Fixed Points of Contraction Mappings on Probabilistic Metric Spaces

by
V. M. SEHGAL
University of Wyoming
and
A. T. BHARUCHA-REID
Wayne State University

ABSTRACT

In this paper the notion of a contraction mapping on a probabilistic metric space is introduced, and several fixed-point theorems for such mappings are proved.

1. Introduction. Probabilistic metric spaces were introduced by Menger [2]. In Menger's theory the concept of distance is considered to be statistical or probabilistic, rather than deterministic; that is to say, given any two points p and q of a metric space, rather than consider a single non-negative real number $d(p, q)$ as a measure of the distance between p and q, a distribution function $F_{pq}(x)$ is introduced which gives the probabilistic interpretation as the distance between p and q is less than x $(x > 0)$. For detailed discussions of probabilistic metric spaces and their applications we refer to Onicescu [3, Chap. VII] and Schweizer [4, 5].

In Section 2 we introduce some basic definitions and concepts from the theory of probabilistic metric spaces which are used in this paper. Section 3 is devoted to the main results of this paper, namely, the proofs of several fixed-point theorems for contraction mappings on probabilistic metric spaces. In a subsequent paper we will utilize theorems of the type considered in this paper to study solutions of operator equations in probabilistic metric spaces.

2. Basic Definitions and Concepts. Let R denote the reals and $R^+ = \{x \in R: x \geq 0\}$.

Definition 1. A mapping $F: R \to R^+$ is called a *distribution function* if it is nondecreasing, left-continuous with $\inf F = 0$ and $\sup F = 1$.

We will denote by $\mathscr{L}$ the set of all distribution functions.

Definition 2. A *probabilistic metric space* (*PM-space*) is an ordered pair $(E, \mathscr{F})$, where E is an abstract set of elements and $\mathscr{F}$ is a mapping of $E \times E$ into $\mathscr{L}$. We shall denote the distribution function $\mathscr{F}(p, q)$ by $F_{p,q}$ and $F_{p,q}(x)$

will represent the value of $F_{p,q}$ at $x \in R$. The functions $F_{p,q}$, $p, q \in E$, are assumed to satisfy the following conditions:

(PM-I) $F_{p,q}(x) = 1$ for all $x > 0$, if and only if $p = q$.
(PM-II) $F_{p,q}(0) = 0$.
(PM-III) $F_{p,q} = F_{q,p}$.
(PM-IV) If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$, then $F_{p,r}(x+y) = 1$.

Remark. Definition 2 suggests that $F_{p,q}(x)$ may be interpreted as the probability that the distance between p and q is less than x.

Definition 3. A mapping Δ: $[0, 1]\times[0, 1] \to [0, 1]$ is a *Δ-norm* if it satisfies

(Δ-I) $\Delta(a, 1) = a$, $\Delta(0, 0) = 0$,
(Δ-II) $\Delta(a, b) = \Delta(b, a)$,
(Δ-III) $\Delta(c, d) \geq \Delta(a, b)$ for $c \geq a$, $d \geq b$,
(Δ-IV) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$.

Let $\mathscr{B}$ denote the set of all Δ-norms, partially ordered by $\Delta_1 \leq \Delta_2$ if and only if $\Delta_1(a, b) \leq \Delta_2(a, b)$ for all $a, b \in [0, 1]$ and $\Delta_1, \Delta_2 \in \mathscr{B}$.

Definition 4. A *Menger space* is a triplet $(E, \mathscr{F}, \Delta)$, where $(E, \mathscr{F})$ is a PM-space and $\Delta \in \mathscr{B}$ satisfies the following triangle inequality:

(PM-IV′) $F_{p,r}(x+y) \geq \Delta(F_{p,q}(x), F_{q,r}(y))$

for all $p, q, r \in E$ and for all $x \geq 0$, $y \geq 0$.

The concept of a neighborhood in a PM-space was introduced by Schweizer and Sklar [6]. If $p \in E$, and ϵ, λ are positive reals, then an (ϵ, λ)-neighborhood of p, denoted by $U_p(\epsilon, \lambda)$ is defined by

$$U_p(\epsilon, \lambda) = \{q \in E: F_{q,p}(\epsilon) > 1-\lambda\}.$$

The following result is due to Schweizer and Sklar [6].

