Local Controllability of a Nonlinear Wave Equation

by

H. O. FATTORINI*

Departamento de Matemática
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
Ciudad Universitaria, Buenos Aires, Argentina
and
University of California
Department of Mathematics
Los Angeles, California 90024

ABSTRACT

We obtain results on local controllability (near an equilibrium point) for a nonlinear wave equation, by application of an infinite-dimensional analogue of the Lee-Markus method of linearization. Controllability of the linearized equation is studied by application of results of Russell, and local controllability of the nonlinear equation follows from the inverse function theorem. We prove that every state that is sufficiently small in a sense made precise in the paper can be reached from the origin in a time T depending on the coefficients of the equation.

1. Introduction. We consider the nonlinear wave equation

$$\rho(x)u_{tt} = (p(x)u_x)_x - F(x, u) + b(x)f(t) \qquad (0 \leq x \leq X, 0 \leq t \leq T). \tag{1.1}$$

This equation governs the small transverse displacement u of a string with density $\rho(x)$ and modulus of elasticity $p(x)$ acted upon by a nonlinear restoring force $-F(x, u)$ and by an external force $b(x)f(t)$. The function $f(t)$, the magnitude of the external force, is thought of as a control or steering function by means of which we try to influence the motion of the string in a way that will be made precise later; the function $b(x)$, the spatial distribution of the external force, is fixed.

We assume that the position of the string satisfies the boundary conditions

$$\begin{cases} A_0u(0, t) + A_1u_x(0, t) = 0 & (0 \leq t \leq T) \\ B_0u(X, t) + B_1u_x(X, t) = 0 & (0 \leq t \leq T), \end{cases} \tag{1.2}$$

where the real numbers A_0, A_1, B_0, B_1 satisfy $A_0^2 + A_1^2 > 0$, $B_0^2 + B_1^2 > 0$.

* This research was supported in part by the National Science Foundation under contract GP-9658.

MATHEMATICAL SYSTEMS THEORY, Vol. 9, No. 1.

We examine in the sequel the problem of bringing the position and speed of the string to arbitrarily prescribed values at time $t = T$ starting at zero position and speed at $t = 0$ (or, equivalently, the problem of bringing the string to zero position and speed starting at some arbitrary position and speed). This problem is handled by applying a method due to Lee and Markus ([6], [7]) in the finite-dimensional case. The Lee-Markus method, adapted to deal with the present infinite-dimensional problem, can be roughly described as follows. Assume that $u = u_t = 0$ is an equilibrium point for the system (1.1)–(1.2), i.e., that

$$F(x, 0) = 0 \qquad (0 \leq x \leq X) \tag{1.3}$$

and let Φ_T be the map

$$f(\cdot) \to \Phi_T(f(\cdot)) = (u(\cdot, T), u_t(\cdot, T)) \tag{1.4}$$

that carries the control f into the values (at $t = T$) of the solution of (1.1), (1.2) with $u(x, 0) = u_t(x, 0) = 0$ $(0 \leq x \leq X)$ and of its t-derivative. Choosing adequately the domain and range of Φ_T and imposing some rather natural conditions on the coefficients of (1.1) (all this will be made precise later), one can show that Φ_T is continuously differentiable everywhere in the sense of Fréchet and that its differential $d\Phi_T(0)$ at $f = 0$ is given by the map

$$h(\cdot) \to d\Phi_T(0)h(\cdot) = (y(\cdot, T), y_t(\cdot, T)) \tag{1.5}$$

defined in the same way as (1.4) but now in reference to the "linearized" system

$$\rho(x)y_{tt} = (p(x)y_x)_x - F_u(x, 0)y + b(x)h(t) \quad (0 \leq x \leq X, 0 \leq t \leq T) \tag{1.6}$$

$$A_0 y(0, t) + A_1 y_x(0, t) = B_0 y(X, t) + B_1 y_x(X, t) = 0 \qquad (0 \leq t \leq T). \tag{1.7}$$

Controllability of the system (1.6) – (1.7)—that is, the range of the map $d\Phi_T$—has been exhaustively studied by Russell in [12]. Precisely, he has shown that $d\Phi_T$ is *onto* if b satisfies certain conditions. This, in view of the results previously sketched, makes possible the application of the inverse function theorem to the nonlinear map Φ_T. We are able to show in this way (as in [6] for a different problem) that the range of Φ_T must contain a neighborhood of the origin, which furnishes a local answer to our controllability problem.

It appears that the Lee-Markus method as outlined above could be applied to more general classes of hyperbolic equations, for instance the genuinely nonlinear wave equations studied in [1], at least in the case of one space variable where controllability results are available in the linear case. Success would then exclusively depend on having the right existence, uniqueness and "differentiable dependence" results for the nonlinear system under consideration. We wish merely to illustrate the method with an example and thus we restrict ourselves to the mildly nonlinear equation (1.1) for which these results are readily available.

The assumptions on the coefficients of (1.1) are as follows. The functions $\rho(x)$ and $p(x)$ are twice continuously differentiable and positive in $0 \leq x \leq X$.

