_{rj}} \right| : j \notin \rho \right\}.$$

Agküi [1] shows that the above conditions are not correct, and hence the proof of Corollary 2 in [2] is not valid.

Corollary 2 (Aucamp and Steinberg [2]). *If the optimal primal is degenerate and the corresponding dual is not, then there exists an alternate dual optimal solution.*

Akgül [1] shows that the correct conditions for the r-th row, $r \in T$, to be a VDPR are:

(a) $\bar{b}_r = 0$

(b) $\bar{a}_{rj} < 0 \Rightarrow \bar{c}_j > 0 \quad$ for all $j \notin \rho$

because of the known rules for the dual simplex method. The definition of k as

$$k = \arg\min \left\{ \frac{\bar{c}_j}{|\bar{a}_{rj}|} : \bar{a}_{rj} < 0 \right\}.$$

Akgül [1] defines shadow prices in another way, which uses the same principles as Shapiro [28] (p. 37 f.). He introduces the function

$$F(b) = \max \{c^T x \mid Ax \leqslant b, x \geqslant 0\}$$

(known as the optimal value function – see e.g. [4, 14, 28]), which is a nondecreasing, continuous, piecewise linear and concave function. The directional derivative of F at b in the direction u is defined as

$$D_u F(b) = \lim_{t \to 0+} \frac{F(b + tu) - F(b)}{t}.$$

Then

$$p_u^+ = D_u F(b), \quad p_u^- = -D_{(-u)} F(b),$$

where Akgül considers the possibility, "a decision maker can face an offer concerning a combination of several resources, i.e. a 'package' deal" (p. 426), which is represented by the vector u.

The positive shadow price p_i^+ of the i-th constraint is then defined by

$$p_i^+ = D_{\mathrm{e}^i} F(b), \quad u := \mathrm{e}^i \text{ -- the } i\text{-th unit vector}$$

and the negative shadow price p_i^- as

$$p_i^- = -D_{(-\mathrm{e}^i)} F(b) \geqslant 0, \quad u := -\mathrm{e}^i.$$

Here p_i^+ is interpreted as the "maximal buying price" for the i-th resource, and p_i^- as the "least selling price" for the i-th resource. Denoting by

$$H(t) = F(b + tu),$$

Akgül [1] shows that p_u^+ and p_u^- are the one-sided derivatives of H at $t = 0$, i.e. "p_u^+ and p_u^- are nothing but slopes of $H(t)$ at $t = 0$" (p. 427).

He also endorses the result of Aucamp and Steinberg [2] that $p_i^+ = \min_k \{\bar{y}_i^{(k)}\}$ and adds

$$p_i^- = \max_k \{\bar{y}_i^{(k)}\},$$

which follows from the above results. With this he proves that "many-sided shadow prices" are not reasonable.

In connection with the Menger-Wieser-Theory of imputation [23, 27, 29, 33], Uzawa [31] shows that, in case the "opportunity costs" are not well defined, two of them – the one from upward and the other from downward – can be defined.

We will not go into Uzawa's theory in detail; we mention only that Uzawa introduces the (optimal value) function

$$h(b) = \max \; \{f(x) \mid x \geqslant 0, g(x) \leqslant b\}$$

and based on the concavity of $h(b)$, he shows that

$$\frac{\partial h}{\partial b_i^+} \leqslant \frac{\partial h}{\partial b_i^-}$$

(compare with Akgül [1]). Thus, Uzawa [31] also considers "two-sided" shadow prices.

In connection with parameterizing the original data of an LP, Saaty [26] also mentions the "two-sided" character of shadow prices when the parameter t takes on a critical value.

A very recent paper on the subject shadow prices is due to A. Ben-Israel and S. D. Flåm [5]. Based on the theory of canonical bases developed by A. Ben-Israel the authors obtain similar results as already described above.

