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A Comment on 'Democratic Theory: A Preliminary Mathematical Model.'

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In a recent paper in this journal, Raphael Kazmann[1] has independently rediscovered a model of group decision-making in a dichotomous choice situation whose implications were first realized in the eighteenth century by the French mathematician and social theorist Nicholas Charles de Condorcet.[2] Condorcet's work was, as far as I know, first brought to the attention of the English speaking world by the noted economist Duncan Black,[3] and the result described by Kazmann (which I shall label the Condorcet Jury Theorem), has been discussed by

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[1]"Democratic Organization: A Preliminary Mathematical Model," *Public Choice* 16 (Fall 1973), 17-26.

[2]Essai sur l'Application de l'Analyse a la Probabilite des Decisions Rendue a la Pluralite des Voix, Paris, 1876.

[3]*The Theory of Committees and Elections,* London: Cambridge University Press, esp. pp. 159-178. As Black has pointed out, Condorcet's discovery of the paradox of cyclical majorities, in significant ways, anticipated Arrow's Impossibility Theorem. (See Kenneth Arrow, *Social Choice and Individual Values,* 2nd Edition, New York: Wiley, 1962, and Duncan Black, "An Examination of Professor Arrow's Impossibility Theorem," Vienna, 1968.) The most extensive treatment of Condorcet's mathematical contributions to political and economic theory is Gilles-Gaston Granger, *La Mathematique Sociale du Marguis de Condorcet,* Paris: Presses Universitaires de France, 1956.

Black,[4] and in publications by two political scientists [5] as well as by myself in some unpublished work.[6]

Consider a group of N members such that each member has some probability p of reaching a correct judgment in some dichotomous choice situation (e.g., with respect to the innocence or guilty of a defendant in a criminal trial). Let us initially assume that this probability is the same for all group members. Also, let us assume that each member arrives at his decision independently of the choice of the other members and that there exists a secret ballot mechanism for ascertaining these choices. Finally, let us assume that the group decides by simple majority vote.

Let m be a majority of the group, defined as $\frac{N+1}{2}$. The probability (for N odd) that a majority of the group will reach a correct judgment is simply (as Kazmann points out)[7]:

$$1)\quad \sum_{h=m}^{N} \binom{N}{h} p^{h} (1-p)^{N-h}$$

Theorem 1: (Condorcet's Jury Theorem): For N odd, $1 > p > \frac{1}{2}$ the larger the group, the *more* likely it is that a majority of the group will reach a correct judgment; while for N odd, $0 < p < \frac{1}{2}$, the larger the group, the *less* likely it is that a majority of the group will reach a correct verdict; and for N odd, $p = \frac{1}{2}$, the likelihood of a majority of the group reaching a correct judgment is independent of N and is equal to ½, i.e.

$$2)\quad \sum_{h=m}^{N} \binom{N}{h} p^{h} (1-p)^{N-h} - \sum_{h=m+1}^{N+2} \binom{N+2}{h} p^{h} (1-p)^{N+2-h}$$

$$\lesseqgtr 0 \text{ as } p \gtreqless \tfrac{1}{2}$$

Moreover, as $N \to \infty$ the probability that the group's judgment will be correct $\to 1$ if $p > \frac{1}{2}$ and 0 if $p < \frac{1}{2}$.

A rather straightforward proof of this theorem has been found by Giles Auchmuty.[8]

[4] Black, *op. cit.*, pp. 163-165.

[5] S. Sidney Ulmer, "Quantitative Analysis of Judicial Processes, Some Practical and Theoretical Applications." In Hans W. Baade (Ed.) *Jurimetrics*, New York: Basic Books, 1963, 179-180; Brian Barry, *Political Argument*, London: Routledge and Kegan Paul, 1963, 293 and "The Public Interest," *Proceedings of the Aristotelian Society*, Supplementary volume 38 (1964), esp. pp. 9-14.