The function $F(x, u)$ is twice continuously differentiable in the strip $0 \leq x \leq X$, $-\infty < u < \infty$, and *odd* in u for each fixed x, so that (1.3) is automatically satisfied. As for $b(x)$, we shall assume that it belongs to $L^2(0, X)$. (Additional conditions on F and b will be needed in the following sections.) Rather than dealing with the equation (1.1) directly we shall make, as in [12], the customary change of independent and dependent variables

$$\xi(x) = \int_0^x \rho(r)^{1/2}p(r)^{-1/2}dr \qquad (0 \leq x \leq X) \tag{1.8}$$

$$\tilde{u}(\xi, t) = (\rho(x(\xi))p(x(\xi)))^{1/4}u(x(\xi), t), \qquad (0 \leq \xi \leq L = \xi(X), 0 \leq t \leq T). \tag{1.9}$$

The function $\tilde{u}(\xi, t)$ is a solution of the simpler equation

$$\tilde{u}_{tt} = \tilde{u}_{\xi\xi} - \tilde{F}(\xi, u) + \tilde{b}(\xi)f(t) \qquad (0 \leq \xi \leq L, 0 \leq t \leq T). \tag{1.10}$$

The coefficients of this equation are $F(\xi, u) = a(x(\xi))u - S(x(\xi))F(x(\xi), R(x(\xi))u)$, where $a(x) = S(x)(p(x)R'(x))'$, $R(x) = \rho(x)^{-1/4}p(x)^{-1/4}$, $S(x) = \rho(x)^{-3/4}p(x)^{1/4}$, and $\tilde{b}(\xi) = S(x(\xi))b(x(\xi))$. The boundary conditions (1.2) become

$$\begin{aligned} \tilde{A}_0\tilde{u}(0, t) + \tilde{A}_1\tilde{u}_\xi(0, t) &= (A_0R(0) + A_1R'(0))\tilde{u}(0, t) \\ + A_1\rho(0)^{1/4}p(0)^{-3/4}\tilde{u}_\xi(0, t) &= 0 \qquad (0 \leq t \leq T), \\ \tilde{B}_0\tilde{u}(L, t) + \tilde{B}_1\tilde{u}_\xi(L, t) &= (B_0R(X) + B_1R'(X))\tilde{u}(L, t) \\ + B_1\rho(X)^{1/4}p(X)^{-3/4}\tilde{u}_\xi(L, t) &= 0 \qquad (0 \leq t \leq T). \end{aligned} \tag{1.11}$$

We note that the coefficients of the new equation (1.10) satisfy all the conditions so far required of the coefficients of (1.1). Likewise the coefficients $\tilde{A}_0$, $\tilde{B}_0$, $\tilde{A}_1$, $\tilde{B}_1$ in the new boundary conditions (1.11) satisfy the same conditions as the A_0, B_0, A_1, B_1.

We establish in Section 2 the necessary existence, uniqueness and differentiability results for (1.10), (1.11). The results on linear controllability in [12] are stated—and some of them proved—in Section 3. Finally, local controllability of the system (1.10)–(1.11) is studied in Section 4. We include there also a detailed discussion on how local controllability of (1.10)–(1.11) is related through the transformation (1.8), (1.9) to local controllability of the original system (1.1)–(1.2).

We note that the problem of controllability of nonlinear hyperbolic systems in one space dimension was already considered by Cirina in [3]. Although the systems considered there are, in a sense, more general than the nonlinear wave equations studied in this paper, Cirina considers *boundary* controllability. His method is totally different from the present one.

I am grateful to M. Artola for some very useful remarks about existence and uniqueness theory for (1.1).

2. Existence, Uniqueness and Differentiability Results. Reverting to the original x, u notation, we consider the equation

$$u_{tt} = u_{xx} - F(x, u) + b(x)f(t) \qquad (0 \leq x \leq L, 0 \leq t \geq T). \tag{2.1}$$

We shall assume that the boundary conditions take one of the three following forms:

$$(2.2)\qquad \begin{array}{llll} \text{(I)} & u(0,t) = u(L,t) = 0 & & (0 \le t \le T) \\ \text{(II)} & u(0,t) = u_x(L,t) = 0 & & (0 \le t \le T) \\ \text{(III)} & u_x(0,t) = u_x(L,t) = 0 & & (0 \le t \le T). \end{array}$$

The assumptions on F and b are the same as those in Section 1.

Let $\mathbf{A}_I$ be the operator in $L^2(0, X)$ defined by

$$(2.3)\qquad (\mathbf{A}_I y)(x) = -y''(x)$$

with domain $\mathbf{H}_I$ consisting of all functions y continuously differentiable in $[0, L]$, such that y' is absolutely continuous, $y'' \in L^2(0, L)$ and $y(0) = y(L) = 0$. The operators $\mathbf{A}_{II}$, $\mathbf{A}_{III}$ are similarly defined, but their domains $\mathbf{H}_{II}$, $\mathbf{H}_{III}$ are determined by the boundary conditions $y(0) = y'(L) = 0$ and $y'(0) = y'(L) = 0$ respectively. We shall ordinarily denote by $\mathbf{A}$ any of the three operators $\mathbf{A}_I$, $\mathbf{A}_{II}$, $\mathbf{A}_{III}$.

It is well known that $\mathbf{A}$ is self-adjoint and possesses a sequence $\{\lambda_n\}$ of distinct real eigenvalues, $0 \le \lambda_0 < \lambda_1 < \cdots$ corresponding to a complete orthonormal set of eigenfunctions $\{\varphi_n\}$. The eigenvalues are given by $\lambda_n = \omega_n^2$, where

$$(2.4)\qquad \begin{cases} \omega_n = (\pi/L)(n+1) & (n \ge 0) \quad \text{for} \quad \mathbf{A}_I, \\ \omega_n = (\pi/L)(n+\frac{1}{2}) & (n \ge 0) \quad \text{for} \quad \mathbf{A}_{II}, \\ \omega_n = (\pi/L)n & (n \ge 0) \quad \text{for} \quad \mathbf{A}_{III}. \end{cases}$$