An extensive work on the effects of degeneracy in the interpretation of shadow prices has been done by G. Knolmayer [19–21]. He uses pure economic arguments to show that Eilon and Flavell's [12] "many-sided" shadow prices are not reasonable. He also comes to the result of the existence of two shadow prices under degeneracy and determines p_i^+ and p_i^- by analyzing the algebraic sign of $\bar{a}_{ij}$ for $i \in T$ and $j \notin \rho$ belonging to slack variables in an optimal (degenerate) solution.

He shows: Let $\check{b}_r$ and $\hat{b}_r$ be the lower and upper bounds of b_r, respectively, in the sense of ordinary sensitivity analysis. Then, if the optimal primal solution is nondegenerate,

$\check{b}_r < b_r < \hat{b}_r$ holds,

if it is degenerate, then 3 cases are possible:

(a) $\check{b}_r = b_r < \hat{b}_r$

(b) $\check{b}_r < b_r = \hat{b}_r$

(c) $\check{b}_r = b_r = \hat{b}_r$

Let x_{n+r} be the r-th slack variable and B the corresponding optimal basis. Then, $\bar{y}_r$ can be interpreted

(i) like in a nondegenerate solution if

$\bar{a}_{i,n+r} = 0$ for all $i \in T$,

(ii) like p_r^+ if

(1) $\bar{a}_{i,n+r} \geqslant 0 \quad \forall\, i \in T$ and

$\bar{a}_{i,n+r} > 0$ for at least one $i \in T$,

(iii) like p_r^- if

(2) $\bar{a}_{i,n+r} \leqslant 0 \quad \forall\, i \in T$ and

$\bar{a}_{i,n+r} < 0$ for at least one $i \in T$,

(iv) there is no interpretation if

(3) there exist $k \in T$ and $l \in T$ such that

$\bar{a}_{k,n+r} \cdot \bar{a}_{l,n+r} < 0$.

Knolmayer [19] has derived these results earlier then Akgül; as a matter of fact, Knolmayer's results can be easily derived from the viewpoint of the function $H(t)$ in Akgül [1] and/or – as Knolmayer mentions in [21] – from the very well known rules for performing sensitivity analysis with respect to b.

Aucamp and Steinberg [2] claim that in order to dermine p_i^+ and p_i^- for all $i = 1, \ldots, m$, it is necessary to generate all alternate optimal dual solutions (tableau). Knolmayer [19, 21] gives an algorithm according to which this is not necessary. This follows also from the theory of linear parametric programming (compare Section 2).

Let us demonstrate the determination of p_i^+ and p_i^- $\forall\ i$ using first Knolmayer's example in [21] (Example 1.2) (degeneracy caused by weak redundancy) and then another Example 1.3 in which degeneracy is not caused by weak redundancy.

Example 1.2 (Knolmayer [21]):

$$\max z = 100x_1 + 150x_2 + 160x_3$$

$$\begin{aligned} \text{s.t.} \quad 0.5x_1 + x_2 + x_3 &\leqslant 125 \\ x_1 + 0.5x_2 + x_3 &\leqslant 100 \\ x_1 + x_2 + x_3 &\leqslant 150 \\ x_2 + x_3 &\leqslant 100 \\ x_j \geqslant 0,\ j = 1, 2, 3 & \end{aligned}$$

The "first" optimal solution is in Table 5.

Table 5

ρ_1	3	4	5	x_B
1	2/3	−2/3	4/3	50
2	2/3	4/3	−2/3	100
6	−1/3	−2/3	−2/3	0
7	1/3	−4/3*	2/3	0
Δz_j	20/3	400/3	100/3	20,000

We note that in row "6" all $\bar{a}_{6j}, j \notin \rho$, are negative and the value of $x_6 = 0$. This implies that constraint No. 3 is weakly redundant [18].