THEOREM 1. *If $(E, \mathscr{F}, \Delta)$ is a Menger space and Δ is continuous then $(E, \mathscr{F}, \Delta)$ is a Hausdorff space in the topology induced by the family $\{U_p(\epsilon, \lambda)$: $p \in E, \epsilon > 0, \lambda > 0\}$ of neighborhoods.*

Note that the above topology satisfies the first axiom of countability. In this topology a sequence $\{p_n\}$ in E *converges* to a $p \in E$ ($p_n \to p$) if and only if for every $\epsilon > 0$ and $\lambda > 0$, there exists an integer $M(\epsilon, \lambda)$ such that $p_n \in U_p(\epsilon, \lambda)$, i.e., $F_{p,p_n}(\epsilon) > 1-\lambda$, whenever $n \geq M(\epsilon, \lambda)$. The sequence $\{p_n\}$ will be called *fundamental* in E if for each $\epsilon > 0$, $\lambda > 0$, there is an integer $M(\epsilon, \lambda)$ such that $F_{p_n,p_m}(\epsilon) > 1-\lambda$ whenever $n, m \geq M(\epsilon, \lambda)$. In conformity with the completion concept in metric spaces, a Menger space E will be called *complete* if each fundamental sequence in E converges to an element in E.

The following theorem is easy to prove and it establishes a connection between metric spaces and Menger spaces.

THEOREM 2. *If (E, d) is a metric space then the metric d induces a mapping $\mathscr{F}: E\times E \to \mathscr{L}$, where $\mathscr{F}(p, q)$ $(p, q \in E)$ is defined by $\mathscr{F}(p, q)\,x = H(x-d(p,q))$, $x \in R$, where $H(x) = 0$ if $x \leq 0$ and $H(x) = 1$ if $x > 0$. Further, if Δ:$[0, 1]\times[0, 1] \to [0, 1]$ is defined by $\Delta(a, b) = \min\{a, b\}$, then $(E, \mathscr{F}, \Delta)$ is a Menger space. It is complete if the metric d is complete.*

The space $(E, \mathscr{F}, \Delta)$ so obtained will be called *induced Menger space*.

3. Some fixed-point theorems for contraction mappings on probabilistic metric Spaces. We first introduce the notion of a contraction mapping on a PM-space.

Definition 5. A mapping T of a PM-space $(E, \mathscr{F})$ into itself will be called a *contraction mapping* if and only if there exists a constant k, with $0 < k < 1$, such that for each $p, q \in E$,

$$F_{Tp,Tq}(kx) \geq F_{p,q}(x) \qquad \text{for all } x > 0. \tag{1}$$

Expression (1) may be interpreted as follows: the probability that the distance between the image points Tp, Tq is less than kx is at least equal to the probability that the distance between p, q is less than x.

THEOREM 3. *Let $(E, \mathscr{F}, \Delta)$ be a complete Menger space, where Δ is a continuous function satisfying $\Delta(x, x) \geq x$ for each $x \in [0, 1]$. If T is any contraction mapping of E into itself, then there is a unique $p \in E$ such that $Tp = p$. Moreover, $T^n q \to p$ for each $q \in E$.*

Proof. We first prove uniqueness. Suppose $p \neq q$ and $Tp = p$, $Tq = q$. Then by (PM-1), there exists an $x > 0$ and an a, with $0 \leq a < 1$, such that $F_{p,q}(x) = a$. However, for each positive integer n, we have by (1)

$$a = F_{p,q}(x) = F_{T^n p, T^n q}(x) \geq F_{p,q}(x/k^n). \tag{2}$$

Since $F_{p,q}(x/k^n) \to 1$ as $n \to \infty$, it follows that $a = 1$. This contradicts the selection of a, and therefore, the fixed point is unique.