We shall denote by $\mathbf{K}_I$ the domain $D(\mathbf{A}_I^{1/2})$ of the (unique) self-adjoint non-negative square root of $\mathbf{A}_I$. The subspaces $\mathbf{K}_{II}$ and $\mathbf{K}_{III}$ are similarly defined. We shall always assume the spaces $\mathbf{H}$ and $\mathbf{K}$ endowed with their graph norms: for $\mathbf{H}$, this norm is equivalent to

$$(2.5)\qquad |u|_{\mathbf{H}} = |\mathbf{A}u|$$

in the cases (I) and (II) or to

$$(2.6)\qquad |u|_{\mathbf{H}} = |(\mathbf{A}+\epsilon I)u| \qquad (\epsilon > 0)$$

in case (III). Similarly, the graph norm of K is equivalent to

$$(2.7)\qquad |u|_{\mathbf{K}} = |\mathbf{A}^{1/2}u|$$

and in cases (I) and (II) to

$$(2.8)\qquad |u|_{\mathbf{K}} = |(\mathbf{A}+\epsilon I)^{1/2}u| \qquad (\epsilon > 0)$$

in case (III).

For any Banach space E we denote by $L^\infty(0, T; E)$ the space of all strongly measurable, essentially bounded functions from $(0, T)$ to E endowed with its usual essential supremum norm ([8], [9]).

THEOREM 2.1. *Assume that for some* $\nu < \lambda_0 =$ *least eigenvalue of* $\mathbf{A}$

$$F(x, u)+\nu u \geq 0 \qquad (u \geq 0), \tag{2.9}$$

$$b(x) = \sum_{n=0}^{\infty} b_n\varphi_n(x), \qquad \limsup_{n\to\infty} n\,|b_n| < \infty, \tag{2.10}$$

$$f \in L^2(0, T), \tag{2.11}$$

$$u_0 \in \mathbf{H}, \qquad u_1 \in \mathbf{K}. \tag{2.12}$$

Then there exists a function $t \to u(t) \in H$ *defined in* $[0, T]$ *such that*

$$u(\cdot) \in L^\infty(0, T; \mathbf{H}),\ u'(\cdot) \in L^\infty(0, T; \mathbf{K}),\ u''(t) \in L^\infty(0, T; L^2) \tag{2.13}$$

$$u''(t)+\mathbf{A}u(t)+F(u(t)) = bf(t) \tag{2.14}$$

and

$$u(0) = u_0, \qquad u'(0) = u_1. \tag{2.15}$$

We comment briefly on Theorem 2.1. We have written L^2 instead of $L^2(0, L)$, which we will keep on doing later. The derivatives in (2.13), (2.14) and (2.15) are understood in the sense of the theory of vector-valued distributions. Equation (2.14) is equation (2.1) plus one of the boundary conditions (2.2) written in operational form; for instance, $u(x, t)$ is thought of as a function $t \to u(\cdot, t)$ of t with values in the function space $\mathbf{H}$, and so on; $F(u(t))\,(x) = F(x, u(x, t))$, $bf(t)\,(x) = b(x)f(t)$, etc. The prime indicates t-derivative; x-derivatives will be indicated, as above, by the subindex x. We shall revert one or two times in this section, however, to the notation used at the beginning for reasons of convenience.

Theorem 2.1 is not, strictly speaking, contained in any of the results in [10]. However, the proof can be carried out essentially in the same way as that of Theorem 1.3 in [10], Chapter 1, that is by the Faedo-Galerkin approximation method combined with a priori estimates and compactness properties of various function spaces. Since these a priori estimates have exactly the same aspect as other estimates we will need to derive later in this section for different purposes, we omit the proof of Theorem 2.1.

We note, finally, that it follows from the theory of Lebesgue-Bochner integration of vector-valued functions that, by modifying u, u' in a null set, we may assume that u' is the indefinite Lebesgue-Bochner integral of u'', both thought of as L^2-valued functions; likewise, u is the indefinite integral of u', both thought of as K-valued functions. This will be important in what follows.

We study in the sequel the nonlinear map

$$\Phi\colon L^2(0, T) \to L^\infty(0, T; \mathbf{H}\times\mathbf{K}) \tag{2.16}$$

defined by

$$\Phi(f)\,(t) = (u(t), u'(t)) \in \mathbf{H}\times\mathbf{K} \qquad (0 \leq t \leq T), \tag{2.17}$$

where $u(\cdot)$ is the solution of (2.14) with $u(0) = u'(0) = 0$ produced by Theorem

2.1. The space $\mathbf{H}\times\mathbf{K}$ is equipped with its Hilbert product norm. In all the results that follow, we assume without explicit mention that the assumptions contained in Theorem 2.1 are fulfilled; however, to simplify somewhat the notation we shall assume that the boundary conditions are either (I) or (II) and that (2.9) is verified with $\nu = 0$. We indicate later how these additional assumptions can be removed. In the various estimates that will be derived, $C, C', \cdots$ denote positive constants—not necessarily the same for different inequalities—which are, in each case, independent of the parameters subject to variation.

LEMMA 2.1. *Let $C > 0$ be given. Then there exists $C' > 0$ such that*

$$\|\Phi(f)(t)\| \leq C' \tag{2.18}$$

whenever $|f| \leq C$.

Proof. We begin by observing that $F(x, u) = uG_u(x, u^2)$ $(0 \leq x \leq L, -\infty < u < \infty)$, where $G(x, u) = 2\int_0^{u^{1/2}} F(x, v)\,dv$ $(0 \leq x \leq L, u \geq 0)$. Since $G_u = u^{-1/2}F(x, u^{1/2})$, it is clear that G is continuously differentiable in the half-strip $0 \leq x \leq L$, $u \geq 0$.

