ABSTENTIONS AND EQUILIBRIUM IN THE ELECTORAL PROCESS

by

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Many consider spatial analyses of party competition, inagurated by Downs's classic *An Economic Theory of Democracy*,[1] as a "take-off" point for political science in the attempt to describe and explain the reciprocal relationships between public policy and the process by which democracies select public policy makers, i.e., elections. Although numerous case studies and empirical generalizations exist which purport to describe these relationships, the obvious complexity of democratic processes necessitates formal deductive analysis for any adequate general understanding of them. Spatial analyses, rooted in the economic concept of rational action, seek to provide the requisite rigor of formulation. This rigor is accompanied by the derivation of inferences concerning the logical imperatives toward the organization of policy outcomes (i.e. what policies are forced by the structure of public opinion and from party competition), and toward stability in such systems. Spatial research, therefore, focuses on two objectives: (1) increasing the deductive rigor of descriptions of democratic processes, and; (2) accommodating the assumptions of spatial models with empirical fact. Neither of these objectives is obtainable, however, without incurring some cost.

Davis and Hinich, for example, extend the Downsian model to include multi-issue elections, and the mathematical rigor of their analyses yield a variety of nonobvious theorems, unobtainable through previous, and less regorous, formulations.[2] The incurred cost, however, is their assumption that all eligible members of the electorate vote – a serious limitation of the applicability of their model to the real world. Similarly, Garvey formalizes the citizen's calculus and provides a more explicit description of the causes of rational abstention with reference to the candidates' strategies.[3] The costs of this formalization, however, are: (1) retention of the unidimensional assumption, and; (2) a decrease in deductive rigor (compared to the Davis and Hinich analyses).[4]

The cause of these incurred costs is easily identified: mathematical reasoning grows more complex and difficult as deductive rigor increases, as assumptions are generalized and stated more explicitly, and as these assumptions are made more consonant with empirical fact. This leads us to the objective in this essay – we attempt to recoup some of the costs incurred by previous analyses. We postulate a calculus of electoral behavior more general than Davis and Hinich's, and more rigorous than Downs's or Garvey's. This calculus assumes that a citizen either votes for a preferred candidate, or he abstains if sufficient incentives to vote fail to exist.

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Theorems deduced from this model describe the equilibrium behavior of candidates (who are assumed to be maximizing plurality), where equilibrium implies that neither candidate has any incentive to adopt an alternative strategy while his opponent remains at an equilibrium point. Thus, equilibrium strategies are minimax in the game theoretic sense. In this essay we determine necessary and sufficient conditions for the existence of pure equilibrium strategies when preferences are distributed unimodally across one or more issues (see T1, T2, and T3). Briefly, a sufficient condition for the existence of an equilibrium is the symmetry and unimodality of the density of preferences, so that the equilibrium point is the mean of the density (see T1). Furthermore, for any unimodal density, if an equilibrium exists it is a singular point, i.e., the candidates should converge (see T4). We then describe a candidate's plurality maximizing strategy when his opponent fails to adopt a dominant strategy, and we conclude that the candidate should adopt a strategy near or at the mean of the density of preferences whenever his opponent is near of "far" from the mean (see T5, and T6). Hence, Tullock's conclusion that "an extremist candidate can pull a vote-maximizing opponent far off toward the extremist's desires" when all eligible citizens vote is not valid when nonvoting in the form of alienation is permitted.[5] Finally, we consider the effects of variations in the cost of voting (see T7). Assuming that the cost of voting either is raised or lowered uniformly throughout the electorate, the values of equilibrium strategies are unaffected if the density of preferences is unimodal and symmetric, but if the density is not symmetric the values of such strategies are affected. Thus, although it is obvious that electoral outcomes can be manipulated if inequities are generated in the costs of voting by such measures as intimidation and poll taxes, electoral outcomes similarly may be sensitive to "democratic" variations in such costs.

We start with the definitions and assumptions employed in this analysis. Let:

$x=(x_1, x_2, ..., x_n)$ denote the real vector of a citizen's preferred policy positions for each of n issues.

$\theta=(\theta_1, \theta_2, ..., \theta_n)$ denote candidate θ's real vector of policy positions for each of n issues.

$\psi=(\psi_1, \psi_2, ..., \psi_n)$ denote candidate ψ's real vector of policy positions for each of n issues.

$f(x) = f(x_1, x_2, \ldots, x_n)$ denote the joint multivariate density function of preferences where $f(x)$ is continuously differntiable and defined for all $x \epsilon R^n$. Without loss of generality we assume $E(x_1) = E(x_2) = E(x_n) = 0$.

$\phi(x - \theta)$ denote the multivariate loss function translating spatial distance between a citizen's preferred policies and the policies of a candidate whose position is θ into an inverse utility function $(0 \leqslant \phi(x - \theta) < \infty)$.

π denote the utility a citizen derives from voting such that, if and only if $\pi > 0$, he votes, and, if and only if $\pi \leqslant 0$, he does not vote.

Although these variables are mathematically exact their substantive interpretation is less precise. First, the vector x is stated to represent "a vector of preferred *policy* positions." Citizens, however, may not know their preferred policies. In order to facilitate the analysis mathematically, however, we assume that the electorate behaves *as if* each citizen estimates a preferred vector and employs it in his calculus of voting behavior. This assumption is somewhat less objectionable if the "policy" dimensions are interpreted as general dimensions of taste so that party identification and personality (image) are included in x as well as the traditional measures of governmental outputs (e.g., tax policy, unemployment, interest rates). Additionally, we assume that each dimension is continuous. Stokes, for example, objects to this assumption because many issues appear to be discrete (i.e., only a finite number of spatial positions are avialable).[6] Obviously our assumption is an abstraction – made, however, to facilitate the analysis.

Discrete dimensions suggest that the candidates either cannot readily change spatial position (e.g., party identification, religion), or that a single, easily identifiable, position is dominant (i.e., Stokes's "valence" issues). The absence of cardinal dimensions, therefore, frequently implies fixed spatial policies – the candidates are unable to vary θ or ψ during a campaign. Generally, candidates overcome a disadvantage or take advantage of fixed spatial positions by employing alternative means of influencing the electorate (e.g., varying uncertainty or the saliency of issues). Since we are concerned solely with the analysis of spatial strategies, however, we assume cardinal measures of policy or taste.

Additionally, we assume that all citizens are identical except for the value of their preferred policy vectors. Hence, we assume that all citizens make identical estimates of θ and ψ. Empirically we know this assumption is incorrect – cognitive dissonance and ignorance generate diverse estimates of θ and ψ. Furthermore, candidates attempt differentiated appeals which, ideally, would consist of convincing each citizen that the candidates advocate his or her preferred policies. Candidates, however, require modern, mass oriented campaigns, and can not attain this ideal. If, therefore, we conceptualize θ and ψ as those strategies which candidates advocate through the mass media (i.e., when differentiated appeals are impossible or impractical), our assumption is less objectionable.[7] Similarly, optimal or equilibrium strategies are interpreted as the policies candidates should advocate publicly when differentiated appeals cannot be made.

