A NOTE ON CONDORCET SETS

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Suppose we have a set Ω of possible political platforms and a number N (preferably odd) of voters. The platforms will also be called *points.* If a platform P is preferred to a platform Q by a majority vote, we write P $\succ$ Q. We suppose that between any pair of points P and Q there is a relation either P $\succ$ Q or Q $\succ$ P. In order that this should be realistic, we must assume that the number of points is at most countably infinite, but for simplicity we shall go further and assume that the number is finite. When P $\succ$ Q we say also that P *preponderates* Q or Q is preponderated by P. In this note, the word *dominates* is reserved for Pareto domination: P *dominates* Q if it is unanimously preferred to Q.

Definition. A C1 set is a set of points each of which preponderates all points (in Ω) outside the set. This definition was informally suggested by Dr. C. J. Goetz to Mr. James F. Crook in discussion. I am indebted to Mr. Crook for bringing it to my attention. Dr. Goetz called it a *Condorcet set,* but I shall propose a somewhat different definition and shall prove that it always exists and is unique.

Theorem 1. A C1 set is not necessarily unique.

Proof. If Ω consists of three points P, Q, and R and if P $\succ$ Q $\succ$ R and P R, then {P} is a C1 set and so is {P, Q}. (For that matter, so is Ω in the "null" sense.) (The reader is recommended to draw his own oriented linear graphs.)

Definition. A C2 set is a C1 set S such that each point is preponderated by at least one point in S.

Theorem 2. A C2 set is not necessarily unique. (See Figure 1.)

Proof. Suppose that Ω consists of the points D, E, F, and G, and that D $\succ$ E $\succ$ F $\succ$ D and D $\succ$ G, E $\succ$ G, F $\succ$ G. Then {D, E, F and Ω are both C2 sets. If the reader does not like the use of the null sense in which Ω is a C2 set, he can add a further point H that is preponderated by all the others.

Definition. A C3 set is a C2 set S such that each point in S preponderates at least one point in S.

Definition. If all the points in a set S preponderate all the points in a set T, then we say that the set S preponderates the set T.

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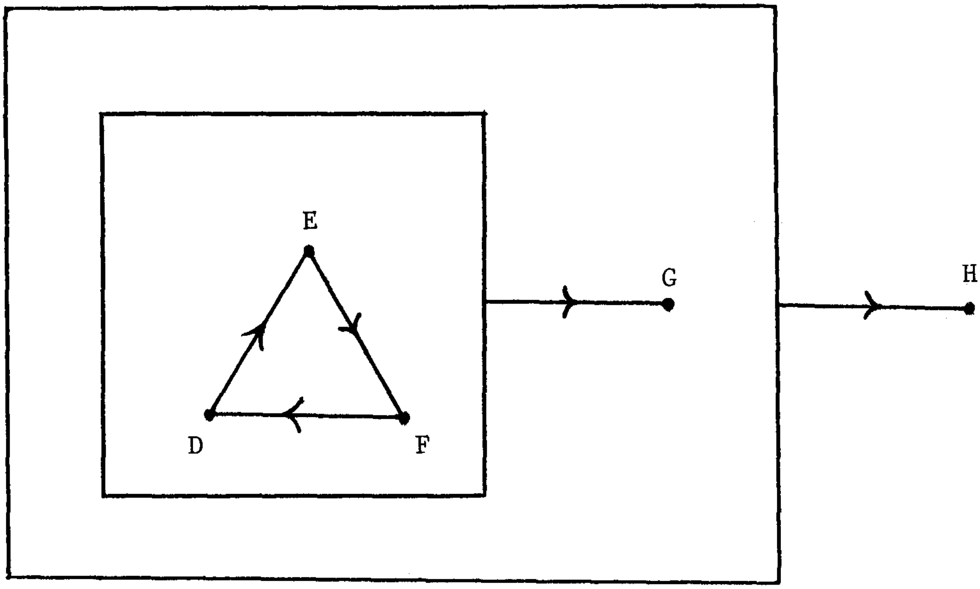
Fig. 1. Illustrating Theorem 2.



Theorem 3. A C3 set is not necessarily unique. (See Figure 2.)

Proof. Suppose that Ω consists of the six points D, E, F, D′, E′, F′ and that D $\succ$ E $\succ$ F $\succ$ D, D′ $\succ$ E′ $\succ$ F′ $\succ$ D′ and that the set {D, E, F} preponderates the set {D′, E′, F′}. Then the sets {D, E, F} and Ω are both C3 sets.

