Simply Transitive Groups of Motions.

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This paper deals with Simply Transitive Groups of Motions of Riemannian spaces of any order which admits an orthogonal system of hypersurfaces. The various normal forms of the fundamental quadratic differential form of the spaces possessing these properties are determined and the expressions for the vectors of the infinitesimal generators of the corresponding group.

1. Consider a simply transitive group G_n in n variables x^i, the vectors of the group being denoted by ξ^i_α, where α indicates the vector and i the component. Quantities ξ^α_i are uniquely determined by

$$(1.1) \qquad \xi^\alpha_i \xi^j_\alpha = \delta^j_i, \quad \xi^\alpha_i \xi^i_\beta = \delta^\alpha_\beta \text{ [1]}.$$

We define functions Λ^i_{jk} by

$$(1.2) \qquad \Lambda^i_{jk} = \xi^i_\alpha \frac{\partial \xi^\alpha_k}{\partial x^j} = -\xi^\alpha_k \frac{\partial \xi^i_\alpha}{\partial x^j},$$

from which we have

$$(1.3) \qquad \frac{\partial \xi^i_\alpha}{\partial x^j} + \xi^k_\alpha \Lambda^i_{jk} = 0, \quad \frac{\partial \xi^\alpha_i}{\partial x^j} - \xi^\alpha_k \Lambda^k_{ji} = 0.$$

If g_{ij} are the components of the fundamental tensor of a space V_n, a necessary and sufficient condition that G_n be a group of motions is that the following equations of Killing be satisfied:

$$(1.4) \qquad \xi^k_\alpha \frac{\partial g_{ij}}{\partial x^k} + g_{ih} \frac{\partial \xi^h_\alpha}{\partial x^j} + g_{jh} \frac{\partial \xi^h_\alpha}{\partial x^i} = 0 \text{ [2]}.$$

By means of (1.3) these may be put in the equivalent form

$$(1.5) \qquad \frac{\partial g_{ij}}{\partial x^k} - g_{ih} \Lambda^h_{jk} - g_{jh} \Lambda^h_{ik} = 0.$$

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[1] The summation convention is used throughout this paper.

[2] C. G., p. 217; a reference of this kind is to the author's Continuous Groups of Transformations.

We denote by ζ^i_α the vectors of the group Γ_n reciprocal to G_n; they satisfy the completely integrable system of differential equations[3])

(1.6) $$\frac{\partial \zeta^i_\alpha}{\partial x^j} + \zeta^h_\alpha \Lambda^i_{hj} = 0.$$

From (1.5) and (1.6) we have

(1.7) $$\frac{\partial}{\partial x^k}(g_{ij}\zeta^i_\alpha \zeta^j_\beta) = 0.$$

Hence if we choose the initial values of a set of solutions of (1.6) to satisfy the conditions

(1.8) $$g_{ij}\zeta^i_\alpha \zeta^j_\beta = 0 \quad (\alpha \neq \beta), \quad g_{ij}\zeta^i_\alpha \zeta^j_\alpha = e_\alpha,$$

where the e's are $+1$ or -1 according to the signature of the fundamental form of V_n, equations (1.8) hold for all values of the x's. Since any solution of (1.6) is a linear combination with constant coefficients of the set, we have

When a V_n admits a simply transitive group G_n, the basis of the reciprocal group can be chosen so its vectors form an orthogonal ennuple of unit vectors.

Since the ζ's are the vectors of a group, we have

(1.9) $$\zeta^i_\alpha \frac{\partial \zeta^j_\beta}{\partial x^i} - \zeta^i_\beta \frac{\partial \zeta^j_\alpha}{\partial x^i} = \bar{c}_{\alpha\beta}^{\ \varepsilon} \zeta^j_\varepsilon,$$

which may be written

(1.10) $$\zeta^i_\alpha \zeta^j_{\beta,\, i} - \zeta^i_\beta \zeta^j_{\alpha,\, i} = \bar{c}_{\alpha\beta}^{\ \varepsilon} \zeta^j_\varepsilon,$$

where a comma followed by an index indicates covariant differentiation with respect to the g's.

If we put

(1.11) $$\zeta^\alpha_j = g_{ij}\zeta^i_\alpha,$$

then

(1.12) $$\zeta^j_\alpha \zeta^\alpha_j = e_\alpha \quad (\alpha \text{ not summed}), \quad \zeta^j_\alpha \zeta^\beta_j = \delta^\beta_\alpha.$$

If by definition

(1.13) $$\gamma_{\alpha\beta\delta} = \zeta^i_{\alpha,j}\zeta^\beta_i \zeta^j_\delta, \quad \gamma_{\alpha\beta\delta} + \gamma_{\beta\alpha\delta} = 0$$ [4])

it follows from (1.10) that

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[3]) C. G., p. 113.