To prove the existence of the fixed point, consider an arbitrary $q \in E$, and define $p_n = T^n(q)$, $n = 1, 2 \cdots$. We show that the sequence $\{p_n\}$ is fundamental in E. Let ϵ, λ be positive reals. Then for $m > n$, we have

$$\begin{aligned} F_{p_n,p_m}(\epsilon) &\geq \Delta(F_{p_n,p_{n+1}}(\epsilon - k\epsilon), F_{p_{n+1},p_m}(k\epsilon)), \\ &\geq \Delta(F_{q,p_1}((\epsilon - k\epsilon)k^{-n}), F_{p_{n+1},p_m}(k\epsilon)). \end{aligned}$$

Set $d = (\epsilon - k\epsilon)k^{-n}$. It follows by (PM-IV′) and (Δ-III) that

$$\begin{aligned} F_{p_n,p_m}(\epsilon) &\geq \Delta(F_{q,p_1}(d), \Delta(F_{p_{n+1},p_{n+2}}(k\epsilon - k^2\epsilon), F_{p_{n+2},p_m}(k^2\epsilon))) \\ &\geq \Delta(F_{q,p_1}(d), \Delta(F_{q,p_1}(d), F_{p_{n+2},p_m}(k^2\epsilon))). \end{aligned}$$

By the associativity of Δ, and the hypothesis $\Delta(x, x) \geq x$, we have

$$F_{p_n,p_m}(\epsilon) \geq \Delta(F_{q,p_1}(d), F_{p_{n+2},p_m}(k^2\epsilon)). \tag{3}$$

Using the same argument repeatedly, we obtain from (3)

$$\begin{aligned} F_{p_n,p_m}(\epsilon) &\geq \Delta(F_{q,p_1}(d), F_{p_{m-1},p_m}(k^{m-n-1}\epsilon)) \\ &\geq \Delta(F_{q,p_1}(d), F_{q,p_1}(k^{-n}\epsilon)) \\ &\geq \Delta(F_{q,p_1}(d), F_{q,p_1}(d)) \geq F_{q,p_1}((\epsilon - k\epsilon)k^{-n}). \end{aligned}$$

Therefore, if we choose N such that $F_{q,p_1}((\epsilon - k\epsilon)k^{-N}) > 1 - \lambda$, it follows that $F_{p_n,p_m}(\epsilon) > 1 - \lambda$ for all $n \geq N$. Hence, $\{p_n\}$ is a fundamental sequence. Since $(E, \mathscr{F}, \Delta)$ is a complete PM-space, there is a $p \in E$ such that $p_n \to p$, that is, $T^n q \to p$. We prove that $T^n q \to Tp$ also. Let $U_{Tp}(\epsilon, \lambda)$ be any neighborhood of Tp. Then $p_n \to p$ implies the existence of an integer N such that $p_n \in U(\epsilon, \lambda)$ for all $n \geq N$. However,

$$F_{Tp_n,Tp}(\epsilon) \geq F_{p_n,p}(\epsilon/k) \geq F_{p_n,p}(\epsilon) > 1 - \lambda$$

for all $n \geq N$. Therefore, $Tp_n \in U(\epsilon, \lambda)$ for all $n \geq N$, that is, $T^n q \to Tp$. We conclude therefore that $Tp = p$. This proves the existence part of the theorem.

We now state and prove two theorems, the proofs of which utilize the results stated in the last section. Theorem 4 is the well-known Banach contraction-mapping theorem; this proof uses the notion of a probabilistic metric.

THEOREM 4. *Let (E, d) be a complete metric space and let $T: E \to E$ satisfy the condition: there exists a constant k, $0 < k < 1$, such that $d(Tp, Tq) \leq kd(p, q)$ for all $p, q \in E$. Then T has a unique fixed point $p_* \in E$ and $T^n q \to p_*$ for each $q \in E$.*

Proof. If $\mathscr{F}: E \times E \to \mathscr{L}$ is the mapping induced by the metric d, then it follows by Theorem 2 that $(E, \mathscr{F}, \Delta)$ is a complete Menger space, where $\Delta(a, b) = \min \{a, b\}$. Since for each $x > 0$,

$$\begin{aligned} F_{Tp,Tq}(kx) &= H(kx - d(Tp, Tq)), \\ &\geq H(kx - kd(p, q)), \\ &= H(x - d(p,q)), \\ &= F_{p,q}(x), \end{aligned}$$

it follows that T is a contraction of E into itself. The conclusion now follows by Theorem 3.