We determine the RHS-ranges in terms of $b_i(\lambda_i) = b_i + \lambda_i, i = 1, \ldots, 4$:

$$\lambda_1 \in \left[-\frac{300}{6}, 0\right]; \quad \lambda_2 \in [0, 0];$$

$$\lambda_3 \in [0, \infty); \quad \lambda_4 \in [0, \infty).$$

Using Akgül's $H(t)$ in terms of λ_i we obtain: $p_1^- = 400/3$, $p_3^+ = 0$, $p_4^+ = 0$. This result corresponds to the "algebraic-signs analysis" due to Knolmayer (e.g., column "4": in rows "6" and "7", i.e. for $i \in T$, we have $\bar{a}_{64} < 0, \bar{a}_{74} < 0$, hence $p_1^- = 400/3$).

We need p_1^+, p_2^+, p_2^-, p_3^- and p_4^-. Therefore, a dual simplex step with respect to $i \in T$ is performed which leads to Table 6.

Table 6

ρ_2	3	5	7	x_B
1	1/2	1	−1/2	50
2	1	0	1	100
6	−1/2	−1	−1/2	0
4	−1/4	1/2	−3/4	0
Δz_j	40	100	100	20,000

Here we can see that constraint No. 1 is also weakly redundant. We have:

$$\lambda_1 \in [0, \infty); \quad \lambda_2 \in [-50, 0]; \quad \lambda_3 \in [0, \infty);$$

$$\lambda_4 \in [-100, 0],$$

hence $p_1^+ = 0, p_2^- = 100, p_3^+ = 0, p_4^- = 100$.

We still need p_2^+, p_3^-; therefore, a dual simplex step ($i \in T$) is performed which leads to Table 7.

Table 7

ρ_3	5	6	7	x_B
1	0	1	−1	50
2	−2	2	0	100
3	2	−2	1	0
4	0	−1/2	−1/2*	0
Δz_j	20	80	60	20,000

We have:

$$\lambda_1 \in [0, \infty); \quad \lambda_2 \in [0, 50]; \quad \lambda_3 \in [-50, 0];$$

$$\lambda_4 \in [0, 0], \quad \text{hence } p_2^+ = 20, p_3^- = 80.$$

Though there exists another alternate optimal solution (replacing x_4 by x_7), it is not necessary to generate the corresponding tableau because all needed information is available.

We note once more that in the above example, like in all others used by the above cited authors, degeneracy is caused by weak redundancy. In the next example this is not the case.

Example 1.3:

$$\max z = x_1 + 2x_2 + x_3 + 0.5x_4$$

$$\begin{aligned} \text{s.t.} \quad & x_1 + x_2 + x_3 \leqslant 2 \\ & x_1 + x_3 + x_4 \leqslant 2 \\ & x_2 + x_3 + x_4 \leqslant 2 \\ & x_1 + x_2 + x_4 \leqslant 2 \\ & x_j \geqslant 0, \; j = 1, \ldots, 4. \end{aligned}$$

The "first" optimal tableau is in Table 8.

Table 8

ρ_1	1	3	5	7	x_B
2	1	1	2	1	2
4	0	−1*	1	−1	0
6	1	0	1	1	2
8	−1	1**	−1	0	0
Δz_j	1.5	0.5	0.5	1	4

We proceed a little more systematically, using Akgül's [1] idea. Determine p_i^+ and p_i^-, $i = 1, \ldots, 4$.

From Table 8 it follows:

$\lambda_1 \in [0, 0] \Rightarrow p_1^+$, p_1^- not available

$\lambda_2 \in [-2, \infty) \Rightarrow p_2 = p_2^+ = p_2^- = y_2^- = 0$

$\lambda_3 \in [-2, 0] \Rightarrow p_3^- = 1, \quad p_3^+$ not available

$\lambda_4 \in [0, \infty) \Rightarrow p_4^+ = 0, \quad p_4^-$ not available.

Determine: p_1^+, p_1^-, p_3^+, p_4^-. Therefore one dual step (Table 9).