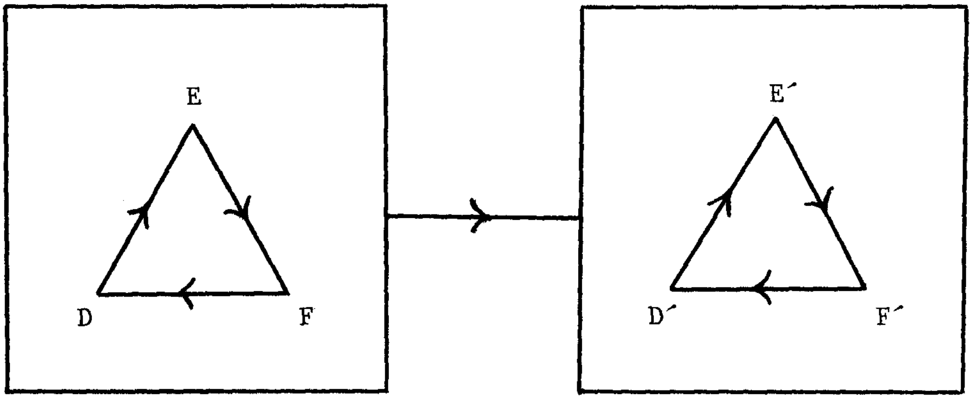
Definition. If a C1 set S contains no subset that preponderates the rest of S, then we say that S is a minimal preponderating set, a C4 set, or a Condorcet set.

Theorem 4. If there is a Condorcet set, it is unique and may therefore be called the *Condorcet set.*

Proof. Suppose there are two distinct Condorcet sets S and T, if possible. If they do not overlap, then each preponderates the other, which is impossible. If they do overlap, let the overlap be denoted by U. Then U preponderates the rest of S and the rest of T, and this contradicts the assumption that S and T are *minimal* preponderating sets. Therefore there cannot be as many as two distinct Condorcet sets.

Theorem 5. Ω has a Condorcet set (which by Theorem 4 is unique).

Fig. 2. Illustrating Theorem 3.



Proof. Let Ω be denoted also by Ω_0. We now define Ω_1, Ω_2 . . . inductively. If Ω_n is not a Condorcet set (n = 0, 1, 2, . . .), and if it consists of more than one point, then it contains a point P_n that preponderates no other point in Ω_n, and then we define Ω_{n+1} as Ω_n - P_n, that is as Ω_n with P_n removed. Since Ω is a finite set, this process must terminate in a set Ω_r which might contain only one point. (In the latter case the process terminates by definition.) Even if this terminating set consists of only one point, we shall still denote the terminating set by Ω_r. Then Ω_r is a Condorcet set, and also our construction is justified, because, for $0 \leqslant n \leqslant r$, the set Ω_n preponderates all the points that had already been removed.

Thus the definition of a Condorcet set as a minimal preponderating set has the merits of existence and uniqueness when (i) Ω is a finite set and, (ii) for each pair of points (P, Q), either P $\succ$ Q or Q $\succ$ P.

When the Condorcet set reduces to a single point, it is called the Condorcet choice. This is a term that has been in use in the Center for Study of Public Choice at Virginia Polytechnic Institute and State University, and it is of course defined as a point that is preferred by a majority to all other points; in other words, it denotes a "majority rule equilibrium."

Discussion. Black (1958, Chapter 7) and Tullock (1967, Chapter 2) give several theorems concerned with majority cycles and these may be regarded as supplementary to the present note.

On a point of terminology, the Condorcet set could naturally be called the *preponderant set* and the Pareto-optimal set the *dominant set.* The definition of a Pareto-optimal set is a set such that every point outside it is dominated by at least one point inside it, whereas no point inside it is dominated by any other point in it (Buchanan, 1968, p. 112). In order to prove the uniqueness of the Pareto-optimal or dominant set, it is not necessary to specify minimality because it is automatic in virtue of dominance's being a transitive relation, unlike preponderance.

Note that a continuous infinity of degrees of preponderance can be interposed between preponderance and dominance: We could say that a point A *p-preponderates* B, or $AB \geqslant p$, or $A \succ_p B$, if A is preferred to B by at least a proportion p of the population. Then (Pareto) dominance is 1-preponderance whereas preponderance is $(N/2 + 1/2)$-preponderance. The p-preponderant set (see below) is always contained in the q-preponderant set if $p < q$. It is only for $p = 1$ that p-preponderance is transitive.

The symbol AB could be interpreted as the largest number x for which $AB \geqslant x$. It can be proved that

$$AC \leqslant AB + BC,$$

which is a kind of triangle inequality. (The proof of this left as an exercise for the reader.) Theorems 1 to 5, and their proofs, are all valid for p-preponderance when $1/2 < p \leqslant 1$, and this justifies the mention above of the p-preponderant set. For $p = 1$, an easier proof is possible because of transitivity of dominance. This assumes of course that each voter's preferences are individually transitive. It can be argued that they might not be because "a neuron is an animal that lives in the head," that is, a person's brain can be regarded as a collection of small individuals, and his preferences might be made by a kind of democratic voting procedure by these small individuals – neurons or perhaps "subassemblies" (Good, 1965). But when a person's attention is brought to an example of intransitivity in his own preference judgments, it seems reasonable to suppose that he would admit he had made a mistake and that he would try to adjust his preferences to eliminate such things. (Compare Savage, 1954, p. 21; Tullock, 1964.) McCulloch (1945) suggested a simple neuro-physiological mechanism that could correspond to a cycle of intransitivity, but did not suggest the analogy of the mind with a democratic process, at least in this context. Note that a person with intransitive preferences can in some circumstances be used as a "money-pump," an expression used I believe by L. J. Savage.

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