[4]) Cf. R. G., p. 97; a reference of this kind is the author's Riemannian Geometry.

$$(1.14)\qquad \gamma_{\delta\alpha\beta}-\gamma_{\delta\beta\alpha}=e_\delta\,\bar{c}_{\alpha\beta}^{\;\delta}\qquad (\delta \text{ not summed}).$$

From this result it follows that [5])

A necessary and sufficient condition that the vectors ζ^i_α *be normal is that the constants of structure* $\bar{c}_{\alpha\beta}^{\;\gamma}$ *for* α, β, γ *different be zero; in this case the* ζ*'s are the normals to the hypersurfaces of an* n*-tuply orthogonal system of hypersurfaces in the* V_n *admitting the* G_n *of vectors* ξ^i_α *as a group of motions.*

If we put

$$(1.15)\qquad c_{\alpha\beta}\equiv\bar{c}_{\alpha\beta}^{\;\alpha}\qquad (\alpha \text{ not summed}),$$

it follows from the Jacobi relations [6])

$$(1.16)\qquad \bar{c}_{\alpha\beta}^{\;\varepsilon}\,\bar{c}_{\varepsilon\gamma}^{\;\delta}+\bar{c}_{\beta\gamma}^{\;\varepsilon}\,\bar{c}_{\varepsilon\alpha}^{\;\delta}+\bar{c}_{\gamma\alpha}^{\;\varepsilon}\,\bar{c}_{\varepsilon\beta}^{\;\delta}=0$$

that when the conditions of the above theorem are satisfied we must have

$$(1.17)\qquad c_{\gamma\alpha}\,c_{\alpha\beta}-c_{\gamma\beta}\,c_{\beta\alpha}=0.$$

When the V_n is referred to the orthogonal system of hypersurfaces to which the vectors ζ^i_α are normal and we put

$$(1.18)\qquad g_{ii}=e_i H_i^2,\quad g_{ij}=0\qquad (i\neq j),$$

we have

$$(1.19)\qquad \zeta^i_i=\frac{1}{H_i},\quad \zeta^i_\alpha=0\qquad (i\neq\alpha).$$

In this coordinate system equations (1.9) reduce to

$$(1.20)\qquad \frac{\partial\log H_i}{\partial x^j}=c_{ij}H_j\qquad (i\neq j).$$

For a general coordinate system we have from (1.3) and (1.6) that each of the vectors ξ^i_α satisfies

$$(1.21)\qquad \xi^j\frac{\partial\zeta^i_\alpha}{\partial x^j}-\zeta^j_\alpha\frac{\partial\xi^i}{\partial x^j}=0.$$

For the case under consideration and the particular coordinate system for which (1.19) holds, it follows from (1.21) that

$$(1.22)\qquad \xi^i_\alpha=X^i_\alpha,$$

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[5]) Cf. R. G., p. 117.

[6]) C. G., p. 26.

where X^i_α are functions of x^i alone, and

$$\frac{d X^i_\alpha}{d x^i} + \frac{\partial \log H_i}{\partial x^j} X^j_\alpha = 0 \qquad (i \text{ not summed}). \tag{1.23}$$

If we differentiate these equations with respect to x^j $(j \neq i)$, we find that the resulting equations are satisfied in consequence of (1.17).

We shall show that the above results apply to any V_n which admits a simply transitive group of motions and an n-tuply orthogonal system of hypersurfaces. In fact, when the latter system is parametric, equations (1.5) for $i = j$ reduce by (1.18) to

$$\frac{\partial H_i}{\partial x^k} - H_i \Lambda^i_{ik} = 0 \qquad (i \text{ not summed}).$$

Consequently equations (1.6) admit the n independent solutions (1.19) and these are the vectors of the reciprocal group. Accordingly we have (1.22) and from (1.2) we find that $\Lambda^i_{jk} = 0$ for $i \neq j$, so that (1.5) are satisfied when $g_{ij} = 0$ for $i \neq j$. Hence the solutions of (1.20) and (1.23) which are obtained in the following sections constitute the most general types of a V_n admitting a simply transitive group of motions and an n-tuply orthogonal system of hypersurfaces.