Taking the scalar product of (2.14) with $u'(t)$ in L^2 we obtain

$$\frac{d}{dt}\tfrac{1}{2}(u'(t), u'(t)) + \frac{d}{dt}\tfrac{1}{2}(\mathbf{A}^{1/2}u(t), \mathbf{A}^{1/2}u(t)) + \frac{d}{dt}\tfrac{1}{2}\int_0^L G(u^2(t))\,dx = f(t)(b, u'(t)) \quad (0 \leq t \leq T). \tag{2.19}$$

Integrating,

$$(u'(t), u'(t)) + (\mathbf{A}^{1/2}u(t), \mathbf{A}^{1/2}u(t)) + \int_0^L G(u^2(t))\,dx = 2\int_0^t f(s)(b, u'(s))\,ds \qquad (0 \leq t \leq T). \tag{2.20}$$

We make now use of (2.5), of the fact that G is non-negative, of Schwarz's inequality and of the elementary inequality $ab \leq (a^2+b^2)/2$. We obtain an inequality of the type

$$|u'(t)|^2 + |u(t)|_{\mathbf{K}}^2 \leq C\left(\int_0^t |f(s)|^2\,ds + \int_0^t |u'(s)|^2\,ds\right), \quad (0 \leq t \leq T). \tag{2.21}$$

It follows from Gronwall's inequality [2] that

$$|u'(t)|^2 \leq C\int_0^T |f(s)|^2\,ds \qquad (0 \leq t \leq T), \tag{2.22}$$

and again from (2.21) that

$$|u(t)|_{\mathbf{K}}^2 \leq C\int_0^T |f(s)|^2\,ds \qquad (0 \leq t \leq T). \tag{2.23}$$

To obtain from these estimates a *uniform* bound for $u(x, t)$ we note that $\mathbf{K}$ may be characterized as the space of all functions $y \in L^2$ with Fourier development

$$y(x) = \sum_{n=0}^{\infty} c_n\varphi_n(x), \qquad \sum_{n=0}^{\infty} |\omega_n c_n|^2 = |y|_{\mathbf{K}} < \infty$$

or, equivalently, as the space of all y absolutely continuous in $[0, L]$, such that $y' \in L^2$ and

$$y(0) = y(L) = 0 \tag{2.24}$$

in the case of boundary condition (I) or

$$y(0) = 0 \tag{2.25}$$

in the case of boundary condition (II). The norm $|\cdot|_{\mathbf{K}}$ is in both cases equivalent to $|\cdot|_1$, where $|y|_1^2 = |y|^2 + |y'|^2$. It is not hard to show, using the norm $|\cdot|_1$, that

$$|y(x)| \le C|y|_{\mathbf{K}} \qquad (0 \le x \le L, y \in \mathbf{K}) \tag{2.26}$$

so that (2.23) implies

$$|u(x, t)|^2 \le C \int_0^T |f(s)|^2 \, ds \qquad (0 \le x \le L). \tag{2.27}$$

We wish now to prove that $F(u(t) \in \mathbf{K}$ and $|F(u(t)|_{\mathbf{K}}^2 \le C \int_0^T |f(s)|^2 \, ds \; (0 \le t \le T)$. Since

$$F(x, u(x, t))_x = F_x(x, u(x, t)) + F_u(x, u(x, t)) u_x(x, t),$$

this follows from (2.23) and from the characterization of $\mathbf{K}$ commented on after that inequality. As a consequence of this characterization, we also obtain that $F(u(t)) = \sum_{n=0}^{\infty} F_n(t)\varphi_n$ with

$$\sum_{n=0}^{\infty} |\omega_n^2 F_n(t)|^2 = |F(u(t)|_{\mathbf{K}}^2 \le C \int_0^T |f(s)|^2 \, ds \qquad (0 \le t \le T). \tag{2.28}$$

We now take the scalar product in L^2 of both sides of (2.14) with each φ_n. Setting $u_n(t) = (u(t), \varphi_n)$, we obtain

$$u_n''(t) + \omega_n^2 u_n(t) = -F_n(t) + b_n f(t) \qquad (0 \le t \le T),$$

where $\{b_n\}$ are the Fourier coefficients of $b(x)$. Since $u_n(0) = u_n'(0) = 0$,

$$\begin{aligned} u_n(t) &= -\frac{1}{\omega_n} \int_0^t \sin \omega_n (t-s) F_n(s) \, ds + \frac{b_n}{\omega_n} \int_0^t \sin \omega_n (t-s) f(s) \, ds \\ &= v_n(t) + w_n(t) \qquad (0 \le t \le T). \end{aligned} \tag{2.29}$$

It follows immediately from (2.28) and (2.4) that

$$\sum_{n=0}^{\infty} |\omega_n^2 v_n(t)|^2 \le C \int_0^T |f(s)|^2 \, ds \qquad (0 \le t \le T). \tag{2.30}$$

On the other hand, because of (2.10), it is not difficult to see that $w_n(t)$ satisfies an estimate of the same type. Accordingly,

$$|u(t)|_{\mathbf{H}}^2 = \sum_{n=0}^{\infty} |\omega_n^2 u_n(t)|^2 \le C \int_0^T |f(s)|^2 \, ds. \tag{2.31}$$

Now differentiating (2.29), we obtain

$$u_n'(t) = \int_0^t \cos \omega_n (t-s) (F_n(s)+b_n f(s))\, ds \qquad (0 \le t \le T). \tag{2.32}$$

Operating with (2.32) in the same way as with (2.31) we deduce that

$$|u'(t)|_{\mathbf{K}}^2 \le C \int_0^t |f(s)|^2\, ds \qquad (0 \le t \le T), \tag{2.33}$$

which ends the proof of Lemma 2.1.

