ON DIVISION OF THE QUESTION

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I. *Introduction*

I believe that the mathematical theory of voting may be used to shed light on traditional rules of parliamentary procedure. For example, Black's [2] theory of voting on issues when the alternatives can be ordered so that each member's preferences are single-peaked highlights the importance of the chairman's casting vote in the case of an even number of voters. In this paper I study how another rule, division of the question on amendments, affects the creation of successful packages.

"Package deals" are an important part of American legislative behavior. Packages are often said to be put together to avoid compromises on separate issues by giving many parties the concession they most want, thus ensuring them support, as in logrolling. Typically a legislator is seen as strongly favoring an expenditure in his district, and fairly neutral, but possibly slightly opposed to, expenditures in other districts. By putting together a package which has projects in a majority of districts, passage of all the projects is assured. If each project voted on separately would have failed, this is an example of cyclic voting. The possibility of this cyclic behavior was pointed to by Anthony Downs [6, p. 55ff] who described the strategy of a "coalition of minorities," comprising persons who feel most strongly about issues on which they are in the minority.

This paper explores the relationship between the cycle of outcomes on issues and the rules of the amendment process. Under certain assumptions, division of the question on amendments resolves the cycle.

The mathematical theory discussed in sections 2 through 4 is applied to state constitutional amendments in section 5. Here the essential assumption is that voters tend to favor "modernization" of state constitutions, but sometimes vehemently oppose a particular provision. Such a preference schedule is the obverse of the state legislator's preferences as hypothesized above. Submission of revisions in a single package allows the vehement opponents of each article to unite, frequently defeating the package. Submission in separate questions, however, divides the opposition and leads to passage. Quite possibly this theory applies elsewhere as well.

*Department of Statistics, Carnegie-Mellon University. I was led to this study by a conversation with James Crook in which he told me of unpublished work he did jointly with Charles Goetz of Virginia Polytechnic Institute which led them to conjecture part (i) of Theorem 1. A suggestion of Bernard Grofman attacted my attention to state constitution amendments as an application I am indebted to William H. Riker and Melvin Hinich for comments on an earlier draft.

II. *Assumptions*

Suppose that there are ℓ voters indexed $i=1,2,\ldots,\ell$ to decide among n issues indexed $j=1,2,\ldots,n$. Suppose that the set of alternatives for issue j is A_j. Then the committee is to choose a member of

$$A = A_1 \times A_2 \times \ldots \times A_n$$

as a platform, that is, the committee is to choose a member of each set A_j.

I make two assumptions on the preferences of voters. First, the issues are assumed to be separable for each voter i: if $a=(a_1,\ldots,a_{j-1},a_j,a_{j+1},\ldots,a_n) \in A$ and $a' - (a_1,\ldots,a_{j-1},a'_j,a_{j+1},\ldots,a_n) \in A$ and voter i prefers a to a', then voter i prefers $b-(b_1,\ldots,b_{j-1},a_m,b_{j+1},\ldots,b_n) \in A$ to $b' = (b_1,\ldots,b_{j-1}, a'_j,b_{j+1},\ldots,b_n)$. A recent article by Gorman [8] discusses utility functions of this type. Separability is essential to this paper. In terms of Davis-hinich model [5], this assumption corresponse to a diagonal matrix A. This causes no loss of generality in their model.

Secondly, the set of preferences of the voters on each issue is assumed to be single-peaked [1]. Thus there is assumed to be an ordering on each set A_j such that if y is between x and z then y is preferred to at least one of x and z by each voter. If A_j has only two elements, preferences are automatically single-peaked. Kramer [11] shows that single-peakedness is very restrictive if the set A is in the interior of a real-space of three or more dimensions, but his notion of single-peakedness is a stronger condition than the notion used here. For a discussion of the differences see Davis, DeGroot and Hinich [4]. Tullock [16] and Simpson [14] also discuss the resolution of cyclical voting in multidimensional situations. Fortunately single-peakedness is not essential to the argument here, but is used for convenience of exposition. The more general case, without single-peakedness, is given in the Appendix.