Table 9

ρ_2	1	4	5	7	x_B
2	1	−1	1	2	2
3	0	−1	−1	1	0
6	1	0	1	1	2
8	−1*	1	0	−1	0
Δz_j	1.5	0.5	1	1.5	4

Here we have:

$\lambda_1 \in [-2, 0] \Rightarrow p_1^- = 1, \quad p_1^+$ not available

$\lambda_3 \in [0, 0] \Rightarrow p_3^+$ not available

$\lambda_4 \in [0, \infty] \Rightarrow p_4^+ = 0$ already known,

p_4^- not available

Determine p_1^+, p_3^+, p_4^-. A dual step leads to Table 10.

Table 10

ρ_3	4	5	7	8	x_B
2	0	1	1	1	2
3	−1	−1*	1	0	0
6	1	1	0	1	2
1	−1	0	1	−1	0
Δz_j	2	1	0	1.5	4

Here we have:

$\lambda_1 \in [-2, 0] \Rightarrow p_1^- = 1$ already known,

p_1^+ not available

$\lambda_3 \in [0, \infty) \Rightarrow p_3^+ = 0$

$\lambda_4 \in [-2, 0] \Rightarrow p_4^- = 1.5$

Determine p_1^+. One dual step leads to Table 11.

Table 11

ρ_4	3	4	7	8	x_B
2	1	−1	2	1	2
5	−1	1	−1	0	0
6	1	0	1	1	2
1	0	−1	1	−1	0
Δz_j	1	1	1	1.5	4

Here we have:

$\lambda_1 \in [0, \infty) \Rightarrow p_1^+ = 0,$

hence p_i^+ and p_i^- are determined for all $i = 1, \ldots, 4$.

Note that though in Table 10 the dual is degenerate, this has no influence on the determination of the p_i's.

2. Sensitivity Analysis Under Primal Degeneracy

Using LP for solving problems in the practice, most commercial LP-software provides RHS-or objective function coefficients (OFC) ranging as a powerfull Decision Support System for the manager. However, as it is in the case of the shadow prices, RHS or OFC-ranging fails to give correct results when primal degeneracy occurs in an optimal solution.

As Evans and Baker [13] say, "This problem, if overlooked, has significant managerial implications" (p. 348). These authors complain about "the apparent lack of mention of these issues in any textbook, even in the treatice of post-optimality analysis by Gal [14]". Let us say that this assertion is simply not true. Knolmayer [19, 20] provides detailed results on this topics and, in the mentioned book by Gal [14], this problem is tackled e.g. on pp. 21f., 40f., 133, 136, 295, 313, 395 etc.

Since there are some differences between sensitivity analysis caused by weak redundancy and not caused by weak redundancy, we shall split this section into two parts.

Before we start the specific considerations, some general notes are necessary.

Namely, in ordinary sensitivity analysis (without degeneracy) the imperative is to compute the critical region of a parameter such that the found *optimal basis* (solution, tableau, basis-index) *does not change.*

In primal degenerate cases there are several bases assigned to and optimal EP x^0. Hence the question is, what should be understood by sensitivity analysis in this case. Clearly, it is not enough to consider one optimal basis associated with x^0. On the countrary, a "parametric analysis" over several bases $B_k \in B^0$, where B^0 is the set bases associated with x^0 [15, 22] ought to be performed. As follows from the theory of parametric programming (see, e.g. [4, 14]) the corresponding "overall" critical region of admissible parameters is the union of the critical regions associated with the single bases B_k. In terms of $b_r(\lambda_r) = b_r + \lambda_r$, $r \in \{1, \ldots, m\}$ fixed and $b_i(\lambda_i) = b_i$ for all $i \neq r$, Knolmayer [21], and in terms of $c_j(t_j) = c_j + t_j$, $j \in \{1, \ldots, n\}$ fixed, Evans and Baker [13] perform sensitivity analysis in about this way.