2. For a V_2 we have from (1.20) the two equations

$$\frac{\partial \log H_1}{\partial x^2} = c_{12} H_2, \quad \frac{\partial \log H_2}{\partial x^1} = c_{21} H_1. \tag{2.1}$$

If $H_1 = \text{const.}$, we have by a suitable choice of coordinates, the two possible quadratic forms

$$e_1 (d x^1)^2 + e_2 (d x^2)^2 \tag{2.2}$$

$$e_1 (d x^1)^2 + e_2 e^{2 a x_1} (d x^2)^2, \tag{2.3}$$

where a is a constant, and from (1.22) and (1.23) by a suitable choice of basis the respective matrices of the vectors ξ^i_α

$$\left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\| \qquad \left\| \begin{matrix} 1 & e^{-a x^2} \\ 0 & 1 \end{matrix} \right\|. \tag{2.4}$$

When neither H_1 nor H_2 is a constant, we have from (2.1)

$$\frac{\partial^2 \log H_1}{\partial x^1 \partial x^2} = c_{21} c_{12} H_1 H_2 = c_{21} \frac{\partial H_1}{\partial x^2}. \tag{2.5}$$

Integrating with respect to x^2 and taking for a new x^1 a suitable function of x^1, we obtain

$$\frac{\partial \log H_1}{\partial x^1} = c_{21} H_1,$$

from which and (2.1) we have the form

$$\frac{1}{(x^1 - x^2)^2}\left(\frac{e_1 (dx^1)^2}{c_{21}^2} + \frac{e_2 (dx^2)^2}{c_{12}^2}\right) \tag{2.6}$$

and the matrix of the vectors $\xi_\alpha^{\,i}$ is

$$\begin{vmatrix} 1 & 1 \\ x^1 & x^2 \end{vmatrix} \tag{2.7}$$

The Gaussian curvature of (2.3) is $-e_1 a^2$ and of (2.6) it is $-(e_1 c_{21}^2 + e_{12}^2)$, so that the curvature is constant, which is negative if the fundamental form is positive definite[7]).

3. In this and the next section we understand that $n > 2$ and in this section we consider the case when one of the H's, say H_2 does not involve one of the variables other than x^2, say x^1; then from (1.20) we have

$$c_{21} = 0. \tag{3.1}$$

From the equation

$$c_{l2}\, c_{21} - c_{l1}\, c_{12} = 0$$

obtained from (1.17) it follows that

$$c_{l1} = 0 \qquad (l = 3, \ldots, n) \quad \text{or} \quad c_{12} = 0. \tag{3.2}$$

When the first of these conditions is satisfied, we have from (1.20)

$$\frac{\partial^2 \log H_1}{\partial x^m\, \partial x^1} = c_{m1} \frac{\partial H_1}{\partial x^m} = 0 \qquad (m = 2, \ldots, n).$$

Consequently $H_1 = X_1\, \varphi_1 (x^2, \ldots, x^n)$, where X_1 is a function of x^1 alone; by a suitable choice of a new x^1 as a function of x^1 we have $X_1 = 1$ in the new coordinate system, in consequence of which and the first of (3.2) all the H's are independent of x^1 and a solution of (1.23) is

$$X_1^i = \delta_1^i. \tag{3.3}$$

Conversely, if equations (1.23) admit a solution involving only one non-vanishing component, say X_1^1, the coordinate x^1 can be chosen

[7]) Cf. C. G., p. 228.

so that we have (3.3) and then from (1.23) it follows that all the H's are independent of x^1. Under these conditions by a suitable choice of basis of the group, we have

$$(3.4) \qquad X^1_\beta = a_\beta x^1, \quad X^j_\beta \frac{\partial \log H_1}{\partial x^j} = -a_\beta \qquad (\beta, j = 2, \ldots, n),$$

where the a's are constants.

If H_1 is a constant, the a's are zero, and for $n > 2$ we have for the matrix of the vectors

$$(3.5) \qquad \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & M \\ \hline \end{array},$$

where M is determined for the V_{n-1} with the fundamental form $g_{ij}\, dx^i\, dx^j\, (i, j = 2, \ldots, n)$ by the various methods we are applying to a V_n.

When H_1 is not constant, by a suitable renumbering of the coordinates, if necessary, we have that H_1 is a function of $x^2, \ldots, x^p$ $(p \leqq n)$. From (1.20) we have

$$(3.6) \qquad \frac{\partial \log H_1}{\partial x^a} = c_a H_a, \qquad c_a \equiv c_{1a} \qquad (a = 2, \ldots, p),$$

from which it follows that the numbers c_a are all different from zero, and any H_a is a function of $x^2, \ldots, x^p$ with the possible exception of x^a.