The citizen's calculus is defined now by the assumptions which relate θ and ψ to the loss and to π. We assume that $\phi(x - \theta)$ is a *monotonic function* of the metric

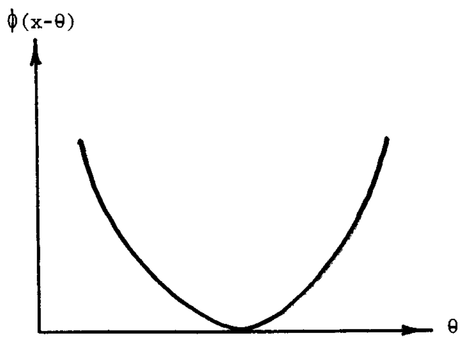


FIGURE 1a

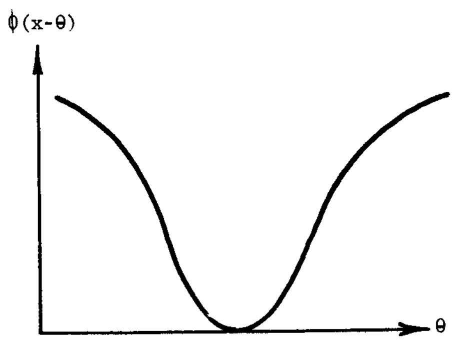


FIGURE 1b

$$\| x - \theta \|_A = [(x - \theta)' A (x - \theta)]^{1/2}$$

where A is a positive definite nxn matrix. If this loss matrix A is identical for all citizens, there exists a linear transformation of the dimensions such that A is the identity matrix I in the transformed space. Thus, with no loss of generality we assume,

$$\phi (x - \theta) = \phi (\sum_{i=1}^{n} (x_i - \theta_i)^2)$$

where ϕ is monotonic (and clearly the loss function satisfies $\phi (x - \theta) = \phi (\theta - x)$). This is a related but weaker assumption that Davis and Hinich's. They assume that the loss function is the convex and quadratic form $\| x - \theta \|_A^2$. In Figure la we represent a unidimensional quadratic loss function. Note that a convex loss function such as the quadratic implies marginally increasing loss with $\| x - \theta \|$. In Figure 1b we represent an alternative loss function satisfying our assumptions.

Hence, our assumptions permit marginally increasing and marginally *decreasing* loss.

With these definitions and assumptions we formulate the citizen's rule of candidate choice:

> If a citizen votes, he votes for θ if and only if $\phi (x - \theta) < \phi (x - \psi)$, or he votes for ψ if and only if $\phi (x - \theta) > \phi (x - \psi)$.

The citizen's calculus, remains incomplete, however, without specification of a decision rule for voting as against abstaining. Specifically, the relationship between π and the candidates' strategies must be considered. We assume that U, the deterministic factor in π, is a decreasing monotonic function of the loss derived from the policies of the preferred candidate. Additionally, we assume that π is affected by stochastic factors, represented by ϵ where $E(\epsilon) = 0$, and the density of ϵ is integrable. Consider a citizen preferring θ to ψ. The citizens's utility from voting, π, is expressed (assuming that θ is the preferred candidate),

(1) $\pi = U (x, \theta) + \epsilon$

where $U (x,\theta)$ is decreasing monotonically with $\phi (x - \theta)$. The variable π is a random variable with expected value $U (x, \theta)$. The probability that $\pi > 0$ equals the probability that $\epsilon > - U (x, \theta)$. Assuming that the density of ϵ is independent of x and θ, this probability is expressed as a function of $\phi (x - \theta)$, or, simply as $g (x - \theta)$.[8] Hence, $g (x - \theta) = Pr [\pi > 0]$.

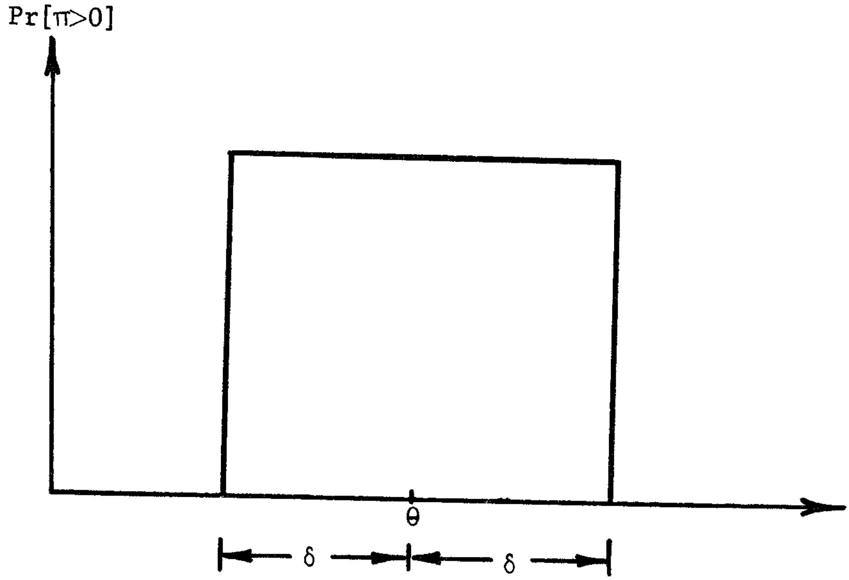


FIGURE 2

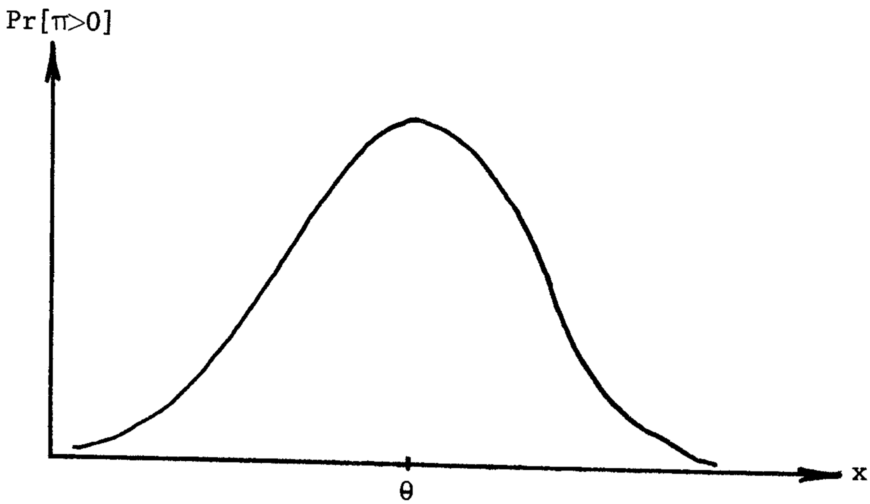


FIGURE 3

It is easily seen that $g(x - \theta)$, the probability that the citizen votes if his preferred policy vector is x, and his preferred candidate is θ, is symmetric about θ. To prove symmetry consider two citizens, the first preferring the policy vector $\theta + \delta$, and the second preferring the vector $\theta - \delta$. Their respective losses from θ are $\phi(\theta + \delta) - \theta) = \phi(\delta)$, and $\phi(\theta - \delta) - \theta) = \phi(-\delta)$. Since ϕ is assumed to be symmetric, however, $\phi(\delta) = \phi(-\delta)$, so that $U(\theta + \delta, \theta) = U(\theta - \delta, \theta)$. Therefore, $g(\theta + \delta - \theta) = g(\delta) = g(\theta - \delta - \theta) = g(-\delta)$, and $g(x\ \theta)$ is symmetric.

This property is clarified by examining explicit formulations of $U(x, \theta)$. Consider the unidimensional case when ϵ is absent (i.e., the variance of ϵ equals zero), and $U(x, \theta)$ is expressed,

$$(2) \qquad U(x, \theta) = A - C\,\phi(x - \theta)$$

(A and C are arbitrary positive constants). Obviously expression (2) satisfies our assumption that $U(x, \theta)$ is decreasing monotonically with $\phi(x - \theta)$. Further, $U(x, \theta)$ is positive for $\phi(x - \theta) < A/C$, equals zero for $\phi(x - \theta) = A/C$, and is negative for all $\phi(x - \theta) > A/C$. For any fixed θ, assume that $\phi(x - \theta) = A/C$ if the distance between x and θ equals $\delta > \theta$. Hence, all citizens preferring policies such that this distance is less than δ vote with probability equal to one, and all citizens preferring policies such that this distance is greater than or equal to δ vote with probability equal to zero. For expression (2) and n=1, $g(x, \theta)$ is represented in Figure 2. Observe that if, for example, $n = 2$, $g(x - \theta)$ is a cylinder with diameter 2δ and center at θ.