Let

$$\tilde{B}^0 := \{\tilde{B}_k^0 \mid \Delta z_j^{(k)} \geq 0 \text{ for all } j \notin \rho_k, k = 1, \ldots, K\} \subseteq B^0 \tag{2.1}$$

be the set of optimal bases associated with x^0. With respect to $b_r(\lambda_r) = b_r + \lambda_r$ let

$$\Lambda_r^{(k)} := \{\lambda_r \mid x_B^{(k)}(\lambda_r) \geq 0\} \text{ for every } k \in \{1, \ldots, K\} \tag{2.2}$$

be the critical region of λ_r with respect to the basis $\tilde{B}_k \in \tilde{B}^0$. Then the "overall critical region for λ_r with respect to x^0 is

$$\Lambda_r := \bigcup_{k=1}^{K} \Lambda_r^{(k)} \tag{2.3}$$

A proposal how to define sensitivity analysis with respect to $b_r(\lambda_r)$ under primal degeneracy is:

Determine Λ_r (see (2.3)) such that for all $\lambda_r \in \Lambda_r$ at least one basis $\tilde{B}_k \in \tilde{B}^0$ remains optimal.

With respect to $c_j(t_j) = c_j + t_j$ the proposal is not the same:

Determine the "overall" critical region $T_j = \bigcup_{k=1}^{K} T_j^{(k)}$ such that for all $t_j \in T_j$ the set $\tilde{B}^0$ does not change.

Note that these definitions can be easily extended to the vectorparametric cases $b(\lambda) = b + F\lambda$, where F is an $(m \times s)$ matrix $\lambda \in \mathbb{R}^s$, or $c(t) = c + Ht$, where H is an $(n \times p)$ matrix, $t \in \mathbb{R}^p$.

2.1 Sensitivity Analysis Under Primal Degeneracy Caused by Weak Redundancy

Evans and Baker [13] investigate cost ranging saying that "we shall take an intuitive point of view, directed primarily at users of LP software. The underlying theory can be found in most standard textbooks".

Let us present Evans' and Baker's example P1 in [13].

$$\max z = 2x_1 + x_2$$

$$\text{s.t.} \quad x_1 + x_2 \leqslant 10$$
$$x_1 \leqslant 5$$
$$x_2 \leqslant 5$$
$$x_1 \geqslant 0,\ x_2 \geqslant 0.$$

In Table 12 an optimal solution is presented.

Table 12 (Evans/Baker [13])

ρ_1	3	4	x_B
2	1	-1	5
1	0	1	5
5	-1	1*	0
Δz_j	1	1	15

We note that replacing x_5 by x_3 it can be seen that the constraint No. 1 is weakly redundant. In terms of $c_j(t_j) = c_j + t_j, j = 1, 2$, we have:

$t_1 \in [-1, \infty)$; since $c_1 = 2$ it follows in terms of c_1 that

$$c_1 \in [1, \infty)$$

$t_2 \in [-1, 1]$; since $c_2 = 1$ it follows in terms of c_2 that

$$c_2 \in [0, 2].$$

Setting $t_1 = -1$, column "4" becomes pivot-column. Pivoting yields Table 13.

Table 13

ρ_2	3	5	x_B
2	0	1	5
1	1	-1	5
4	-1*	1	0
Δz_j	2	1	15

Here we have $t_1 \in [-2, -1]$, i.e. $c_1 \in [0, 1]$, which constradicts the result of Evans and Baker who claim $c_1 \in [0, \infty)$.

Setting $t_1 = -2$, column "3" becomes pivot-column. Evans and Baker use $\bar{a}_{43} = -1$ as pivot which contradicts the generation of alternate optimal solutions (in case the dual is degenerate). However, in order to stay in the set $\tilde{B}^0$, this step is necessary. In this way they obtain Table 14.

Table 14

ρ_3	4	5	x_B
2	0	1	5
1	1	0	5
3	-1	-1	0
Δz_j	2	1	15

Here we have $t_1 \in [-2, \infty)$, i.e. $c_1 \in [0, \infty)$, and it follows that constraint No. 1 is weakly redundant.

Evans and Baker conclude correctly that the "true range" (p. 351) is $c_1 \geqslant 0$.