Expressing the conditions ot integrability of (1.20), we have in particular

$$(3.7) \qquad c_{ia} \frac{\partial H_a}{\partial x^c} - c_{ic} \frac{\partial H_c}{\partial x^a} = 0 \qquad \begin{pmatrix} i = 1, \ldots, p; \\ a, c = 2, \ldots, p; a \neq c \end{pmatrix},$$

from which it follows that

$$c_{ba} = t_b c_a \qquad (a, b = 2, \ldots, p; \ a \neq b).$$

From these equations and (1.17) in which $\gamma = 1$, we find that all the numbers t_b are equal, so that we write

$$(3.8) \qquad c_{ba} = t c_a, \quad t \neq 0 \qquad (a, b = 2, \ldots, p; \ a \neq b).$$

Consequently we have from (1.20), (3.6) and (3.8)

$$(3.9) \qquad \frac{\partial \log H_b}{\partial x^a} = t\, c_a H_a = t \frac{\partial \log H_1}{\partial x^a},$$

from which for a given b and $a=2,\ldots,p$ $(a\neq b)$, we have by a suitable choice of x^b

$$H_b = H_1^t \qquad (b=2,\ldots,p). \tag{3.10}$$

From this result and (3.9) we obtain

$$\frac{1}{H_1^t} = -t\, c_a x^a \qquad (a=2,\ldots,n), \tag{3.11}$$

the possible additive constant being removed by a suitable choice of one of the x's.

From (3.4) and (3.9) we have

$$X_\beta^1 = a_\beta x^1,\ X_\beta^b c_b H_1^t = -a_\beta \qquad (b,\ \beta=2,\ldots,n), \tag{3.12}$$

in consequence of which we have from (1.23)

$$(X_\beta^b)' = a_\beta t. \tag{3.13}$$

By a suitable choice of basis we have $a_2=1$, $a_s=0$ $(s>2)$, and the solutions of (3.12) and (3.13) are

$$\begin{aligned} &X_2^1 = x^1,\ X_2^a = t x^a + d_2^a, && c_a d_2^a = 0 \\ &X_s^1 = 0,\ \ X_s^a = d_s^a, && c_a d_s^a = 0 \quad (a=2,\ldots,p;\ s>p), \end{aligned} \tag{3.14}$$

When $p=n$, the matrix of the vectors of the group is

$$\left\| \begin{matrix} 1 & 0 & \ldots\ldots & 0 \\ x^1 & t x^2 & \ldots\ldots & t x^n \\ 0 & d_3^2 & \ldots\ldots & d_3^n \\ \ldots & \ldots & \ldots\ldots & \ldots \\ \ldots & \ldots & \ldots\ldots & \ldots \\ 0 & d_n^2 & \ldots\ldots & d_n^n \end{matrix} \right\| \qquad c_a d_b^a = 0 \begin{pmatrix} a=2,\ldots,n; \\ b=3,\ldots,n \end{pmatrix}. \tag{3.15}$$

When $p<n$ from the equations

$$c_{sa} c_{ab} - c_{sb} c_{ba} = 0 \quad (a,\ b=2,\ldots,p;\ s>p), \tag{3.16}$$

we see that any H_s involves all or none of the coordinates $x^2,\ldots,x^p$. If none of them involve these coordinates, the fundamental form consists of two distinct parts

$$\begin{gathered} e_1 H_1^2 (d x^1)^2 + \ldots + e_p H_p^2 (d x^p)^2, \\ e_{p+1} H_{p+1}^2 (d x^{p+1})^2 + \ldots + e_n H_n^2 (d x^n)^2, \end{gathered} \tag{3.17}$$

and the matrix of the ξ's is

$$\begin{array}{|c|c|} \hline M_1 & 0 \\ \hline 0 & M_2 \\ \hline \end{array} \tag{3.18}$$

where M_1 is of the form (3.15) and M_2 is any possible matrix for the second part (3.17) of the fundamental form.