Assuming that ϵ has a non-zero variance smoothes $g(x - \theta)$. A general form for $g(x - \theta)$ when $n = 1$ is illustrated in Figure 3.

Including ϵ in our expression for the value of voting, therefore, permits us to use the continuous calculus without loss of generality.

Expression (1) requires that $g(x - \theta) = 1$ for $x = \theta$, and $\lim g(x - \theta) = 0$, as $||x - \theta|| \rightarrow \infty$. Alternative expressions, however, can constrain $g(x - \theta)$ to more limited ranges. In our analysis, therefore, we assume simply that the range of $g(x - \theta)$ conforms to the convention of probability numbers and does not exceed the interval [0, 1].

Before turning to the analysis of the candidates' strategies a few comments are in order concerning the assumption that $U(x, \theta)$ varies inversely with $\phi(x - \theta)$. Observe that, in Figure 3, the greater the discrepancy between a citizen's preferred policy and the policy advocated by the citizen's preferred candidate, the less likely he is to vote. In the limiting case where $Var(\epsilon) = 0$ (Figure 2), the citizen is interpreted as having a range of "tolerance," 2δ, which, if the discrepancy between x and θ is greater than δ, he abstains. Thus, our assumption concerning non-voting captures the intuitive meaning of the word "alienation."

Obviously, alternative assumptions are available, such as assuming that U varies monotonically with $| \phi (x - \theta) - \phi (x - \psi) |$. This latter assumption conforms closely with our intuitive understanding of the term "cross-pressures." While the relative importance of alienation and cross-pressures is unknown, both affect electoral behavior, and, therefore, optimal campaign strategies. Riker and Ordeshook, for example, conclude that the subjective differential in utility between two candidates remains an important determinant of turnout after other discernable effects are controlled.[9] Alternatively, Pool *et al.* observe that the cross-pressure hypothesis fails to account fully for fluctuations in turnout.[10] Consider, for example, two citizens, both perceiving little or no difference between θ and ψ. The first citizen, however, suffers no loss from θ or ψ while the second is greatly dissatisfied with both. It is unreasonable to suppose that the second citizen votes with a probability equal to that of the first. Neither alienation or cross-pressures, therefore, can be eliminated as a cause of variation in turnout.

We select a single, identifiable, relationship between turnout and strategy, however, for two reasons. First, alienation is an important strategic consideration. The rise of third parties, the appeals of demagogic candidates, and the success of these appeals, for example, can be traced to the effects of alienation. Second, our assumption permits manageable analysis. If both causes of non-voting are posited simultaneously, additional a priori assumptions are required concerning their interrelationship. We posit, therefore, a somewhat restrictive model, but we derive from it important and nonobvious results.[11]

Turning now to the analysis of strategies, the candidates' objective functions must be specified. Downs assumes that candidates and parties wish to attain office, and, therefore, maximize votes. The conclusion that candidates maximize votes, however, does not follow from the assumed objective of retaining or obtaining office. Specifically, a candidate must consider how many votes his opponent receives in addition to the number of votes he receives. We assume, therefore, that candidates maximize plurality.

Vote maximization and plurality maximization are distinct criteria for selecting strategies. The first does not imply a zero sum game while plurality maximization, in effect, interprets electoral competition as two person, and zero sum. Hence, equilibrium and optimal strategies may be sensitive to this assumption. In fact, it can be shown that, although preferences are distributed symmetrically and unimodally, vote maximizing candidates should not converge necessarily to the mean, but plurality maximizing candidates should converge. The relevant criterion for selecting strategies depends on the context of the election.

In one party dominated states, for example, the payoff of electoral competition to minor parties is patronage. Frequently, the amount of patronage is a function of the total vote claimed by the candidates of the party. Similarly, precinct and ward leaders of major as well as minor parties might maximize votes in their districts to secure their positions within the party hierarchy. However, in closely contested elections, or in elections where both candidates have some chance of winning, simple vote maximization loses its relevance. Whenever the rewards of a campaign are determined by the closeness of the vote and by who wins, a candidate's objective is to maximize his plurality.

The objective can be expressed in terms of our assumptions about individual voting behavior. First, the probability that a randomly selected citizen votes for θ is, in vector notation,

$$(3) \qquad V(\theta, \psi) = \int_S f(x)\, g(x - \theta)\, dx$$

where $S = \{ x: \phi(x - \theta) < \phi(x - \psi) \}$. If candidates maximize plurality, they should maximize the probability that a randomly selected citizen votes for θ, minus the probability that the citizen votes for ψ. This objective function is expressed,

$$(4) \qquad P(\theta, \psi) = \int_S f(x)\, g(x - \theta)\, dx - \int_{\bar{S}} f(x)\, g(x - \psi)\, dx$$

where $\bar{S} = \{ x: \phi(x - \theta) > \phi(x - \psi) \}$.

In this essay we are concerned primarily with determining necessary and sufficient conditions for equilibrium. With our assumptions about abstention, we prove, for the class of *symmetric, unimodal, multivariate* f (x), the following:

(T1) a unique pure equilibrium exists at the mean of f (x),

where by symmetry we mean f (x) = f (-x). The general proof of this statement is contained in Appendix A1. For clarification, we present here a proof of this statement when n, the number of issues, equals 1.

Without loss of generality assume that $\theta \leqslant \psi$. Since the loss functions are assumed to be symmetric, all citizens preferring policies to the left of $(\theta + \psi)/2$ prefer θ while all citizens preferring policies to the right of $(\theta + \psi)/2$ prefer ψ. Hence, equation (4) becomes,

$$(5) \qquad P(\theta, \psi) = \int_{-\infty}^{(\theta+\psi)/2} f(x)\, g(x-\theta)\, dx - \int_{(\theta+\psi)/2}^{\infty} f(x)\, g(x-\psi)\, dx$$

Consider now the rate of change of $P(\theta, \psi)$ with respect to a change in θ, and ψ constant. Differentiating equation (5) with respect to θ and integrating by parts,

$$(6) \qquad \partial P(\theta, \psi)/\partial\theta = \int_{-\infty}^{(\theta+\psi)/2} f'(x)\, g(x-\theta)\, dx$$

where $f'(x) = df(x)/dx$.

Let candidate ψ select the mean, 0, of $f(x)$ as his strategy. Since $f(x)$ is symmetric the mean equals the mode, and $f'(x)$ is positive everywhere in the range of integration of equation (6). Therefore, since $g(x-0)$ is everywhere positive, $\partial P(\theta, \psi)/\partial\theta > 0$, and candidate θ increases his plurality by moving towards ψ. Furthermore, $\lim_{\theta \to 0} P(\theta, 0) = 0$ from the symmetry of $f(x)$ and $g(x-\theta)$. Therefore, if both candidates converge to the mean, neither has any incentive to adopt an alternative strategy. Stated differently, the mean dominates all alternative strategies. Furthermore, since the candidates tie after converging to the mean, and since $f(x)$ has but a single mean, the mean is a *unique* pure equilibrium strategy.

This result, and the more general result concerning multivariate $f(x)$, emphasizes the importance of the mean on the ideological organization of electoral conflict. Davis and Hinich demonstrate the dominance of the mean when all eligible citizens vote. Our analysis proves that the mean continues to dominate the strategic considerations of the candidate when abstentions from alienation are permitted.