We will show now that, since degeneracy is caused by weak redundancy, it is possible to omit the weakly redundant constraint No. 1 in Table 14 obtaining the same result. Omitting constraint No. 1, we obtain Table 15.

Table 15

	4	5	x_B
2	0	1	5
1	1	0	5
Δz_j	2	1	15

Here we have: $t_1 \in [-2, \infty) \Rightarrow c_1 \in [0, \infty)$, hence the result is the same when the weakly redundant constraint is included.

It is clear that omitting a weakly redundant constraint in an optimal solution has no influence on the cost ranging because the weakly redundant constraint does not influence the "admissible rotation" of the objective function.

Concerning RHS-ranging, Evans and Baker say that "we do not specifically address changes in the RHS coefficients since these are simply dual to sensitivity analysis of the objective function coefficients". This is, especially with respect to weak redundancy as causing degeneracy, not correct, because, with respect to the RHS, it is not possible to omit weakly redundant constraints.

To show this, turn back to Table 12. Here we have in terms of $b_i(\lambda_i) = b_i + \lambda_i$:

$$\lambda_1 \in [-5, 0]; \quad \lambda_2 \in [0, 5]; \quad \lambda_3 \in [0, \infty).$$

Replacing x_5 by x_3 we obtain Table 14. Hence, we have:

$$\lambda_1 \in [0, \infty); \quad \lambda_2 \in [-5, 0]; \quad \lambda_3 \in [-5, 0].$$

Thus $\lambda_1 \in [-5, \infty)$, $\lambda_2 \in [-5, 5]$ and $\lambda_3 \in [-5, \infty)$ taking the corresponding set-unions. Leaving constraint No. 1, we have Table 15. Here λ_1 becomes senseless, $\lambda_2 \in [-5, \infty)$ what obviously differs from the above result. Consider an optimal EP in $\mathbb{R}^n$ which is defined by n constraints plus a weakly redundant one. Introducing λ_k in the sense $b_k(\lambda_k) = b_k + \lambda_k$ means geometrically to move the k-th constraint (more exactly: the boundary hyperplane of the k-th constraint) parallel to itself. Let $r \neq k$ be the index of the weakly redundant constraint.

Then it is obvious that "moving" the k-th constraint the r-th weakly redundant constraint influences the "admissible moving" of the k-th constraint (see also what was cited above from the paper by Aucamp and Steinberg [2]). Thus the "admissible moving" of the k-th constraint depends heavily on whether the r-th constraint is omited or not.

Knolmayer [20] has found that investigating sensitivity analysis with respect to the RHS b, one has to generate a part of the optimal tableaux assigned to the degenerate EP, determine for each of them the critical region $\Lambda_i^{(k)} = [\underline{\lambda}_i^{(k)}, \bar{\lambda}_i^{(k)}]$, $k = 1, \ldots, K$, and take the union (2.3).

He claimes that the same applies to the objective function coefficients sensitivity analysis, what – as shown above – is not correct in cases when degeneracy is caused by weak redundancy.

2.2 Sensitivity Analysis Under Primal Degeneracy not Caused by Weak redundancy

The main difference in performing sensitivity analysis under degeneracy between degeneracy caused and not caused by weak redundancy is with respect to the OFC. As shown above in Sect. 2.1, in case degeneracy is caused by weak redundancy the corresponding weakly redundant constraint can be simply omitted and sensitivity analysis with respect to OFC is performed like in a nondegenerate case.

This is not possible to do with respect to the RHS. As also shown above, omitting the weakly redundant constraints, the results differ in this case crucially from those obtained with the weakly redundant constraint included.

However, when degeneracy is not caused by weak redundancy, there is nothing to omit and the above described approach applies. To show this, we shall use our Example 1.3.

Let us start with Table 8 and consider as an example $c_3(t_3) = c_3 + t_3$. From Table 8 it follows that $t_3 \in (-\infty, 0.5]$ and the pivot-column is column "3"; the pivot-element is labeled by 2 stars. Pivoting we obtain Table 16.