[6] Bernard Grofman, "Optimal Jury Rules," State University of New York at Stony Brook, dittoed, December 1971, Giles Auchmuty and Bernard Grofman, "Some Theorems on Optimal Jury Rules," State University of New York at Stony Brook, Xeroxed, March 1972.

[7] Kazmann, *op. cit.*, p. 20.

[8] Auchmuty and Grofman, *op. cit.*

Theorem 1 enables us to shed light on various seemingly contradictory proverbs, e.g., "Too many cooks spoil the broth," "A camel was a horse designed by a committee," "Two heads are better than one," and "Vox populi, vox dei." If a group's members' probabilities of correct judgment are all less than ½, then the majority group judgment must be inferior to the judgment of the group's best member and the voice of the people is apt to be quite wrong. Indeed, the more people the more wrong it is likely to be. If, on the other hand, the group's probabilities of correct judgment are all even slightly better than >½ (and indeed, as we shall see, if the group's mean judgmental capability is ½, even if some members have judgmental capabilities below ½) then the group verdict approaches infallibility as the group size approaches infinity.

Kazmann goes on to generalize this theorem by considering what happens if instead of the group being homogeneous in p, the group is characterized by a mean value of p, normally distributed, with variance equal to $\frac{p(1-p)}{N}$.[9] Of course a normal distribution with mean p and variance $\frac{p(1-p)}{N}$ approximates a binomial distribution of mean p, and this approximation is quite good even for relatively small N. Thus, what Kazmann calls his "advanced model"[10] is formally identical to a normal approximation to the binomial expression given in the Condorcet Jury Theorem above. Kazmann then goes on to consider what happens if we drop the least competent members of the group. Though he does not state this, it follows readily from the Condorcet Jury Theorem that the judgmental (majority vote) competence of the group arrived at by dropping the least competent half of the group approaches 1 as the original group size approaches infinity, for initial $p \geqslant ½$.[11]

We may ask for what values of x and y do groups of size N+y and competence p-x have expected group (majority verdict) competence identical to that of a group of size N and mean competence p:

[9]Kazmann, *op. cit.*, p. 22.

[10]*Ibid*, pp. 21-23.

[11] Kazmann does not answer the question of how the least competent members of the group are to be determined. One possibility is as follows: Consider a group of size N with mean competence p binomially distributed. The group takes a vote and some external feedback mechanism determines what is the "Correct" answer. Those who get it wrong are then dropped from the group. This process is then repeated. If p_K is the mean competence of the group after the Kth round of this weeding out then I believe (although I haven't formally proved) that

$$p_K \approx \frac{(K+1)p}{Kp+1}$$

Similarly, if n_K is the size of the group after the Kth round of this weeding out process, then I believe that

$$n_K \approx \frac{NK!\quad p^K}{\pi\ [(K-1)p+1]\qquad [(K-2)p+1]\ \ldots}$$

$$(K-i) \geq 0$$

where $\underset{K=0}{\pi} = 1.$

Theorem 2: Groups of size N+y and mean (bionomially distributed) competence p-x are identical in expected group (majority) verdict to a group of size N and mean competence p iff

$$3)\quad y=N\left[\frac{.25x\ (2p-1-x)}{p(1-p)\ (p-x-.5)^2}\right]$$

Proof: By the normal approximation to the binomial, we wish to find x,y such that

$$4)\quad \Phi\left(\frac{p-.5}{\sqrt{\frac{p(1-p)}{N}}}\right)=\Phi\left(\frac{p-x-.5}{\sqrt{\frac{(p-x)\ (1-p+x)}{N+y}}}\right)$$

The desired result follows from some simple algebraic manipulation.

We show in Figure 1 isocompetence curves for groups of various sizes for p=.55, .6, .7, .8, and .9,

These isocompetence curves shed interesting light on the relative attractiveness (judgmental competence) of democracy and dictatorship (or oligarchy) as a function of the mean competence of the dictators (oligarchs) versus the mean competence of the larger (and presumably less competent, on the average) democratic mass. If the mean competence of the democratic electorate is $>1/2$, then majority rule (for N large enough) may indeed be regarded as "divinely" inspired, and to be preferred to the judgments of any dictator or any band of oligrachs who are not themselves infallible.