Previously, we assumed that $\partial\phi/\partial\theta$ is identical for all citizens. Since $\partial\phi/\partial\theta$ can be interpreted as the relative saliency of each issue (recalling that $\partial\phi/\partial\theta$ is a vector in the multidimensional case), this appears to be a highly objectionable assumption. We know that citizens do not all weight the issues in an identical fashion. However, it is easily shown that if $\partial\phi/\partial\theta$ is not identical for all citizens, the mean remains the dominant strategy under certain general conditions. If $\partial\phi/\partial\theta$ is distributed independently of $f(x)$ so that, for any specific value of $\partial\phi/\partial\theta$, the relevant density of preferences is symmetric, unimodal, and with zero expected value, the mean of $f(x)$ is the dominant strategy for the entire population. Stated

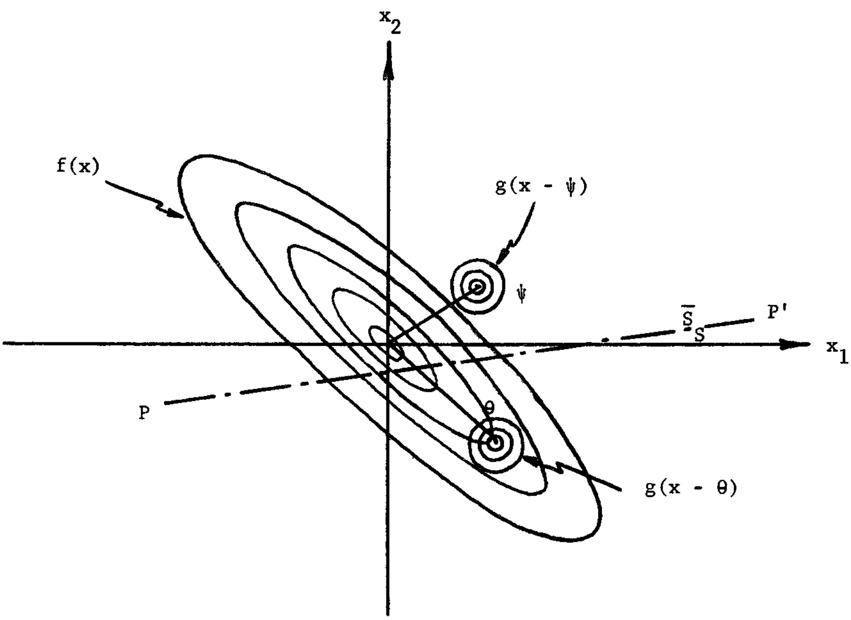


FIGURE 4

differently, if we classify citizens on the basis of the value of $\partial\phi/\partial\theta$, and if the density of preferences for each class of citizens is unimodal and symmetric, the mean of each density is an equilibrium strategy for that density. Whenever these means are identical, the common mean is the equilibrium strategy for the entire population.

When abstention from alienation is introduced into the analysis, however, one conclusion derived from the Davis and Hinich model is reversed. Specifically, it is not necessarily the case that a strategy close to the mean dominates a strategy further from the mean.[12] A simple counter example is sufficient. The contour lines for a bivariate, symmetric, unimodal density are represented graphically in Figure 4. The line PP′ bisects the space of competition into S and $\bar{S}$, with θ further from the origin, 0, than ψ (i.e., candidate θ is further from the mean than candidate ψ). Finally the contour lines of $g(x-\theta)$ and $g(x-\psi)$ are represented such that turnout falls rapidly as one moves from θ or ψ.

It is obvious from this diagram that candidate θ wins even though the mean, 0, is in $\bar{S}$. Candidate ψ cannot take advantage of his relative closeness to the mean as a consequence of the correlation between x_1 and x_2, and the sensitivity of turnout to spatial distance. Candidate ψ's dilemma is rendered more apparant if we assume that only those citizens preferring ψ or θ vote. Obviously θ wins since he is at a higher contour line. We conclude, therefore, that a strategy close to the mean dominates a strategy further from the mean, in general, if $g(x-\theta)$ and $g(x-\psi)$ equal constants (e.g., all eligable citizens vote), or if the issues are uncorrelated (i.e., if the contour lines of $f(x)$ form concentric circles).

Turning now to the general class of unimodal densities, (i.e., nonsymmetric $f(x)$), we are forced to restrict our focus. Specifically, we consider unidimentional competition only (i.e., $n = 1$). For this class of densities, however, we define necessary and sufficient conditions for equilibrium and some general results about optimal strategies. First we introduce some additional notation and definitions.

Let

(7)
$$P_o(\theta < \psi) = \partial P(\theta, \psi) / \partial \theta, \ \theta < \psi$$
$$P_o(\theta > \psi) = \partial P(\theta, \psi) / \partial \theta, \ \theta > \psi$$

We define the interval $[p_1, p_2]$ such that,

$$P_o(\theta < \psi) > 0, \qquad \text{for all } \theta < \psi \leqslant p_2$$

$$(8) \qquad P_o(\theta > \psi) < 0, \qquad \text{for all } \theta > \psi \geqslant p_1$$

and p_2 (p_1) is the maximum (minimum) value satisfying (8). This interval is known to exist for any unimodal f(x); at least the mode satisfies equation (8). Furthermore, if,

$P_o(\theta < \psi) > 0$ for all $\theta < \psi$, take $p_2 = \infty$, and if

$P_o(\theta > \psi) < 0$ for all $\theta > \psi$, take $p_1 = -\infty$.

Additionally, we define the set of points $\{x^*\}$ such that

$$(9) \qquad \int_{-\infty}^{x_o^*} f(x)\, g(x - x_o^*)\, dx = \int_{x_o^*}^{\infty} f(x)\, g(x - x_o^*)\, dx$$

where $x^*_o \in \{x^*\}$. Thus, from (9), $\lim_{\theta \to \psi} P(\theta, \psi) = 0$, if and only if $\psi \in \{x^*\}$. Stated differently, if $\psi \notin \{x^*\}$, $\lim_{\theta \to \psi} P(\theta, \psi) \neq 0$. Hence, if candidate ψ adopts a strategy in the set $\{x^*\}$, $P(\theta, \psi)$ goes to zero as θ converges to ψ, and if ψ is not in the set $\{x^*\}$, a sharp discontinuity appears in $P(\theta, \psi)$ at $\theta = \psi$. This is illustrated in Figure 5 for a symmetric density and the $g(x - \theta)$ described in Figure 2. For $\psi = 0$, the mean of f(x), $\lim_{\theta \to \psi} P(\theta, \psi) = 0$ by symmetry. However, for $\psi \neq 0$ (ψ' in our illustration), $\lim_{\theta \to \psi} P(\theta, \psi) \neq 0$ by symmetry.

Observe that, if f(x) is symmetric the median is always a point in the set $\{x^*\}$. It is easily shown that at most one point in the set $\{x^*\}$ can exist in the interval $[p_1, p_2]$. We now can prove that,

(T2) if $p_1 \leqslant x_9^* \leqslant p_2$, x_o^* is a pure strategy equilibrium.

Finally we conjecture that $\partial P(\theta, \psi)/\partial\theta$ equals zero at most once for any fixed ψ, and $\theta < \psi$ (or for any $\theta > \psi$). This conjecture is shown to be true for the case where $\epsilon = 0$ in Appendix A2. We now state the necessary condition for equilibrium for the class of *unimodal, univariate* f(x).