We consider next the case when every H_s for $s > p$ involves $x^2, \ldots, x^p$. From the equations

$$c_{s\alpha} c_{\alpha t} - c_{st} c_{t\alpha} = 0 \qquad (s,\ t = p+1, \ldots, n)$$

it follows that none of the H's involve $x^{p+1}, \ldots, x^n$. From (3.16) and (3.8) we have

$$c_{s\alpha} = t_s c_\alpha,$$

and consequently from (1.20)

$$\frac{\partial \log H_s}{\partial x^\alpha} = t_s c_\alpha H_\alpha = t_s \frac{\partial \log H_1}{\partial x^\alpha},$$

so that

$$H_s = H_1^{t_s}. \tag{3.19}$$

Then from (1.17), (3.9), (3.19), (3.10) and (3.12) we obtain

$$(X_\beta^s)' = a_\beta t_s,$$

and the matrix is by a suitable choice of basis

$$\left|\begin{array}{cccccc|cccc} 1 & 0 & 0 & \ldots & . & 0 & 0 & & 0 \ldots & 0 \\ x^1 & tx^2 & tx^3 & \ldots & . & tx^p & t_{p+1}x^{p+1} & & t_{p+2}x^{p+2} . & t_n x^n \\ 0 & c_3 & -c_2 & 0 & \ldots & 0 & 0 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & c_p & 0 & \ldots & 0 & -c_2 & 0 & \ldots & \ldots & 0 \\ \hline & & & & & & 1 & 0 & \ldots & 0 \\ & & & & & & 0 & 1 & \ldots & 0 \\ & & & 0 & & & \ldots & \ldots & \ldots & \ldots \\ & & & & & & \ldots & \ldots & \ldots & \ldots \\ & & & & & & \ldots & \ldots & \ldots & \ldots \\ & & & & & & 0 & \ldots & \ldots & 1 \end{array}\right| \tag{3.20}$$

If only one of the H's, say H_n, does not involve $x^2, \ldots, x^p$ from the equation

$$(3.21) \qquad c_{us}\, c_{sa} - c_{ua}\, c_{as} = 0$$

for $u=n$, $s=p+1, \ldots, n-1$ it follows that H_n is a function of x^n alone and this leads to a matrix analogous to (3.5), where M is of the form (3.20) of order $n-1$.

If certain of the H's do not involve $x^2, \ldots, x^p$ say H_u for $u=r+1, \ldots, n$ (by a suitable renumbering), then from (3.21) for $u=r+1, \ldots, n$ and $s=p+1, \ldots, r$, it follows that H_u are independent of $x^{p+1}, \ldots, x^r$ and we have a matrix (3.18), where M_1 is of the form (3.20) and M_2 is any possible matrix for $\sum_u e_u H_u^2 (dx^u)^2$.

The case when $c_{us}=c_{su}=0$ is of the type not yet fully discussed, that is, when (3.1) and the second of (3.2) are satisfied. We consider this case now and observe that H_1 does not involve x^2 and H_2 not x^1. If H_1 involves only x^1, we have the case (3.5). Consequently we assume that H_1 involves $x^3, \ldots, x^p$, by suitable numbering of the coordinates, so that we have equations (3.6) for $a=3, \ldots, p$. Also H_1 must involve x^1; otherwise the H_a given by (3.6) do not involve x^1 and then from

$$(3.22) \qquad c_{s1}\, c_{1a} - c_{sa}\, c_{a1} = 0$$

we have $c_{s1}=0$, so that all the H's do not involve x^1, and this is the case previously considered. From

$$c_{12}\, c_{2a} - c_{1a}\, c_{a2} = 0$$

we have $c_{a2}=0$. Hence $p<n$, otherwise some of the H's involve x^2, which is the case previously considered with the roles of 1 and 2 interchanged.

From (3.22) for $s=2$ we have $c_{2a}=0$ and consequently H_2 does not involve $x^1, x^3, \ldots, x^p$. Since $p<n$, we have $c_{1s}=0$ for $s>p$. If $c_{s1}\neq 0$ for any s, we have the case (3.1) and the first of (3.2) with s and 1 in place of 1 and 2 respectively. Hence $c_{s1}=0$ for $s>p$ and from (3.22) we have $c_{sa}=0$, that is H_s for $s>p$ are independent of $x^1, x^3, \ldots, x^p$. Hence the fundamental form consists of the two distinct parts

$$\sum_l e_l H_l^2 (dx^l)^2, \quad \sum_t e_t H_t^2 (dx^t)^2 \quad (l=1, 3, \ldots, p;\ t=2, p+1, \ldots, n)$$

such that the coefficients of either part involve only the variables of that part. For the first of these all of the c's are different from zero, and for the second no conditions have been established. Hence when in the next section we consider the case where all the c's are different from zero, this result and the consequences of (3.1) and the first of conditions (3.2) are the only possible types. Consequently for any V_n we have one of these types, or a combination of them, each applying to an isolated part of the fundamental form. In the latter case the group for V_n is the direct product of the groups for these parts as follows from (1.23).