THEOREM 5. *If (E, ϵ, Δ) is a complete Menger space where Δ satisfies any one of the following conditions*: (a) Δ_1: $\Delta(a, b) = \min \{a, b\}$, (b) Δ_2: $\Delta(a, b) = \max \{a, b\}$, (c) A_3: $\Delta(a, b) = a+b-ab$, (d) Δ_4: $\Delta(a, b) = \min \{a+b, 1\}$, *then Theorem* 3 *holds.*

Proof. Each of these Δ functions is continuous and satisfies the condition $\Delta(x, x) \geq x$. The conditions of Theorem 2 remain valid.

It is natural to ask if mappings on a PM-space which are local, or pointwise, contractions admit a fixed point. We first introduce some definitions.

Definition 6. Let ϵ and λ be positive reals. A mapping T of a PM-space into itself will be called an (ϵ, λ)-*local contraction* if there exists a constant k, $0 < k < 1$, such that if $p \in E$ and $q \in U_p(\epsilon, \lambda)$, then

$$F_{Tp,Tq}(kx) \geq F_{p,q}(x) \qquad \text{for } x > 0. \tag{4}$$

Definition 6 has the following probabilistic meaning: Whenever the probability of the distance between p and q being less than ϵ is greater than $1-\lambda$, then T acts as a contraction map for the pair of points p and q in the sense of Definition 5.

We now show that for certain types of PM-spaces, each (ϵ, λ)-contraction mapping has the fixed-point property.

Definition 7 (Edelstein [1]). Let (E, d) be a metric space, and let $\epsilon > 0$. A finite sequence $x_0, x_1, \cdots, x_n$ of elements of E is called an ϵ-*chain* joining x_0 and x_n if $d(x_i, x_{i+1}) < \epsilon$, $i = 0, 1, \cdots, n-1$. The metric space (E, d) is ϵ-*chainable* if for every $x, y \in E$, there is an ϵ-chain joining x and y.

We introduce the following definition.

Definition 8. Let $(E, \mathscr{F})$ be a PM-space and ϵ, λ positive reals. The space $(E, \mathscr{F})$ is called (ϵ, λ)-*chainable* if for each $p, q \in E$ there exists a finite sequence

$p = p_0, p_1, \cdots, p_n = q$ of elements in E such that $p_{i+1} \in U_{p_i}(\epsilon, \lambda)$, i.e., $F_{p_{i+1}, p_i}(\epsilon) > 1-\lambda$ for $i = 0, 1, \cdots, n-1$.

THEOREM 6. *If (E, d) is a ϵ-chainable metric space then the induced Menger space $(E, \mathscr{F}, \Delta)$ is an (ϵ, λ)-chainable space in the sense of Definition* 8.

Proof. Let $p, q \in E$, and let $p = p_0, p_1, \cdots, p_n = q$ be the ϵ-chain joining p and q. Then $d(p_i, p_{i+1}) < \epsilon$, $i = 0, 1, \cdots, n-1$. However, $F_{p_{i+1}, p_i}(\epsilon) = H(\epsilon - d(p_i, p_{i+1})) = 1 > 1-\lambda$ for all $\lambda > 0$. Therefore the Menger space $(E, \mathscr{F}, \Delta)$ is (ϵ, λ)-chainable.

THEOREM 7. *Let $(E, \mathscr{F}, \Delta)$ be a complete (ϵ, λ)-chainable Menger space, where Δ is continuous and satisfies $\Delta(x, x) \geq x$. If $T: E \to E$ is an (ϵ, λ)-contraction, then T has a unique fixed point $p_* \in E$ and $T^n p \to p_*$ for each $p \in E$.*

We first prove the following lemma.