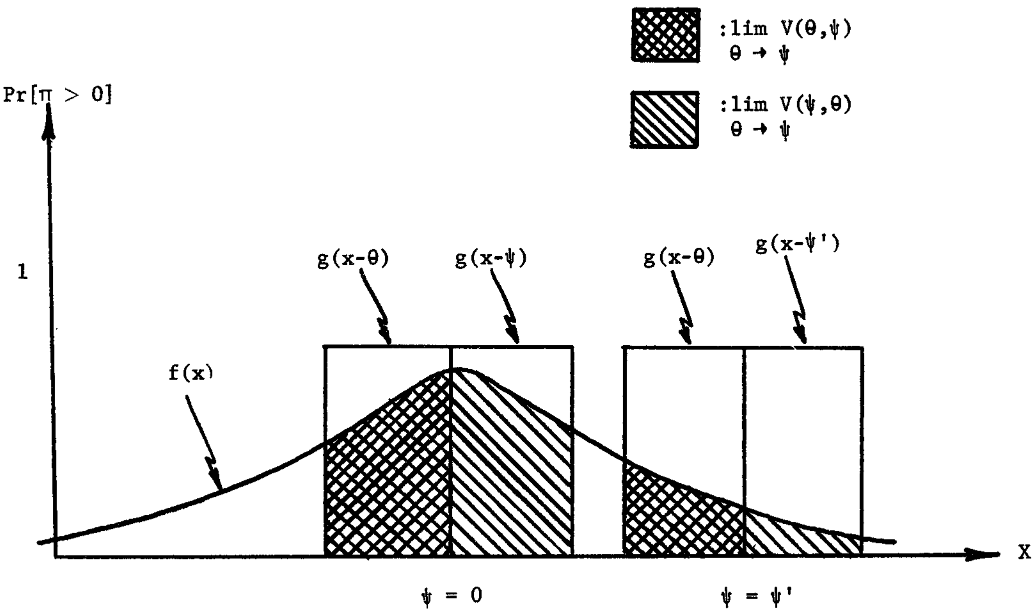


FIGURE 5

(T3) a *necessary* condition for the existence of a pure strategy equilibrium is $p_1 \leqslant x_o^* \leqslant p_2$. The equilibrium strategy is x_o^*, and is unique.

The proofs of these two theorems are presented in Appendix A3. Observe that it is possible for the set of all unimodal, univariate f (x) failing to satisfy the condition $p_1 \leqslant x_o^* < p_2$ to be a null set. Until this possibility is eliminated, however, the researcher or the campaign strategist can conduct parametric analyses by computing p_1, p_2 and x_o^* for any specific density and compare their values.

An important corollary to the previous result is the following:

(T4) if a pure equilibrium exists it is an identical point for both candidates.

Stated differently, if an equilibrium strategy exists, the candidates converge to the same strategy. This corollary follows from the definition of the interval $[p_1, p_2]$ – one and only one x_o^* can exist in this interval. Downs's assertion, therefore, that the candidates should converge if preferences are distributed unimodally appears valid. Note, however, that the set of densities for which $\{ x^* \}$ lies outside the interval $[p_1, p_2]$ may not be empty. Hence, the generality of Downs's assertion is subject to further analysis.

Assume now that one candidate, say ψ, fails to adopt any $x_o^* \epsilon \{ x^* \}$. Candidate θ may now attempt to maximize his plurality instead of minimaxing. However, we prove the following:

(T5) if $p_1 \leqslant x_o^* \leqslant p_2$, $p_1 \leqslant \psi \leqslant p_2$, $\psi \neq x_o^*$, then no maximizing strategy exists for candidate θ.

Thus, if candidate ψ adopts a policy in the interval $[p_1, p_2]$, and $\psi \neq x_o^*$, for any strategy θ there exists a strategy θ' such that $P(\theta' \psi) > P(\theta, \psi)$, i.e., θ' is better than θ. To prove this note that from the definition of $[p1, p2]$, candidate θ should converge towards ψ. However, from the definition of x_o^*, and the condition that $\psi \neq x_o^*$, $\lim P(\theta, \psi) \neq 0$, as $\theta \to \psi$. Hence, $\theta = \psi$ dominates all strategies either to the right or left of ψ (Figure 5). Without loss of generality assume $\theta = \psi$ dominates all $\theta > \psi$. Candidate θ, therefore, should adopt some strategy to the left of ψ. Assume he adopts the strategy $\theta = \psi - \delta$, where δ is some arbitrarily small positive number, such that $\psi - \delta$ dominates ψ. But $\psi - \delta/2$ dominates $\psi - \delta$, since $P_o(\theta < \psi) > 0$. In fact, for any δ, a better strategy exists between $\psi - \delta$ and ψ. Therefore, no strategy exists which maximizes plurality.

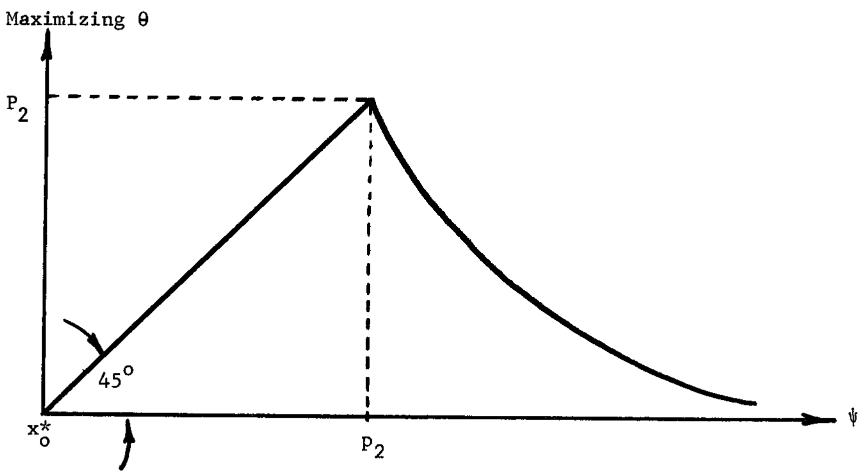


FIGURE 6*

This conclusion, of course, results from the mathematical properties of the limit of P (θ, ψ) as θ approaches ψ. Observe that if g $(x - \theta)$ equals a constant (e.g., all eligable citizens vote), $p_1 = -\infty$ and $p_2 = \infty$. In principle, therefore, if abstentions are not permitted such as in the Davis and Hinich analyses, no maximizing strategies exist. We should be cautious, however, and avoid placing too much significance on this result. In the real world, the uncertainties and inertia of the electoral process probably prohibit such fine mathematical arguments. Instead, candidates are likely to attempt to "get close to" their opponent and leave it at that.

If, however, $\psi > p_2$ or $< p_1$, maximizing strategies exist, and an interesting and important relationship between ψ and the maximizing value of of θ is observed. Specifically, we prove

(T6) if $p_1 \leqslant x_o^* \leqslant p_2$, for $\psi > p_2$, a maximizing strategy exists for candidate θ. If ψ' and ψ'' are two strategies such that $\psi' > \psi'' > p_2$, with θ' and θ'' the corresponding maximizing strategies, then $\theta' < \theta'' < p_2$.

A parallel statement can be constructed for $\psi < p_1$. While the proof of this assertion is relegated to Appendix A4 its interpretation is of interest. Briefly, for all ψ in the interval $[x_o^*, p_2]$, candidate θ should converge towards – but not identically to – ψ. As ψ increases beyond p_2, however, candidate θ can now maximize his plurality by decreasing θ from p_2. (It can be shown that in the limit as $\psi \to \infty$, candidate θ's maximizing strategy equals the mean if f (x) is symmetric.)

In Figure 6 candidate θ's maximizing strategy is plotted against ψ, for $\psi \geqslant x_o^*$, $\theta \leqslant \psi$ (this figure should be interpreted as suggestive only).[13]

This description of the behavior of the maximizing θ with respect to ψ is not without analogues in the real world. It is generally assumed, for example, that Senator Goldwater alienated many segments of the American electorate in 1964 (and his percentage of the vote suggests this). Additionally, President Johnson was not simply attempting to win, but to build a concensus and win by a landslide (i.e., maximize plurality). The combination of Goldwater's strategy and Johnson's goal explains the logic of a consensus strategy which consists of advocating x_o^*. By becoming estranged from the bulk of the electorate, Goldwater permitted Johnson to maximize his plurality by appealing to the bulk of the electorate. If Goldwater's policies had not been so radical, and if Johnson still sought to maximize his plurality, it seems safe to say that Johnson would have been drawn towards Goldwater's positions. Adoption of x_o^*, therefore, maximizes plurality either when the candidates are closely matched or when one candidate advocates extreme policies.