A third assumption is that voters vote according to their preferences. Although seemingly innocuous, this assumption is quite restrictive. For example it forbids informal "logrolling" of the type studied by Wilson [17]. Additionally, it requires "sincere voting.' as opposed to "sophisticated" voting (that is, voters are not permitted to anticipate the votes of others and vote accordingly [12]). While one might expect sophisticated voting in small committees, sincere voting seems an appropriate assumption for large electorates. See [7].

III. *No Division of the Question*

Suppose that the committee is to consider the entire set of issues in a single motion, and is to try to arrive at a resolution, by majority vote, of all of them together. One possible platform, here designated P*, is the issue-by-issue majority view. That is,

$$P^* = (0^1_{med}, 0^2_{med}, \ldots, 0^n_{med}),$$

where 0^j_{med} is the median most preferred position on issue j. Much of the remainder of this paper concerns the special role played by P* in the committee's consideration.

A major difficulty in the theory of committees is the possibility fo cycles: that a majority will prefer B to A, C to B and A to C. Either there is a platform which defeats all others, or there is one cycle, possibly of more than three platforms, which defeats all platforms not in this cycle. (See Kadane [10] for an exposition of the relevant graph theory.) The following theorem describes the role of P* in each case:

Theorem 1: There may be

(i) a single platform which defeats all others. In this case P* is that platform.

(ii) a cycle of platforms which defeat all platforms not in the cycle. Then P* is one of the platforms in the cycle.

Proofs of theorems can be found in the Appendix.

Although theorem 1 does give a nice property for P*, it is still not very satisfactory. Examples can easily be given in which the cycle in (ii) includes all possible platforms, thus not distinguishing P* in any way.

IV. *Division of the Question*

In order to rectify the above situation, some additional condition need be imposed. Just as Black used the chairman's casting vote to resolve the problem of an even number of voters, a common parliamentary rule, called "division of the question" is used here. Robert's *Rules of Order* [13, p.91] says

> If a series of independent resolutions relating to different subjects is included in one motion, it must be divided upon the request of a single member. . . .

Clearly if division of the question is applied to the entire motion, the committee reverts to issue-by-issue voting. However the committee might apply division of the question just to amendments. Thus if an amendment to the platform on the floor makes changes in several issues, division of the question on that amendment would require the changes to be considered, and voted upon, separately.

Notice that Roberts requires that a member must request division of the question. The following theorem describes what happens when division of the question is imposed by the chairman automatically, without requiring a request:

> Theorem 2: If division of the question on amendments is imposed automatically, P* defeats all amendments and no other platform does so. From any initial platform as the starting motion, a sequence of successful amendments can be specified which leads to P*.

Theorem 2 leaves open the question of whether some member would always want to have the question on the amendment divided (and thus defeated) if P* is the motion on the floor. The following example shows that it is possible that a committee might *unanimously* favor some amendment to P*. In such a case, no one would find it in his interest to have the question on this amendment divided.

Suppose there are three voters and three issues. The first issue, if the committee approves it, would tax voter one a dollar, and distribute thirty cents to each of the other two voters. Similarly issue two would tax voter two a dollar and distribute thirty cents to voters one and three. And issue three similarly. Issue-by-issue, the committee would favor all three. Yet unanimously the committee would prefer the amendment defeating all three. Perhaps the best solution for this committee is to adjourn, which takes parliamentary precedence over every other motion!

Further examples of this type are considered by Hillinger [9].

V. State Constitutional Amendments

While a mathematical voting theory is of some interest by itself, applications to real political problems help give it substance. Such an application is discussed below.

In recent years public interest in reform of state constitutions has increased. Many states have convened Constitutional Conventions, with results shown below:

An outstanding feature of Table 1 is that all revisions submitted in a single package were defeated, while those submitted in parts were passed nearly entirely. How might this phenomenon be explained?