LEMMA. *Under the hypothesis of Theorem 7, for each $p \in E$ and for positive real x, there exists a positive integer $N(p, x)$ such that $F_{T^m p, T^{m+1} p}(x) > 1-\lambda$ for all $m \geq N(p, x)$.*

Proof. Let $p = p_0, p_1, \cdots, p_n = Tp$ be a finite sequence such that $F_{p_{i+1}, p_i}(\epsilon) > 1-\lambda$, $i = 0, 1, \cdots, n-1$. It follows by (4) that $F_{Tp_{i+1}, Tp_i}(\epsilon) \geq F_{p_{i+1}, p_i}(\epsilon/k) > 1-\lambda$, that is, the sequence of elements $Tp_0, Tp_1, \cdots, Tp_n$ is an (ϵ, λ)-chain for Tp and $T^2 p$, and hence by induction, $T^r p_0, T^r p_1, \cdots, T^r p_n$ is an (ϵ, λ)-chain for $T^r p$ and $T^{r+1} p$ for each positive integer r. Therefore, for $x > 0$, and for each integer $r > 0$,

(5) $$F_{T^r p_{i+1}, T^r p_i}(x) \geq F_{T^{r-1} p_{i+1}, T^{r-1} p_i}(x/k) \geq \cdots \geq F_{p_{i+1}, p_i}(x/k^r).$$

It follows by the triangle inequality (PM-IV′) and by (5) that

$$\begin{aligned} F_{T^r p_0, T^r p_n}(x) &\geq \Delta(F_{T^r p_0, T^r p_1}(x/2), F_{T^r p_1, T^r p_n}(x/2)), \\ &\geq \Delta(F_{p_0, p_1}(x/2k^r), F_{T^r p_1, T^r p_n}(x/2)). \end{aligned}$$

By the triangle inequality, by (PM-III) and by (5) we have

$$F_{T^r p_0, T^r p_n}(x) \geq \Delta(F_{p_0, p_1}(x/2k^r), \Delta(F_{p_1, p_2}(x/2^2 k^r), F_{T^r p_2, T^r p_n}(x/2^2))),$$

Setting $d = x/2^n k^r$, we have by (PM-III),

$$F_{T^r p_0, T^r p_n}(x) \geq \Delta(F_{p_0, p_1}(d), \Delta(F_{p_1, p_2}(d), F_{T^r p_2, T^r p_n}(x/2^2))).$$

Therefore, repeated use of the above argument yields

$$F_{T^r p_0, T^r p_n}(x) \geq \Delta(F_{p_0, p_1}(d), \Delta(F_{p_1, p^t}(d), \cdots, \Delta(F_{p_{n-2}, p_{n-1}}(d), F_{p_{n-1}, p_n}(d)))).$$

Since n is a fixed finite integer, there exists an integer $m_i > 0$ such that $F_{p_i, p_{i+1}}(x/2^n k^r) > 1-\lambda$ for each $r \geq m_i$, $i = 0, 1, \cdots, n-1$. Let $N(p, x) = \max\{m_0, m_1, \cdots, m_{n-1}\}$. Then $F_{T^r p, T^{r+1} p}(x) > 1-\lambda$ for all $r \geq N(p, x)$. This proves the lemma.

Proof of Theorem 7. Let $p \in E$ be arbitrary. By the above lemma, for $\epsilon > 0$, there is an integer $N(p, \epsilon)$ such that $F_{T^n p, T^{n+1} p}(\epsilon) > 1-\lambda$ for all $n \geq N$. Set $T^N p = q$; then we have $F_{T^n q, T^{n+1} q}(\epsilon) > 1-\lambda$ for all integers $n \geq 0$. Therefore, it follows by (4) that for all $x > 0$, we have the inequality