Table 16

ρ_5	1	5	7	8	x_B
2	0	1	1	1	2
4	–1	0	–1	1	0
6	1	1	1	0	2
3	–1	–1*	0	1	0
Δz_j	2	1	1	–0.5	4

Here we have $t_3 \in [0.5; 1]$ and column "5" is pivot column. Pivoting with the labeled pivot we obtain Table 17.

Table 17

ρ_6	1	3	7	8	x_B
2	–1	1	1	2	2
4	–1	0	–1	1	0
6	0	1	1	1	2
5	1	–1	0	–1	0
Δz_j	1	1	1	0.5	4

Here $t_3 \in (-\infty, 1] = (-\infty, 0.5] \cup [0.5; 1]$. The "parametric analysis" with respect to $\tilde{B}^0$ is over because using the possible pivot $\bar{a}_{53} = -1$ in Table 17 yields ρ_5, which is already investigated in Table 16.

We shall now concisely deal with the paper by Williams [34] who deals with conditions under which the marginal value of an LP with respect to some perturbation exists. Given (LP) and (LP_D) he introduces the matrix

$$\tilde{A} = \begin{pmatrix} A & b \\ c & 0 \end{pmatrix}$$

which generates (LP) and (LP_D). He then introduces the feasible sets for (LP) and (LP_D) as

$$S(\tilde{A}) := \{x \in \mathbb{R}^n \mid x \geqslant 0, Ax \leqslant b\} \quad \text{and}$$

$$T(\tilde{A}) := \{y \in \mathbb{R}^m \mid y \geqslant 0, A^T y \geqslant c\},$$

respectively, as well as the corresponding sets of optimal vectors x^0 and y^0 as

$$S^0(\tilde{A}) := \{x^0 \in \mathbb{R}^n \mid x^0 \in S(\tilde{A}), c^T x^0 \geqslant c^T x \text{ for all } x \in S(\tilde{A})\}$$

$$T^0(\tilde{A}) := \{y^0 \in \mathbb{R}^m \mid y^0 \in T(\tilde{A}), b^T y^0 \leqslant b^T y \text{ for all } y \in T(\tilde{A})\}.$$

The value of the objectives implied by $\tilde{A}$ is $b^T y^0 = c^T x^0$ and is denoted by $\varphi(\tilde{A})$. Williams [34] introduces the pertubation matrix

$$\tilde{H} = \begin{pmatrix} H & \tilde{b} \\ \tilde{c} & 0 \end{pmatrix}$$

in the sense of

$$(LP_H) \quad \max\, (c + \tilde{c}\alpha)^T x$$
$$\text{s.t.} \quad (A + H\alpha)\, x \leqslant (b + \tilde{b}\alpha)$$
$$x \geqslant 0$$

and correspondingly for (LP_D). He then considers the LP generated by $(\tilde{A} + \alpha\tilde{H})$ and defines the function

$$f(\alpha) = \varphi(\tilde{A} + \alpha H),$$

which is the known optimal value function (mentioned already above – cf. Akgül [1], see also [4, 14]) with respect to the perturbation $\tilde{H}$. The domain of $f(\alpha)$ is the set of values of α for which $S^0(\tilde{A} + \alpha\tilde{H})$ and $T^0(\tilde{A} + \alpha\tilde{H})$ are not empty.

The problem to be solved is to determine conditions on $\tilde{A}$ such that for every $\tilde{H}, f(\alpha)$ possesses a one-sided (in particular, a right) derivative at $\alpha = 0$. "Such a derivative exists if

(i) there is a (nondegenerate closed) interval $[0, \alpha_0]$ in the domain of $f(\alpha)$,

(ii) $f(\alpha)$ is continuous to the right of $\alpha = 0$,

(iii) the limit

$$f'(0) = \lim_{\alpha \to 0+} \frac{\varphi(\tilde{A} + \alpha\tilde{H}) - \varphi(\tilde{A})}{\alpha}$$

exists" ([34], p 84).