4. In this section we consider the case when each of the H's involves all the coordinates, that is none of the constants $c_{\alpha\beta}$ is zero. From (2.5) we have in this case

$$\text{(4.1)} \qquad \frac{\partial \log H_1}{\partial x^1} = c_{21} H_1 + \psi, \qquad \frac{\partial \psi}{\partial x^2} = 0.$$

Differentiating this equation with respect to x^l for $l > 2$ and making use of the equation obtained from (2.5) on replacing 2 by l, we have

$$\text{(4.2)} \qquad (c_{l1} - c_{21}) \frac{\partial H_1}{\partial x^l} = \frac{\partial \psi}{\partial x^l},$$

and consequently

$$(c_{l1} - c_{21}) \frac{\partial^2 H_1}{\partial x^2 \partial x^l} = 0.$$

From this equation and (4.2) it follows that if ψ contains x^l then

$$\frac{\partial^2 H_1}{\partial x^2 \partial x^l} = 0,$$

from which and (1.20) we have

$$\text{(4.3)} \qquad c_{1l} + c_{2l} = 0,$$

and from these equations and (1.17) we obtain

$$\text{(4.4)} \qquad c_{l1} + c_{21} = 0, \quad c_{l2} + c_{12} = 0.$$

From the first of these equations and (4.2) we obtain

$$\text{(4.5)} \qquad -2 c_{21} H_1 = \psi + \theta,$$

where θ involves the x's other than x^1 not involved in ψ. When this expression is substituted in (4.1), we obtain

$$\text{(4.6)} \qquad \frac{\partial \psi}{\partial x^1} - \frac{1}{2}\psi^2 = -\left(\frac{\partial \theta}{\partial x^1} + \frac{1}{2}\theta^2\right) = \frac{\varphi}{2}.$$

From the above statement about θ and the form of this equation it follows that φ is at most a function of x^1.

From (4.5) and (1.20) we have

$$c_{1l} H_l = \frac{\frac{\partial \psi}{\partial x^l}}{\psi + \Theta}. \tag{4.7}$$

Substituting this expression and the similar one for H_m in

$$\frac{\partial \log H_l}{\partial x^m} = c_{lm} H_m,$$

we obtain

$$\frac{\partial}{\partial x^m} \log \frac{\partial \psi}{\partial x^l} = 2 c_{1m} \frac{\frac{\partial \psi}{\partial x^m}}{\psi + \Theta} \qquad (l \neq m),$$

since from equations analogous to (4.4) and from (1.17) we find that $c_{lm} = c_{1m}$. Since ψ does not contain x^2 and Θ does, the above equation is not possible and consequently ψ can involve at most one x other than x^1, say x^3. Accordingly Θ involves x^1, x^2 and x^s for $s > 3$, so that from (4.2) and (1.17) it follows that

$$c_{s1} = c_{21}, \quad c_{s2} = c_{12}, \quad c_{2s} = c_{1s} \qquad (s = 4, \ldots, n). \tag{4.8}$$

In consequence of (4.2), (4.3) and (4.8) we have that the equations

$$c_{s3} c_{31} - c_{s1} c_{13} = 0, \quad c_{s3} c_{32} - c_{s2} c_{23} = 0$$

are reducible to

$$-c_{21}(c_{s3} - c_{23}) = 0, \quad -c_{12}(c_{s3} + c_{23}) = 0,$$

which are evidently inconsistent with the assumption that ψ involves x^3 and Θ involves an x other than x^1 and x^2. Hence if ψ involves an x other than x^1, then $n = 3$; otherwise ψ is at most a function of x^1.

We consider first the case when ψ is a function of x^1 and x^3 and $n = 3$, and we begin by assuming that equations (1.23) admit a solution such that one of the components is zero; the case when two are zero was considered in § 3. By a suitable choice of the coordinates without changing the coordinate hypersurfaces, we have the three possible cases

$$1, 1, 0; \quad 1, 0, 1; \quad 0, 1, 1. \tag{4.9}$$

For the first case we have from (1.23)

$$\frac{\partial H_i}{\partial x^1} + \frac{\partial H_i}{\partial x^2} = 0 \qquad (i = 1, 2, 3).$$