We define now

$$\Phi_T(f) = \Phi(f)(T) = (u(T), u'(T)) \in \mathbf{H} \times \mathbf{K} \tag{2.34}$$

for $f \in L^2(0, T)$.

THEOREM 2.2. *The map Φ_T is Fréchet continuously differentiable everywhere in $L^2(0, T)$. Its differential $d\Phi_T(f)$ at $f \in L^2(0, T)$ is given by*

$$d\Phi_T(f)h = (y(T), y'(T)), \tag{2.35}$$

where $y(\cdot)$ is the solution of

$$y''(t) + \mathbf{A}y(t) + F_u(u(t))y(t) = bh(t) \tag{2.36}$$

that satisfies $y(0) = y'(0) = 0$.

We note that equation (2.36) obeys an existence and uniqueness theorem of the same form as Theorem 2.1.

Proof of Theorem 2.3. Let $\tilde{u}$, u, y be, respectively, the solutions (with both initial data zero) of

$$\tilde{u}''(t) + \mathbf{A}\tilde{u}(t) + F(\tilde{u}(t)) = b(f(t)+h(t)), \tag{2.37}$$

$$u''(t) + \mathbf{A}u(t) + F(u(t)) = bf(t), \tag{2.38}$$

$$y''(t) + \mathbf{A}y(t) + F_u(u(t))y(t) = bh(t). \tag{2.39}$$

If we let $v(t) = \tilde{u}(t) - u(t)$, v satisfies

$$v''(t) + \mathbf{A}v(t) + F(\tilde{u}(t)) - F(u(t)) = bh(t). \tag{2.40}$$

If we take f fixed and h, say, obeying a constraint $|h| \le C$, then, by virtue of Lemma 2.1, $\tilde{u}$ and u remain bounded in K and, after (2.26), uniformly bounded in $0 \le x \le L$, $0 \le t \le T$. Since F, being differentiable, is locally Lipschitz continuous, we see that $|F(\tilde{u}(x, t)) - F(u(x, t))| \le C|\tilde{u}(x, t) - u(x, t)|$ so that

$$|F(u(t)) - F(u(t))| \le C|v(t)|. \tag{2.41}$$

Operating with (2.40) in the same way as in Lemma 2.1 with (2.14), we obtain

$$\frac{d}{dt} \tfrac{1}{2} (v'(t), v'(t)) + \frac{d}{dt} \tfrac{1}{2} (\mathbf{A}^{1/2}v(t), \mathbf{A}^{1/2}v(t))$$
$$= -((F(\tilde{u}(t)) - F(u(t))), v'(t)) + (bh(t), v'(t)).$$

Taking into account the obvious inequality $|v(t)|^2 \le t \int_0^t |v'(s)|^2\,ds$ and imitating the reasoning leading to (2.21), we obtain the inequality

$$|v'(t)|^2+|v(t)|_{\mathrm{K}}^2 \le C\left(\int_0^t |v'(s)|^2\,ds+\int_0^t |h(s)|^2\,ds\right),$$

wherefrom we deduce, by Gronwall's inequality, that

$$|v(t)|_{\mathrm{K}}^2 = |\tilde{u}(t)-u(t)|_{\mathrm{K}}^2 \le C\int_0^t |h(s)|^2\,ds. \tag{2.42}$$

Let now $\delta(t) = \tilde{u}(t)-u(t)-y(t)$. Then δ satisfies

$$\delta''(t)+A\delta(t) = -F_u(u(t))\delta(t)-P(t), \tag{2.43}$$

where

$$P(t) = F(\tilde{u}(t))-F(u(t))-F_u(u(t))\,(\tilde{u}(t)-u(t)) \tag{2.44}$$

Operating with (2.43) once again as with (2.14) and (2.40) we obtain easily the inequality

$$|\delta(t)|_{\mathrm{K}}^2 \le C\int_0^T |P(s)|^2\,ds \qquad (0 \le t \le T). \tag{2.45}$$

Making use of an argument similar to the one that led to (2.41) we obtain

$$P(t) = o(|\tilde{u}(t)-u(t)|), \qquad |\tilde{u}(t)-u(t)| \to 0 \tag{2.46}$$

uniformly in $[0, T]$, so that, in view of (2.45),

$$|\delta(t)|_{\mathrm{K}} = o(|u(t)-u(t)|), \qquad |u(t)-u(t)| \to 0 \tag{2.47}$$

also uniformly in $[0, T]$. Observe now that

$$\begin{aligned} P(t)_x = {} & F_x(\tilde{u}(t))-F_x(u(t))-F_{xu}(u(t))\,(\tilde{u}(t)-u(t)) \\ & +[F_u(\tilde{u}(t))-F_u(u(t))-F_{uu}(u(t))\,(\tilde{u}(t)-u(t))]u(t)_x \\ & + (F_u(\tilde{u}(t))-F_u(u(t))\,(\tilde{u}(t)-u(t))_x. \end{aligned}$$