$$F_{T^n q, T^{n+1} q}(x) \geq F_{q, Tq}(x/k^n), \qquad n = 0, 1, \cdots.$$

It follows as in the proof of Theorem 3 that the sequence $\{T^n q\}$, and hence the sequence $\{T^n p\}$, is fundamental in E. Let $T^n p \to p_* \in E$. We show that the sequence $\{T^n p\}$ also converges to Tp_*. Let $U_{Tp_*}(\delta, \mu)$ be a neighborhood of Tp_*. Since $T^n p \to p_*$, there is an integer $M \geq 0$ such that $T^n p \in U_{p_*}(\delta, \mu) \cap U_{p_*}(\epsilon, \lambda)$ for all $n \geq M$, that is, $F_{T^n p, p_*}(\epsilon) > 1-\lambda$, and also $F_{T^n p, p_*}(\delta) > 1-\mu$. Now by (4). we have

$$F_{T^{n+1}p, Tp_*}(\delta) \geq F_{T^n p, p_*}(\delta/k) > 1-\mu, \qquad n \geq M.$$

Therefore, $T^n p \to Tp_*$, and hence $Tp_* = p_*$. This proves the existence of a fixed point of T.

To prove uniqueness, let $Tp = p$, $Tq = q$ and $p \neq q$. Then by (PM-1), there is a real $x > 0$ such that $F_{p,q}(x) = a$ for some a with $0 \leq a < 1$. Let $p_0 = p, p_1, \cdots, p_n = q$ be a (ϵ, λ)-chain for p and q. Then, since for each positive integer m, $T^m p_0, T^m p_1, \cdots, T^m p_n$ is an (ϵ, λ)-chain for $T^m p$ and $T^m q$, it follows as in the proof of the existence part, that $a = F_{p,q}(x) = F_{T^m p, T^m q}(x) > a$ for m sufficiently large. Thus $p = q$.

COROLLARY (Edelstein [1]). *Let (E, d) be a complete ϵ-chainable metric space and $T: E \to E$ satisfy the condition that $d(p, q) < \epsilon$ implies $d(Tp, Tq) \leq kd(p, q)$ for some k, $0 \leq k < 1$, and for all $p, q \in E$. Then T has a unique fixed point $p_* \in E$ and $T^n p \to p_*$ for each $p \in E$.*

Proof. Let $(E, \mathscr{F}, \Delta)$ be the induced Menger space. Then $(E, \mathscr{F}, \Delta)$ is an (ϵ, λ)-chainable space for each $\lambda > 0$ (see Theorem 6). Choose $\lambda < 1$. We show that T is an (ϵ, λ)-contraction. If $q \in U_p(\epsilon, \lambda)$, then $F_{q,p} > 1-\lambda$, that is, $H(\epsilon - d(p, q)) > 0$, and therefore $d(p, q) < \epsilon$. Thus $d(Tp, Tq) \leq kd(p, q)$, and hence for $x > 0$

$$F_{Tp, Tq}(kx) = H(kx - d(Tp, Tq)) \geq H(x - d(p, q)) = F_{p,q}(x).$$

The result now follows by Theorem 7.

REFERENCES

[1] M. EDELSTEIN, (i) On fixed and periodic points under contractive mapping, *J. London Math. Soc.* **37** (1962), 74–79; (ii) An extension of Banach's contraction principle, *Proc. Amer. Math. Soc.* **12** (1961), 7–10.

[2] K. MENGER, Statistical metrics, *Proc. Nat. Acad. Sci. U.S.A.* **28** (1942), 535–537.

[3] O. ONICESCU, *Nombres et Systèmes Aléatoires*, Éditions de l'Académie de la R. P. Roumaine, Bucarest, 1964.

[4] B. SCHWEIZER, Probabilistic metric spaces—the first 25 years, *The N.Y. Statistician* **19** (1967), 3–6.

[5] B. SCHWEIZER, Probabilistic metric spaces, *Probabilistic Methods in Applied Mathematics* (A. T. Bharucha-Reid, ed.), Vol. 4, Academic Press, New York (to appear).

[6] B. SCHWEIZER and A. SKLAR, Statistical metric spaces, *Pacific J. Math.* **10** (1960), 313–334.

[7] B. SCHWEIZER, A. SKLAR and E. THORP, The metrization of statistical metric spaces, *Pacific J. Math.* **10** (1960) 673–675.

(Received 1 March 1971)