For $i=1$ we have from (4.5)

$$\frac{\partial\psi}{\partial x^1}+\frac{\partial\Theta}{\partial x^1}+\frac{\partial\Theta}{\partial x^2}=0,$$

from which and (4.6) we have

$$\psi^2=\Theta^2+2\frac{\partial\Theta}{\partial x^2}.$$

This equation is possible only when ψ does not involve x^3 contrary to the hypothesis. Similar results follow from the other two cases in (4.9). Consequently, if there exists a solution none of the components are zero and by a suitable choice of the x's we have as one solution 1, 1, 1 so that from (1.23) we must have

$$\frac{\partial H_i}{\partial x^1}+\frac{\partial H_i}{\partial x^2}+\frac{\partial H_i}{\partial x^3}=0 \qquad (i=1,\ 2,\ 3). \tag{4.10}$$

For $i=1$ this condition is from (4.5)

$$\frac{\partial\psi}{\partial x^1}+\frac{\partial\psi}{\partial x^3}=-\left(\frac{\partial\Theta}{\partial x^1}+\frac{\partial\Theta}{\partial x^2}\right),$$

and consequently

$$\frac{\partial\psi}{\partial x^1}+\frac{\partial\psi}{\partial x^3}=f(x^1),\quad \frac{\partial\Theta}{\partial x^1}+\frac{\partial\Theta}{\partial x^2}=-f(x^1).$$

Expressing the consistency of these equations and (4.6), we find that $f=0$ and $\varphi=$const. Consequently ψ is a function of x^3-x^1 and Θ. of x^2-x^1, and we have from (1.20)

$$c_{12}H_2=\frac{\Theta'}{\psi+\Theta},\quad c_{13}H_3=\frac{\psi'}{\psi+\Theta}, \tag{4.11}$$

where the prime indicates differentiation with respect to the argument These expressions satisfy (4.10) for $i=2,\ 3$ identically. When they are substituted in (1.23), we obtain

$$\frac{dX^1_\alpha}{dx^1}=A,\quad \frac{dX^2_\alpha}{dx^2}-\frac{\Theta''}{\Theta'}(X^1_\alpha-X^2_\alpha)+A=0, \tag{4.12}$$

$$\frac{dX^3_\alpha}{dx^3}-\frac{\psi''}{\psi'}(X^1_\alpha-X^3_\alpha)+A=0,$$

where

$$A=\frac{1}{\psi+\Theta}\left[(\psi'+\Theta')X^1_\alpha-\Theta'X^2_\alpha-\psi'X^3_\alpha\right].$$

From (4.6) we have

$$\psi'' = -\psi\psi', \quad \theta'' = \theta\theta'. \tag{4.13}$$

Adding the first two of equations (4.12), we have

$$\frac{1}{\theta}[(X^1_\alpha)' + (X^2_\alpha)'] - X^1_\alpha + X^2_\alpha = 0. \tag{4.14}$$

Differentiating this equation successively with respect to x^1 and x^2 and making use of it and (4.6) in the result, we obtain

$$(X^1_\alpha)'' + \varphi X^1_\alpha = (X^2_\alpha)'' + \varphi X^2_\alpha. \tag{4.15}$$

Proceeding in like manner with the first and third of (4.12) we obtain

$$(X^1_\alpha)'' + \varphi X^1_\alpha = (X^3_\alpha)'' + \varphi X^3_\alpha. \tag{4.16}$$

We consider first the case when the constant φ is zero. From (4.6) we have

$$\frac{1}{\psi} = \frac{1}{2}(x^3 - x^1), \quad \frac{1}{\theta} = \frac{1}{2}(x^1 - x^2),$$

and from (4.15), (4.16) and (4.12) we obtain

$$X^i_\alpha = a\, x^{i^2} + b\, x^i + c,$$

where a, b, c are constants. Hence the matrix of the ξ's is

$$\begin{vmatrix} 1 & 1 & 1 \\ x^1 & x^2 & x^3 \\ (x^1)^2 & (x^2)^2 & (x^3)^2 \end{vmatrix}. \tag{4.17}$$

From (4.5) and (4.11) with the aid of (4.4) we find for the H's the expressions

$$c_{ji}\, H_i = \frac{x^k - x^j}{(x^i - x^j)(x^i - x^k)}, \tag{4.18}$$

where i, j, k take the values 1, 2, 3 in cyclic order.