Consider now the effects of raising or lowering the cost of voting uniformly throughout the electorate.[14] If a citizen's cost of voting is raised the utility derived from voting decreases, and, *ceteris paribus*, his probability of voting, $g(x - \theta)$, is decreased. Hence, a citizen's probability of voting is decreased (increased) if his cost of voting is raised (lowered). In terms of the function $g(x - \theta)$ illustrated in Figure 2, δ bears an inverse relationship to the cost of voting, *ceteris paribus*. With this relationship between the cost of voting and the probability of voting we prove:

(T7) If the cost of voting either is raised or lowered uniformly throughout the electorate,

(i) the equilibrium strategy is unaffected if $f(x)$ is a symmetric unimodal density, and,

(ii) the equilibrium strategy is affected, in general, if $f(x)$ is not a symmetric density.

The proof of (i) follows directly from (T1). If $f(x)$ is symmetric and unimodal, the mean is the equilibrium point for *all* $g(x - \theta)$ satisfying our assumptions. To prove (ii) observe that the definition of x_o^* (equation 9) is a function of $f(x)$, *and* $g(x - \theta)$. It is easily seen that, in general, x_o^* varies as $g(x - \theta)$ varies for non-symmetric $f(x)$. And, from (T2) and (T3), if an equilibrium exists it is x_o^*. Hence, if $f(x)$ is not symmetric and if an equilibrium exists, this equilibrium varies as the cost of voting varies. What remains unclear at this point is whether or not the existance of an equilibrium is sensitive to the cost of voting.

Conclusions

The results reported in this paper, obviously, do not represent a complete model of electoral competition. Other than unimodal densities require analysis, and alternative causes of nonvoting must be considered (e.g., cross-pressures). Deductive theories, however, are developed incrementally, and, hopefully, our results contribute to the theory of social choice. Determining conditions under which equilibrium or dominant spatial strategies exist is equivalent to determining conditions under which individual preferences yield social welfare functions.[15] Arrow's Inpossibility Theorem states that, in general, no such function exists.[16] Similarly, Black demonstrates that, for $n > 1$, quasi-convex preferences are not sufficient for dominance.[17] Davis and Hinich prove, however, that if preferences are distributed symmetrically and if loss functions are symmetric and quadratic, a dominant strategy exists. We augment the theory by weakening their assumptions about loss functions and assume that losses are symmetric and quasi-convex. We then prove that, if abstentions are permitted, if they are assumed to be caused by alienation, and if the density of preferences is symmetric and unimodal,the mean remains the dominant strategy.

APPENDIX A1

Suppose that we restrict the domain of competition between θ and ψ to the line $\theta = c\psi$ which passes through the origin. By a rotation of the axis it is possible to transform this line onto one of the axis, say x_1. Both candidates, therefore, choose the median preferred position of the population of issues 2 through n (which have been normalized to zero for convenience).

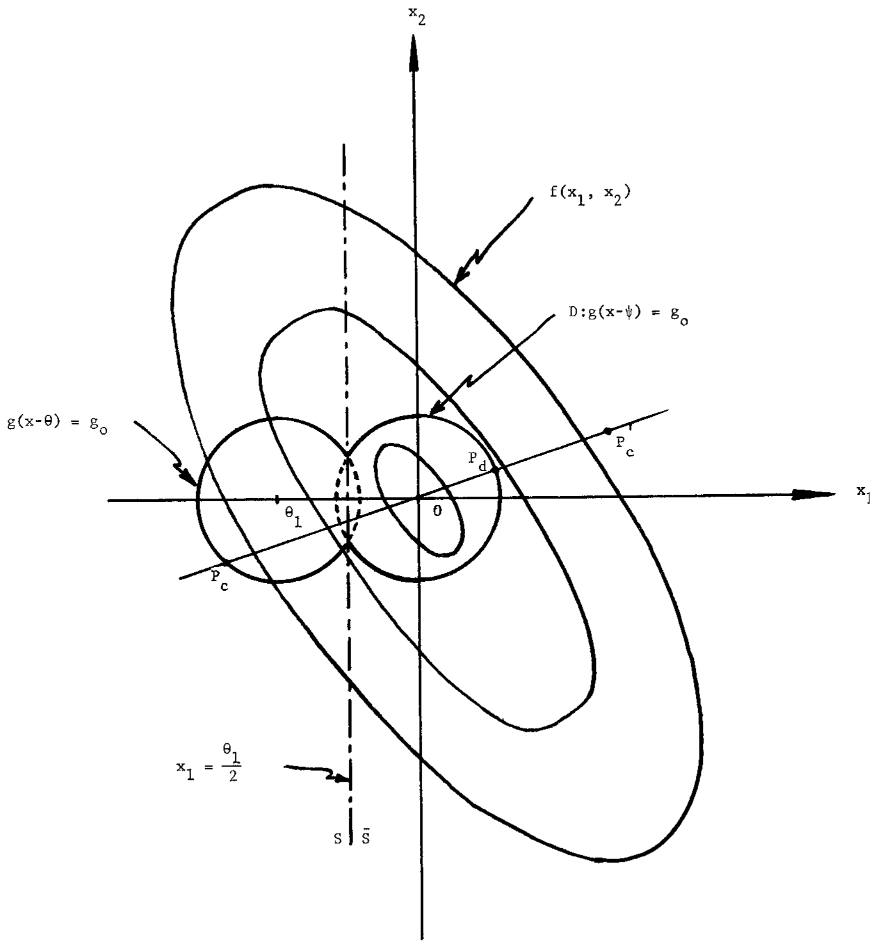
Since the loss matrix is the identity, the rotation of the axis leaves the loss function ϕ invarient, i.e., also in the new coordinate system, $\phi(x - \theta) = \phi(\sum_{i=1}^{n} (x_i - \theta_i)^2)$. Thus, since $g(x - \theta)$ is a function of ϕ, $g(x - \theta)$ depends only on $\sum_{i=1}^{n} (x_i - \theta_i)^2$, where x_i and θ_i are measured in the rotated coordinate system. Additionally, assume that ψ_1 is also equal to zero (the mean). Assuming without loss of generality that $\theta_1 < 0$, equation (4) becomes,

$$\text{(A1).}\quad P(\theta, \psi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\theta_1/2} f_o(x_1, x_2, \ldots, x_n)\, g_o(x_1 - \theta_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n$$

$$- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{\theta_1/2}^{\infty} f_o(x_1, x_2, \ldots, x_n)\, g_o(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n$$

where f_o is the density function of preferences in the new coordinate system and g_o is the respective abstention function. Obviously if f and g are symmetric, f_o and g_o are symmetric.

Simplifying discussion, consider now the bivariate case. Furthermore, consider citizens who derive an identical loss from θ, say $\phi_o(x - \theta)$, or an identical loss from ψ, say $\phi_o(x - \psi)$, where $\phi_o(x - \theta) = \phi_o(x - \psi)$. The preferences of these citizens are characterized by two circles on the $x_1\, x_2$ plane, one, say C, who's center is θ, and the other, say D, who's center is $\psi = 0$ (these preference contours are characterized by circles since A is the identity matrix). This situation is depicted in



x_2
$f(x_1, x_2)$
$D: g(x-\psi) = g_o$
$g(x-\theta) = g_o$
P_c'
P_d
x_1
θ_1
0
P_c
$x_1 = \frac{\theta_1}{2}$
S
$\bar{S}$

FIGURE 7

Figure 7, where some contour lines for f (x) are also represented. (The circles are represented as intersecting but an identical argument applies for nonintersecting preference contours.)