Clearly, this implies that

$$|P(t)_x| = o(|\tilde{u}(t)-u(t)|), \qquad |\tilde{u}(t)-u(t)| \to 0 \tag{2.48}$$

uniformly in $0 \le t \le T$. It is easy to see that, for each t, $P(t)$ satisfies (2.24) in the case of boundary condition (I) and (2.25) in the case of boundary condition (II). This, combined with (2.48) and the comments following (2.23), yields

$$|P(t)|_{\mathrm{K}} = o(|\tilde{u}(t)-u(t)|), \qquad |\tilde{u}(t)-u(t)| \to 0 \tag{2.49}$$

uniformly in $0 \le t \le T$; moreover, in the same way we deduce that the relation (2.47) holds as well for $F_u(u(t))\delta(t)$. Putting now together this relation with (2.49) and making use of (2.42) we finally obtain

$$|-F_u(u(t))\delta(t)-P(t)|_{\mathrm{K}} = o(|h|), \qquad |h| \to 0$$

uniformly in $[0, T]$. Using now the Fourier series techniques applied in Lemma 2.1 we see that

$$|\delta(t)|_{\mathrm{H}}+|\delta'(t)|_{\mathrm{K}} = o(|h|), \qquad |h| \to 0, \tag{2.50}$$

again uniformly in $[0, T]$. This shows that Φ is Fréchet differentiable and that its differential $d\Phi(f)$ at $f \in L^2(0, T)$ is given by

$$d\Phi(f)h = (y(\cdot), y'(\cdot)), \tag{2.51}$$

where y is the same function used in the expression (2.35).

Let, finally, $\tilde{f}, f \in L^2\,(0, T)$, f fixed and $\tilde{f}$ such that $|\tilde{f}\,| \leq C$. If $(\tilde{u}(\cdot), \tilde{u}'(\cdot)) = \Phi(\tilde{f}\,)$, $(u(\cdot), u'(\cdot)) = \Phi(f)$, $(\tilde{y}(\cdot), \tilde{y}'(\cdot)) = d\Phi(\tilde{f}\,)h$ and $(y(\cdot), y'(\cdot)) = d\Phi(f)h$, the function $z(t) = \tilde{y}(t) - y(t)$ satisfies

$$z''(t) + Az(t) = -F(u(t))z(t) - (F(\tilde{u}(t)) - F(u(t)))y(t) \quad (0 \leq t \leq T). \tag{2.52}$$

Equation (2.52) can be treated exactly in the same way as (2.43). Taking (2.42) into account it is not hard to show that the right-hand side of (2.52) is $O(|\tilde{u}(t) - u(t)|) = O(|\tilde{f} - f|)$ as $\tilde{f} \to f$ in $L^2(0, T)$, uniformly in $0 \leq t \leq T$, and then that $|z(t)|_{\mathbf{H}} + |z'(t)|_{\mathbf{K}} \leq C|\tilde{f} - f|$. This shows that $d\Phi(f)$, as a function from $L^2(0, T)$ into the space of bounded linear operators from $L^2(0, T)$ into $L^\infty(0, T; \mathbf{H} \times \mathbf{K})$, is continuous.

Since all the estimates that were used to show continuous differentiability of Φ hold *everywhere* (and not just almost everywhere) in $0 \leq t \leq T$, the statements in Theorem 2.2 about Φ_T follow. This completes the proof.

The assumption that the boundary conditions are (I) or (II) and that $\nu = 0$ in (2.9) can be easily eliminated by writing (2.1) in the operational form

$$u''(t) + (\mathbf{A} - \nu I)u(t) + F(u(t)) + \nu u(t) = bf(t)$$

rather than in the form (2.14). One can then reason exactly as in the proof of Theorem 2.2, replacing $\mathbf{A}$ by $\mathbf{A} - \nu I$, $F(x, u)$ by $F(x, u) + \nu u$.

3. Russell's Results on Linear Controllability. Let $a(x) = F_u(x, 0)$ $(0 \leq x \leq L)$ and define

$$(\mathbf{B}_I y)\,(x) = -y''(x) + a(x)y(x) = (\mathbf{A}_I y)\,(x) + a(x)y(x), \tag{3.1}$$

where $D(\mathbf{B}_I) = D(\mathbf{A}_I) = \mathbf{H}_I$. The operators $\mathbf{B}_{II}$, $\mathbf{B}_{III}$ are similarly defined. As in the case of $\mathbf{A}$, $\mathbf{B}$ will denote any of the three operators defined above when distinction is not necessary.

The operator $\mathbf{B}$ shares most of the properties of $\mathbf{A}$; in particular, $\mathbf{B}$ is self-adjoint and possesses a sequence $\{\mu_n\}$ of distinct real eigenvalues such that $\mu_0 < \mu_1 < \mu_2 < \cdots$ corresponding to a complete orthonormal set of eigenfunctions $\psi_0, \psi_1, \psi_2, \cdots$. We have $\mu_0 > 0$. Indeed, because of assumption (2.9) of Theorem 2.1 and of the fact that F is odd in u, we must have $a(x) \geq -\nu$, where ν lies below the least eigenvalue λ_0 of $\mathbf{A}$. Accordingly, $(\mathbf{B}u, u) \geq (\mathbf{A}u, u) - \nu(u, u) = (\lambda_0 - \nu)\,(u, u)$. Setting $\mu_n = \sigma_n^2$, the sequence $\{\sigma_n\}$ obeys the asymptotic relationships

$$\begin{cases} \sigma_n = (\pi/L)\,(n+1) + O(1/n) & \text{as} \quad n \to \infty \quad \text{for} \quad \mathbf{B}_I \\ \sigma_n = (\pi/L)\,(n+\frac{1}{2}) + O(1/n) & \text{as} \quad n \to \infty \quad \text{for} \quad \mathbf{B}_{II} \\ \sigma_n = (\pi/L)n + O(1/n) & \text{as} \quad n \to \infty \quad \text{for} \quad \mathbf{B}_{III}. \end{cases} \tag{3.2}$$

For proofs of these and other related facts see [14], especially Chapter IV.