Williams proves then the following theorems:

Theorem 2.1 (Williams [34]). *Let the* LP $\tilde{A}$ *be given. Then necessary and sufficient conditions that the marginal value* $f'(0)$ *of the* LP $\tilde{A}$ *with respect to the perturbation* $\tilde{H}$ *exists for every* $\tilde{H}$ *is that both the primal and dual optimal sets of* $\tilde{A}$, *i.e. the sets* $S^0(\tilde{A})$ *and* $T^0(\tilde{A})$ *be bounded, or equivalently, that the regularity conditions*

R1 $\quad w \geqslant 0, \quad A^T w \leqslant 0 \Rightarrow c^T w < 0$

for every vector w

R2 $\quad v \geqslant 0, \quad v^T A \geqslant 0 \Rightarrow v^T b > 0$

for every vector v

be satisfied by $\tilde{A}$.

Theorem 2.2 (Williams [34]). *Let* $\tilde{A}$ *satisfy the regularity conditions* R1 *and* R2, *and let* $\tilde{H}$ *be any perturbation matrix. Then the marginal value* $f'(0)$ *of* $\tilde{A}$ *with respect to* $\tilde{H}$ *is given by*

$$f'(0) = \max_{x^0 \in S^0(\tilde{A})} \; \min_{y^0 \in T^0(\tilde{A})} \; \Psi(\tilde{H}, x^0, y^0)$$

where $\Psi(\tilde{H}, x^0, y^0)$ *is the Lagrangian defined by*

$$\Psi(A, x, y) = c^T x + y^T b - y^T A x,$$

transformed accordingly to (LP_H) *above.*

These results are comparable with those by Akgül [1] who derived them using convex analysis and applied them directly to the degenerate case, and – as a particular case – with those by Aucamp and Steinberg [2].

Conclusions

Shadow prices and sensitivity analysis results are important information for a decision making process, particularly when the implementation of an optimal result using LP fail in the case of primal degeneracy in an optimal solution. Commercial codes do not provide the corresponding information correctly because they are developed for nondegenerate cases only.

Fortunately, several authors noticed these circumstances and it turned out that, in case of primal degeneracy in an optimal solution to an LP, there are two shadow prices for each resource ("selling" and "buying" one unit of the corresponding resource) which can be determined. Various approaches have been proposed to determine such shadow prices. One of them, which seems to be the most reasonable one and which is a modification and combination of Akgül's [1] and Knolmayer's [19, 21] methods, is described in this paper (Sect. 1).

There remains, however, some open questions, particularly with respect to sensitivity analysis under degeneracy.

From the theory of degenerate EP's and the associated degeneracy graphs [15, 22] it follows that if U is the number of all bases (nodes) associated with a degenerate EP x^0, only a part $\tilde{U} \leqslant U$ is associated with optimal tableaux.

Now the question arises whether the "true" critical region Λ_i for λ_i is really determined by analyzing optimal tableaux $\tilde{B}^0$ (see (2.2)) associated with the degenerate EP x^0, or whether it is necessary to investigate all nodes in the so called N-tree in the degeneracy graph [22] or that a subset of the set of all nodes associated with a degenerate EP should be investigated in order to determine Λ_i correctly.

Another question is how to define the critical region with respect to changes of b_i or c_j under primal degeneracy. Namely, in terms of $b_i(\lambda_i) = b_i + \lambda_i, i \in \{1, \ldots, m\}$, in nondegenerate cases the imperative is to determine the critical region Λ_i of λ_i such that for all $\lambda_i \in \Lambda_i$ *the* optimal basis does not change. In degenerate cases there are, however, several bases associated with the optimal extreme point. A proposal how to define Λ_i under primal degeneracy is formulated above (Sect. 2). Note that in degenerate cases there are differences between sensitivity analysis with respect to b or to c and that there remain still open questions.

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