When φ is positive, say $4a^2$, we have from (4.6)

$$\psi = 2a \cot a(x^3 - x^1), \quad \theta = 2a \cot a(x^1 - x^2).$$

In this case by a suitable chöice of basis the respective members of (4.15) and (4.16) may be taken equal to zero for the vectors other than 1, 1, 1. Then from (4.15), (4.16) and (4.12) we obtain

$$X^i_\alpha = b \sin 2a x^i + c \cos 2a x^i \quad (\alpha = 2, 3;\ i = 1, 2, 3),$$

where b and c are constants. Hence the matrix is

$$(4.19)\qquad \left|\begin{array}{ccc} 1 & 1 & 1 \\ \sin\, 2ax^1 & \sin\, 2ax^2 & \sin\, 2ax^3 \\ \cos\, 2ax^1 & \cos\, 2ax^2 & \cos\, 2ax^3 \end{array}\right|,$$

and the H's are of the form

$$(4.20)\qquad c_{ji}H_i = \frac{\sin\, a(x^k - x^j)}{\sin\, a(x^i - x^j)\, \sin\, a(x^i - x^k)}.$$

When φ is negative, say $-4a^2$, we have from (4.6)

$$\psi = 2a \coth a(x^3 - x^1), \quad \theta = 2a \coth a(x^1 - x^2).$$

In this case the matrix is

$$(4.21)\qquad \left|\begin{array}{ccc} 1 & 1 & 1 \\ \sinh\, 2ax^1 & \sinh\, 2ax^2 & \sinh\, 2ax^3 \\ \cosh\, 2ax^1 & \cosh\, 2ax^2 & \cosh\, 2ax^3 \end{array}\right|.$$

and the H's are of the form

$$(4.22)\qquad c_{ji}H_i = \frac{\sinh\, a(x^k - x^j)}{\sinh\, a(x^i - x^j)\, \sinh\, a(x^i - x^k)}.$$

We consider finally the case when ψ in (4.1) is a function of x^1 or a constant. If we put $\bar{x}^1 = f(x^1)$, where $\frac{f''}{f'} = \psi$, and note that $\overline{H}_1 = H_1 f'$, in the new coordinate system $\psi = 0$ and the solution of (4.1) is

$$(4.23)\qquad \frac{1}{H_1} = -c_{21}(x^1 + \varphi),$$

where φ is a function of all the x's except x^1, in accordance with the hypothesis of this section. From (1.20) we have

$$(4.24)\qquad c_{1m}H_m = -\frac{\dfrac{\partial \varphi}{\partial x^m}}{x^1 + \varphi} \qquad (m = 2, \ldots, n),$$

and from (4.2) and (1.17)

$$(4.25)\qquad c_{l1} = c_{21}, \quad c_{lm} = c_{1m} \qquad (l = 3, \ldots, n;\ l \neq m).$$

Substituting from (4.24) in (1.20) for $i > 1$, because of (4.25) we obtain

$$\frac{\partial^2 \varphi}{\partial x^l\, \partial x^m} = 0,$$

and consequently

$$\varphi = X^2 + \ldots + X^n,$$

where X^m is a function of x^m alone. If we put

$$c_{21} = -e_1 b_1, \quad c_{1m} = -e_m b_m \qquad (m = 1, \ldots, n),$$

and effect the change of variables given by

$$e_1 b_1 \bar{x}^1 = x^1, \quad e_m b_m \bar{x}^m = X^m,$$

in the new coordinate system we have

$$H_1 = \ldots = H_n = \frac{1}{\sum_i e_i b_i x^i} \qquad (i = 1, \ldots, n), \tag{4.26}$$

and consequently V_n is of constant curvature $-\sum_i e_i b_i^2$ [8]. The basis may be chosen so that the matrix of the ξ's is

$$\left\| \begin{array}{cccccc} x^1 & x^2 & x^3 & \ldots & & x^n \\ e_2 b_2 & -e_1 b_1 & 0 & \ldots & & 0 \\ e_3 b_3 & 0 & -e_1 b_1 & \ldots & & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ e_n b_n & 0 & 0 & \ldots & & -e_1 b_1 \end{array} \right\| \tag{4.27}$$

As a result of this investigation we have:

A conformally flat space admitting a simply transitive group of motions has constant curvature.

In fact, if the fundamental form is taken as

$$H^2 (e_1 (dx^1)^2 + \ldots + e_n (dx^n)^2),$$

and H involves $x^1, \ldots, x^n$, we have the case (4.26) and (4.27). If H involves only some of the x's, say $x^2, \ldots, x^p$, we have (3.10) and (3.19) with $t = t_s = 1$, and from (3.11) it follows that the curvature is constant.

[8]) Cf. R. G., p. 85.

(Eingegangen: 25. X. 1935.)