One final point: Kazmann quite rightly has called attention to the potential improvement in the accuracy of group verdicts which occurs when less competent members of a group are expelled. However, as Figure 1 makes clear, sometimes adding new members to a group who are less competent on average than the group's previous average member, may actually increase the group's probability of a correct judgment. This rather counterintuitive phenomenon occurs as long as the increase in group size compensates for the decrease in mean group competence. (See Equation (3)).

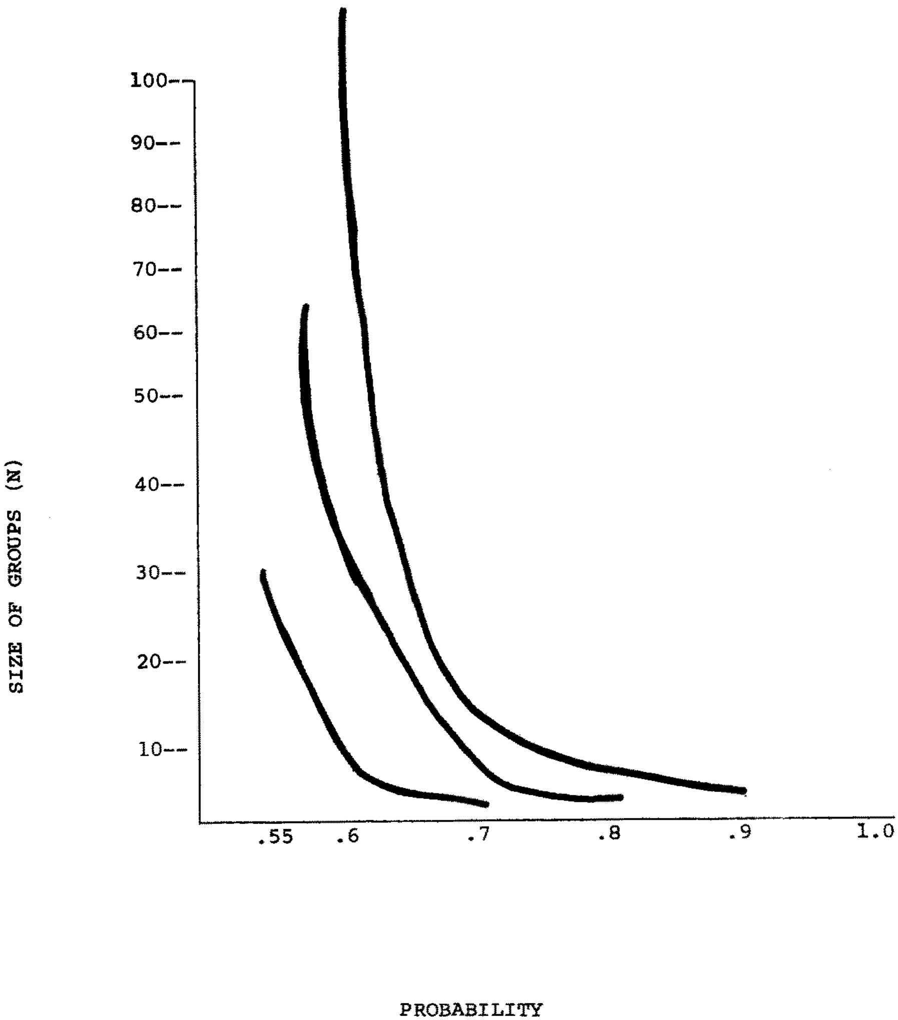

FIGURE 1: ISOCOMPETENCE CURVES FOR VARIOUS VALUES OF MEAN GROUP COMPETENCE ($p>\frac{1}{2}$)