From the definition of our abstention function all citizens on the solid portion of these circles vote with identical probabilities, say g_o. Obviously, these are the only citizens who vote with probability g_o. Now consider the point P_c on C. If we draw a straight line from P_c through the origin 0 this line eventually intersects D at P_d. Extend this line to P'_c such that the distance P_c0 equals the distance $0P'_c$. From the unimodality and the symmetry of f (x) we know that $f(P_c) = f(P'_c)$. Furthermore, from the circularity of the abstention function contours, the distance P_c0 (or $0P'_c$) is greater than the distance $0P_d$. Finally $f(P_d) > f(P_c)$ from the unimodality and the symmetry of f (x), and the fact that P_c, P'_c, and P_d lie on a straight line passing through the origin. If $\theta \neq \psi = 0$, $f(P_d) > f(P_c)$ for any P_c and P_d lying on a line passing through 0, and P_c a point on C for citizens preferring θ and P_d a point on D for citizens preferring 0. With respect to citizens voting with probability g_o, therefore, the mean, 0, dominates all $\theta \neq 0$. Obviously this is true for any (i.e., all) g_o, $0 \leqslant g_o \leqslant 1$. Hence, as in the univariate case, the median dominate all alternative strategies, and, being unique, is a unique equilibrium strategy.

APPENDIX A2

As evidence supporting our conjecture consider, first, the two limiting cases for g: (1) g = c, a constant for all x, and (2) g = 0 for all $x \neq \psi$ or θ, and g = c, for x = ψ and θ. For case 1, with $\theta < \psi$.

$$\partial P(\theta, \psi)/\partial\theta = c \int_{-\infty}^{(\theta+\psi)/2} f'(x)\,dx = cf\left(\frac{\theta+\psi}{2}\right)$$

Since f (x) is everywhere positive, $\partial P/\partial\theta$ is always positive. For case (2) $\partial P/\partial\theta$ becomes $cf'(\theta)$. Since $f'(x) = 0$ only at the mode, $\partial P/\partial\theta$ vanishes at most once.

Now consider a rectangular abstention function such as that shown in Figure 2, namely,

$$g(x-\theta) = a, \quad \text{for all} \quad x < -\delta,\ \delta > 0$$

$$g(x-\theta) = c, \quad \text{for all} \quad \theta - \delta \leqslant x \leqslant \theta + \delta$$

$$g(x-\theta) = a, \quad \text{for all} \quad x > \theta + \delta$$

where $0 \leqslant a < c \leqslant 1$, and $2\,\delta$ is the width of $g\,(x - \theta)$.

The function $\partial P\,(\theta,\, \psi)\,/\partial\theta$ is expressed,

$$\partial P\,(\theta,\, \psi)/\partial\theta \;=\; a \int_{-\infty}^{\theta-\delta} f'(x)\,dx + c \int_{\theta-\delta}^{\theta+\delta} f'(x)\,dx + a \int_{\theta+\delta}^{\infty} f'(x)\,dx$$

$$= af\,(\theta - \delta) - c\,f\,(\theta - \delta) + c\,f\,(\theta + \delta) - a\,f\,(\theta + \delta)$$

$$\text{(A3).} \qquad = (c - a)\;[f\,(\theta + \delta) \,-\, f\,(\theta - \delta)\,]$$

whenever $\theta + \delta < (\theta+\psi)\,/\,2$. Or, whenever $\theta + \delta \geqslant (\theta + \psi)\,/\,2$,

$$\partial P\,(\theta,\, \psi)/\partial\theta \;=\; a \int_{-\infty}^{\theta-\delta} f'(x)\,dx + c \int_{\theta-\delta}^{(\theta+\psi)/2} f'(x)\,dx$$

$$\text{(A4).} \qquad = cf\,(\,(\theta + \psi)/2) - (c - a)\,f\,(\theta - \delta)$$

We need only consider θ where $(\theta - \delta) < 0$ (assume the mode of $f\,(x)$ equals 0), and $(\theta + \delta)$ (or $(\theta + \psi)\,/2) > 0$. Otherwise $\partial P/\partial\theta$ is either strictly positive or negative.

Consider first equation (A3). The sign of $\partial P/\partial\theta$ is determined by the relative magnitudes of $f\,(\theta + \delta)$ and $-\,f\,(\theta - \delta)$. The term $f\,(\theta + \delta)$ is monotonically decreasing as θ increases since $f'\,(x)$ is everywhere negative in the relevant range of $f(\theta + \delta)$. Similarly, $-\,f\,(\theta - \delta)$ is monotonically decreasing as θ increases since $f'\,(x)$ is everywhere positive in the relevant range of $f\,(\theta - \delta)$. Hence $\partial P/\partial\theta$ is monotonically decreasing as θ increases.

Similarly, for equation (A4), both $f((\theta + \psi\,/\,2)$, and $-\,f\,(\,\theta - \delta\,)$ are monotonically decreasing as θ increases. Hence, $\partial P/\partial\theta$ is positive for all $\theta + \delta < 0$ and decreasing thereafter, and can vanish at most once.

APPENDIX A3

First, it is obvious from the definition of the interval $[p_1,\, p_2]$ that such an interval exists and is unique. Now, to prove that $p_1 \leqslant x^*_o \leqslant p_2$, $x^*_o \in \{\, x^* \,\}$, is a *sufficient* condition for equilibrium assume that the condition is satisfied. If $\psi = x^*_o$ candidate θ should converge identically to ψ. This follows from the definition of $[p_1, p_2]$ and x^*_o in that if either candidate adopts a strategy in this interval his opponent should converge towards him. Furthermore, convergence should be absolute whenever this strategy is x^*_o since the limit of candidate θ's plurality approaches zero as $\theta \to \psi = x^*_o$. Stated differently, if $\psi = x^*_o$, and $\theta \neq \psi$, candidate θ's plurality is less than zero. Hence, x^*_o dominates all alternative strategies and is the equilibrium.

To prove that $p_1 \leqslant x^*_o \leqslant p_2$ is a *necessary* condition for equilibrium assume that this condition is not satisfied for any $x^*_o \in \{x^*\}$. Specifically, assume that some x^*_o exists in the interval (p_2, ∞). Obviously x^*_o dominates all strategies greater than x^*_o. But from the assumption that P_o goes to zero at most once and that $x^*_o > p_2$, P_o $(\theta < \psi = x^*_o) < 0$ for an interval to the left of x^*_o. Hence some strategy to the left of x^*_o dominates x^*_o. Assume candidate θ adopts one of these strategies so that $\theta < x^*_o$.

We now have two cases: (1) $\theta \geqslant p_1$, (2) $\theta < p_1$. For case (1), candidate ψ should converge towards θ. If, in the limit as $\psi \to \theta$, plurality goes to zero, candidate θ by the previous argument should adopt some other strategy to the left. If, however, plurality does not approach zero as $\psi \to \theta$, the candidate leapfrog back to x^*_o and the cycle resumes.

For case (2) it is obvious that candidate θ should not shift to the left so that $(\theta + \psi)/2$ is less than the mode of f (x). Otherwise, $f'(x)$ and $g(x - \theta)$ are positive everywhere in the range of integration of equation (6) and he decreases plurality by decreasing θ. For equivalent reasons, if $(\theta + \psi)/2$ is greater than the mode of f (x), candidate ψ should converge towards θ at least until $(\theta+\psi)/2$ equals the mode. Now it is easily shown that if $(\theta + \psi)/2$ equals the mode of f (x) at least one candidate has an incentive to converge some distance towards his opponent so that $(\theta + \psi)/2$ no longer equals the mode. To prove this, consider equation (6). Since $\partial P(\theta, \psi)/\partial\theta$ is positive for $(\theta + \psi)/2$ equal to the mode and decreasing thereafter, as well as continuous, it goes to zero and becomes negative for some $(\theta + \psi)/2$ greater than the mode. Hence, if the candidates are originally located symmetrically about the mode of f (x), candidate θ (and, from an equivalent argument, candidate ψ) should shift toward ψ (θ). The candidates continue this process of adjustment until $\theta = \psi \pm \delta$. If the plurality goes to zero as $\delta \to 0$, from the previous argument, some strategy to the right or left of ψ dominates ψ and the cycle resumes (if only one x^*_o satisfies equation (9) the range of intransitive strategies is to the left of x^*_o in this example). If plurality does not approach zero as $\delta \to 0$, the candidates leapfrog to some x^*_o and the cycle resumes. Hence, if x^*_o is not in the interval $[p_1, p_2]$ and $x^*_o > p_2$, no equilibrium exists. A parallel proof can be constructed for $x^*_o < p_1$.