Let $\mathbf{B}^{1/2}$ be the unique non-negative self-adjoint square root of $\mathbf{B}$. Then, since $\mathbf{B}$ is a bounded perturbation of $\mathbf{A}$, $D(\mathbf{B}^{1/2}) = D(\mathbf{A}^{1/2}) = \mathbf{K}$. Let $y(\cdot)$ be a solution of

$$y''(t)+\mathbf{B}y(t) = bh(t) \qquad (0 \le t \le T) \tag{3.3}$$

with

$$y(0) = y'(0) = 0. \tag{3.4}$$

Then it is easy to see, taking the scalar product of both sides of (3.3) with each ψ_n, that if $y(t) = \sum_{n=0}^{\infty} y_n(t)\psi_n$, $b = \sum_{n=0}^{\infty} b_n\psi_n$, then

$$y_n(t) = \frac{b_n}{\sigma_n}\int_0^t \sin \sigma_n (t-s)f(s)\, ds \qquad (0 \le t \le T, n \ge 0). \tag{3.5}$$

Let now $y_T \in \mathbf{H}$, $y'_T \in \mathbf{K}$; then we have

$$y_T = \sum_{n=0}^{\infty} \alpha_n\psi_n, \qquad \sum_{n=0}^{\infty} |\sigma_n^2\alpha_n|^2 < \infty \tag{3.6}$$

$$y'_T = \sum_{n=0}^{\infty} \beta_n\psi_n, \qquad \sum_{n=0}^{\infty} |\sigma_n\beta_n|^2 < \infty. \tag{3.7}$$

According to (3.5) and the formula obtained from it by differentiating both sides, the solution $y(\cdot)$ will satisfy

$$y(T) = y_T, \qquad y'(T) = y'_T \tag{3.8}$$

if and only if the function $g(s) = f(T-s)$ satisfies

$$\int_0^T g(s) \sin \sigma_n s\, ds = \sigma_n\alpha_n/b_n \qquad (n \ge 0), \tag{3.9}$$

$$\int_0^T g(s) \cos \sigma_n s\, ds = \beta_n/b_n \qquad (n \ge 0). \tag{3.10}$$

The two moment problems (3.9), (3.10) can be subsumed into

$$\int_0^T g(s)\, e^{i\sigma_n s}\, ds = c_n \qquad (n = \cdots, -1, -0, 0, 1, \cdots), \tag{3.11}$$

where we have set $\sigma_n = -\sigma_{-n}$ for $n \le 0$ (note that we differentiate between 0 and -0 as subindices) and

$$c_n = b_{|n|}^{-1} (\beta_{|n|} + is_n\sigma_{|n|}\alpha_{|n|}), \tag{3.12}$$

where n varies over the same set as in (3.11) and $s_n = 1$ for $n \ge 0$, $s_n = 1$ for $n \le -0$. A moment problem more general than (3.12) was solved by Russell [12]. We state next (and prove, for the most part) the results in [12] that are relevant to our situation.

We assume that

$$b_n \neq 0, \qquad \liminf_{n\to\infty} n|b_n| > 0. \tag{3.13}$$

Then, in view of (3.2), if $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying (3.6) and (3.7) respectively, the sequence $\{c_n\}$ given by (3.12) belongs to l^2, that is,

$$\sum_{n=0}^{\infty} |c_n|^2 < \infty. \tag{3.14}$$

The moment problem (3.11) is a particular case of the abstract moment problem

$$(g, f_n) = c_n, \tag{3.15}$$

where $\{f_n\}$ is a sequence in a Hilbert space H and $\{c_n\}$ is a sequence of complex numbers. Assume that there exists $C > 0$ such that

$$\sum |c_n|^2 \leq C |\sum c_n f_n|^2 \tag{3.16}$$

for any finite sequence $\{c_n\}$. Then, as

$$\Big|f_n - \sum_{m \neq n} c_m f_m\Big|^2 \geq C^{-1} \Big(1 + \sum_{m \neq n} |c_m|^2\Big) \geq C^{-1},$$

the distance d_n from f_n to the subspace K_n generated by $\{f_m;\ m \neq n\}$ is positive. It is then an elementary exercise in Hilbert space theory to show that, if $r_n \in K_n$ is the unique element of K_n that lies closest to f_n, the functions $g_n = d_n^{-2}(f_n - r_n)$ provide a sequence $\{g_n\}$ biorthogonal to $\{f_n\}$ (i.e., such that $(g_m, f_n) = \delta_{mn}$. Clearly, any other sequence biorthogonal to $\{f_n\}$ contained in K (the subspace generated by $\{f_n\}$) must coincide with $\{g_n\}$.

Let now $\{c_n\}$ be a finite sequence of complex numbers. We wish to show that

$$|\sum c_n g_n|^2 \leq C \sum |c_n|^2. \tag{3.17}$$

Indeed, let $u = \sum c_n g_n$ and choose another finite sequence $\{d_m\}$ of complex numbers such that, if $v = \sum d_m f_m$, we have $|u - v| \leq \epsilon$ for some $\epsilon > 0$. Then

$$\begin{aligned}
|u|^2 &= (u, v) + (u, u - v) \leq (u, v) + \epsilon\, |u| \\
&= \sum c_n d_n + \epsilon\, |u| \leq (\sum |c_n|^2)^{1/2} (\sum |d_n|^2)^{1/2} + \epsilon\, |u| \\
&\leq C^{1/2} (\sum |c_n|^2)^{1/2}\, |v| + \epsilon\, |u| \\
&\leq C^{1/2} (\sum |c_n|^2)^{1/2} (|u| + \epsilon) + \epsilon\, |u|.
\end{aligned}$$

Letting $\epsilon \to 0$, (3.17) follows.

Inequality (3.17) shows that, if $\{c_n\} \in l^2$ the series $g = \sum c_n g_n$ converges in H and clearly furnishes a solution of the moment problem (3.15). Moreover, it is clear that g is the only solution of (3.15) that belongs to K (for if $\tilde{g}$ were another such solution, $\tilde{g} - g$ would have null scalar product with all of the f_n, thus would be zero itself).