Suppose the voters generally favor reform of state constitutions, but that some voters, a few, take strong exception to each proposed revision. Then submission in a single package would permit opponents of each reform to unit and defeat the package. Division of the question, however, divides the opponents and thus would aid passage of more of the reforms.

This observation is not new to experts on state constitutional reforms. For example, Albert Sturm reports "Success is more likely if highly controversial issues are submitted separately. . . .Submission in a single package consolidates and strengthens the effect of opposition to particular parts of a proposed document; presentation of highly controversial issues separately tends to fragment and weaken the opposition" [15].

The theory given in this paper underscores this practical political advice.

Table 1

Results of Votes on State Constitutional Amendments Submitted to the People by Constitutional Conventions

	State	No. of Parts Proposed	Result
1.	California (1968)	1	Defeated
2.	Florida (1967) (Legislature as Convention)	3	All passed
3.	Hawaii (1968)	23	All but one passed
4.	Maryland (1968)	1	Defeated
5.	New Hampshire (1968)	6	All but one passed
6.	New Mexico (1969)	1	Defeated
7.	New York (1967)	1	Defeated
8.	Pennsylvania (1967)	5	All passed
9.	Rhode Island (1968)	1	Defeated

Sources: *Recent Constitutional Revision Activities,* 1967-1968, The Council of State Governments, March 1969.

Constitutional Revision Activities, 1968-1969, The Council of State Governments, April 1970.

Appendix. Statement and Proof of theorems 1 and 2 in the more general case in which single-peakedness is not required.

Let B_j be the smallest set of positions on issue j which defeat every position not in B_j for each issue j. That such a set exists is an implication of the graph theory in [10]. When single-peakedness applies, B_j has a single element, namely ω^j_{med}. Furthermore, also from [10], there is a cycle which includes all the members of B_j. Let

$$B^* = B_1 \times B_2 \times \ldots \times B_n,$$

be the set of all possible platforms consisting of members of B_j. When single-peakedness applies, B^* has the single element P^*.

Now let B be the smallest set of platforms which defeat every platform not in B.

Theorem 1: $B^* \subseteq B$.
Theorem 2: If Division of the Question is imposed automatically,

$$B^* = B.$$

From any initial platform as the starting motion, a sequence of successful amendments can be specified which leads to any member of B^*.

Before giving the proofs, the following lemma is useful:
Lemma (Improvement Algorithm)

Let P be a platform in B^*, and let P_0 be any platform different from P. Then there is a sequence of platforms P^0, P^1, P^2, . . . , P such that P^{i-1} and P^i differ in only one issue and P^i defeats P^{i-1}.

Proof

Let P_j^i be the position on issue j of platform i. Since P_0 and p are different platforms, they must differ say, on r issues, where r is at least one. Suppose j is one of those issues, so $P_j^0 \neq P_j$. Then there is a sequence of elements of A_j, say $a^1, \ldots, a^k$, such that a_1 defeats P_j^0, a_2 defeats $a_1, \ldots, a_k = P_j$ defeats a_{k-1}. If $P_j^0 \notin B_j$, k can be taken to be 1, that is, P_j defeats P_j^0 by definition of B_j. If $P_j^0 \notin B_j$, the existence of the a's follow because B_j is a cycle. Using separability, there is now a sequence of platforms $A^1, \ldots, A^k$, identical with P^0 except for issue j and there $A_j^{\ell} = a_{\ell}$, each of which defeats its immediate predecessor, that leads from P^0 to A_k, where A^k differs from P on r-1 issues. This process is repeated r times to form the composite sequence required.

QED

Proof of theorem 1

Let $P \in B^*$, and let $P_0 \neq P$. Then by the improvement algorithm, there is a sequence of platforms such that P defeats P_0; P_2 defeats P_1, . . . ,and P defeats the last but one. Hence $\in$ B. Therefore $B^* \subseteq B$.

Proof of theorem 2

Division of the Question requires that amendments change only a single issue. Each attempt to change a single issue of an element of B^* to some position not in B_j fails, by property of B_j. Hence $B = B^*$. The rest is a consequence of the improvement algorithm.

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