APPENDIX A4

The proof of this statement rests on the fact that if $P_o(\theta < \psi)$ goes to zero at most once for any ψ, then $P_o(\theta < \psi)$ goes to zero as $\theta \to \psi = p_2$. To prove this assume the converse, specifically that $P_o(\theta < \psi)$ goes to zero as $\theta \to \psi$ for some $\psi > p_2$. By definition of the interval $[p_1, p_2]$, $P_o(\theta < \psi)$ must now be less than zero for some $\theta \not< \psi$ (otherwise ψ would be said to satisfy $\psi \leq p_2$). It must also be the case, however, that for some $\theta < \psi$, $P_o(\theta < \psi) > 0$, such as a θ for which $(\theta + \psi)/2$ is less than the mode of f(x). And since P_o is continuous it must equal zero for some $\theta < \psi$. Hence, it follows that P_o vanishes at least twice, in violation of our conjecture. A parallel proof can be constructed for p_1.

Now consider two ψ, $\psi' > \psi'' > p_2$. From equation (6) it is easily seen that $P_o(\theta < \psi') < P_o(\theta < \psi'')$. Stated differently, as ψ increases, P_o decreases for any fixed θ greater than the mode of f(x). Furthermore, from the previous discussion we know that $P_o(\theta = p_2 < \psi) < 0$. Hence for some smaller values of θ, say θ' and θ'', we find,

$$P_o \ (\theta' < \psi') = 0$$

$$P_o \ (\theta'' < \psi'') = 0$$

By our conjecture we know that only one such θ' and θ'' exist. Furthermore, since P_o decreases as ψ increases,

$$P_o(\theta'' < \psi') < P_o(\theta'' < \psi'') = 0.$$

Hence θ must be *reduced* from θ'' to θ' as ψ increases from ψ'' to ψ' to make P_o = 0. Parallel arguments can be developed for $\psi < p_1$. QED

FOOTNOTES

[1]Anthony Downs, AN ECONOMIC THEORY OF DEMOCRACY, (New York: Harper and Row, 1957). Also, the recent theoretical developments based on Downs's analysis include: Gordon Tullock, TOWARD A MATHEMATICS OF POLITICS, (Ann Arbor: University of Michigan Press, 1968); Otto A. Davis and Melvin J. Hinich, "A Mathematical Model of Policy Formation in a Democratic Society," in MATHEMATICAL APPLICATIONS IN POLITICAL SCIENCE II, J. L. Bernd, ed. (Dallas: Arnold Foundation Monograph XVI, Southern Methodist University Press, 1966); "Some Results Related to a Mathematical Model of Policy Formulation in a Democratic Society," in MATHEMATICAL APPLICATIONS IN POLITICAL SCIENCE III, J. J. Bernd, ed. (Charlottesville: University Press of Virginia, 1967); "On the Power and Importance of the Mean Preference in a

Mathematical Model of Democratic Choice," PUBLIC CHOICE (Fall 1968), 59-72; "Some Extensions to a Mathematical Model of Democratic Choice," forthcoming in conference volume, Social Choice Conference, University of Pittsburgh, September 1968, B. Lieberman and R. D. Luce, eds.; with P. C. Ordeshook, "An Expository Development of a Mathematical Model of the Electoral Process," AMERICAN POLITICAL SCIENCE REVIEW, (forthcoming, June 1970); David E. Chapman, "Models of Working of a Two-Party Electoral System," PAPERS ON NON-MARKET DECISION MAKING III, and PUBLIC CHOICE (Fall 1968), 19-38; Peter C. Ordeshook, THEORY OF THE ELECTORAL PROCESS (Unpublished Ph.D. dissertation, University of Rochester); Gerald Garvey, "The Theory of Party Equilibrium," AMERICAN POLITICAL SCIENCE REVIEW, LX, (1966), 29-38.

[2]IBID.

[3]IBID.

[4]The necessity for multidimensional models is noted by Philip E. Converse, "The Problem of Party Distances in Models of Voting Charge," in THE ELECTORAL PROCESS, M. Kent Jennings and L. Harmon Zeigler, eds. (Englewood Cliffs: Prentice-Hall, 1966), 175-207; Donald E. Stokes, "Spatial Models of Party Competition," AMERICAN POLITICAL SICENCE REVIEW, 57 (June, 1963), 368-377.

[5]OP. CIT., p. 52. Tullock considers vote maximizing candidates but his implication is plain for plurality maximizing candidates.

[6]OP. CIT., 368-377.

[7]Our assumption also implies that $\partial\theta/\partial\underset{\smile}{\theta}$ is identical for all citizens. Presently we accept this assumption although we weaken it considerably later.

[8]Assume, furthermore, that $\lim_{|x-\theta|\to\infty} U(x,\theta) + \epsilon < O$. This insures that $g(x-\theta)$ is integrable.

[9]William H. Riker and Peter C. Ordeshook, "A Theory of the Calculus of Voting," AMERICAN POLITICAL SCIENCE REVIEW, LXII (March, 1968), 25-42

[10]Ithiel de Sola Pool, Robert P. Abelson, and Samuel L. Popkin, CANDIDATES, ISSUES, AND STRATEGIES (Cambridge: MIT Press, 1964), pp. 74-78.

[11]For an analysis of strategies when abstentions are caused by cross-pressures see: Peter C. Ordeshook, "Extensions to a Mathematical Model of the Electoral Process and Implications for the Theory of Responsible Parties," MIDWEST JOURNAL OF POLITICAL SCIENCE, (forthcoming, February, 1970).

[12]"A Mathematical Model . . . . ," OP. CIT., pp. 184-189.

[13]Strictly speaking, of course, no maximizing θ exists for $\psi \leqslant p_2$.

[14]We require a rigorous definition of uniformly raising or lowering the cost of voting. Let the set X_i be defined so that for any $x_{i1}, x_{i2} \in X_i$, $\|x_{i1}\| = \|x_{i2}\| = a_i, 0 \leqslant a_i < \infty$ and $x' \notin X_i$, $\|x'\| \neq a_i$. Hence, X_i, X_j are disjoint if and only if $a_i \neq a_j$, and for $\phi(x-\theta) = \phi(\sum_{i=1}^{n} (x_i - \theta_i)^2)$,

$\phi(x_{i1} - \theta) = \phi(x_{i2} - \theta)$ for all θ. Finally, let C be the cost of voting, $-\infty < 0 \leqslant 0$. The costs of voting are said to be raised or lowered uniformly if, for all admissable Δ_C, $\Delta g(x_{i1} - \theta) = \Delta g(x_{i2} - \theta)$ for all i.

[15] See Davis, Hinich and Ordeshook, "An Expository Development . . . . ," OP. CIT.

[16] Kenneth Arrow, SOCIAL CHOICE AND INDIVIDUAL VALUES, (New York: Wiley, 1957).

[17] Duncan Black, THE THEORY OF COMMITTEES AND ELECTIONS, (Cambridge: Cambridge University Press, 1958); and with R. A. Newing, COMMITTEE DECISIONS WITH COMPLEMENTARY VALUATION, (London: Hodge, 1951). See also, W. H. Riker, "Voting and the Summation of Preferences, an Interpretive Bibliographical Review of Selected Developments During the Last Decade," AMERICAN POLITICAL SCIENCE REVIEW, LV (December, 1961), 900-911.