We apply now these simple observations to the case $H = L^2(0, T)$ and to $f_n(s) = \{e^{i\sigma_n s}\}$, $n = \cdots, -1, -0, 0, 1, \cdots$. If $T > 2L$ and the σ_n are given by any of the three asymptotic formulas in (3.2) then there exists a n_0 such that

$$\sigma_n - \sigma_{n-1} \geq 2\pi/T \qquad (|n| \geq n_0). \tag{3.18}$$

It was shown by Ingham (see [12]) that (3.18) implies that the sequence $\{e^{i\sigma_n s}\}$ satisfies (3.16) and then the moment problem (3.15) has a solution for any square-summable sequence $\{c_n\}$. If the sequence $\{c_n\}$ is given by (3.12), then, as α_n and β_n are real numbers, $c_{-n} = \bar{c}_n$. This implies that the solution g of (3.15) must be real-valued. Indeed, both g and $\bar{g}$ are solutions of (3.15) and, since K is invariant through complex conjugation, $\bar{g} \in K$ and must then coincide with g.

Assume next that $T = 2L$. It has been proved by Paley and Wiener ([11], Chapter V) that there exists a constant $\kappa > 0$ such that, if

$$|\sigma_n - n| \leq \kappa \; (n = \cdots, -1, 0, 1, \cdots), \tag{3.19}$$

then the sequence $\{e^{i\sigma_n s}\}$ satisfies (3.16) in $L^2(0, T)$. (Moreover, it also satisfies an inequality of the type of (3.17) and is *total* in H). It was shown by Russell that these properties still hold if $\{\sigma_n\}$ satisfies only the weaker condition

$$|\sigma_n - n| \leq \kappa, \qquad |n| \geq n_0. \tag{3.20}$$

Then it is clear that in the case of boundary condition (I) the sequence $\{e^{i\sigma_n s}\} \cup \{1\}$ satisfies (3.20); accordingly, the moment problem (3.15) has a solution g for any square-summable $\{c_n\}$. This solution is not unique, but it is uniquely determined if we specify

$$\int_0^T g(s)\, ds. \tag{3.21}$$

The case of boundary condition (II) is even simpler; here the sequence $\{e^{i\pi s/2L}\, e^{i\sigma_n s}\}$ satisfies (3.20). Accordingly, it also satisfies (3.16) in $L^2(0, T)$ and a fortiori so does the sequence $\{e^{i\sigma_n s}\}$, which is obtained from (3.22) through the (unitary) operator of multiplication by $e^{-i\pi s/2L}$. We deduce that (3.15) has a unique solution g for any $\{c_n\}$ in l^2. By means of an argument similar to the one following (3.18) we deduce that g must be real: the same applies to the case of boundary condition (I) commented on above.

In the case of boundary condition (III), the sequence $\{e^{i\sigma_n s}\}$ does not verify (3.20). However, it can be made to satisfy that condition through removal of any of its elements. It follows that (3.15) cannot have, in general, a solution for arbitrary square-summable $\{c_n\}$.

It remains to settle the case $T < 2L$, which clearly we only have to do for boundary conditions (I) and (II). We begin by observing that not all g_n can vanish in the interval $(T, 2L)$. For, if this were true, the problem (3.15) in $L^2(0, 2L)$ for boundary condition (II) or the problem (3.15) plus (3.21) in the case of boundary condition (I) would have an infinity of solutions (we may define g arbitrarily in $(T, 2L)$) which contradicts the results obtained in the case $T = 2L$. Assume then that the moment problem (3.15) has a solution in $L^2(0, T)$, $T < 2L$, for all square-summable $\{c_n\}$, and let g_m be such that it does not vanish (a.e.) in $(T, 2L)$. Let $\{c_n\} = \{\delta_{mn}\}$. Then the solution of (3.15) extended to $(0, 2L)$ by setting $g(s) = 0$ in $(T, 2L)$ must coincide (by uniqueness in the case $T = 2L$) with g_m in $(0, 2L)$, which is absurd. We conclude that the moment problem (3.15) does not have in general a solution in the case $T < 2L$ no matter what boundary conditions we use.

4. Local Controllability of (2.1). We consider again the nonlinear equation (2.1). The assumptions on the coefficients are the same as those in Section 2, that is, F is twice continuously differentiable and odd in u in the strip $0 \leq x \leq L$, $-\infty < u < \infty$ and b belongs to $L^2(0, L)$.

THEOREM 4.1. *(I) Assume that for some* $\nu < \pi/L$

$$F(x, u)+\nu u \geq 0 \qquad (0 \leq x \leq L, u \geq 0), \tag{4.1}$$

and that, if $\{\psi_n\}$ *are the eigenfunctions of* $\mathbf{B}_I$, *then*

$$b(x) = \sum_{n=0}^{\infty} b_n \psi_n(x) \tag{4.2}$$

with $b_n \neq 0$ $(n \geq 0)$ *and*

$$0 < \liminf_{n\to\infty} n\,|b_n|, \qquad \limsup_{n\to\infty} n\,|b_n| < \infty. \tag{4.3}$$

Then if $T \geq 2L$, *there exist* $\epsilon > 0$, $\delta > 0$ *such that, for any* $u_T \in \mathbf{H}_I$, $u'_T \in \mathbf{K}_I$ *satisfying*

$$|u_T|_{\mathbf{H}} \leq \epsilon, \qquad |u'_T|_{\mathbf{K}} \leq \epsilon, \tag{4.4}$$

there exists $f \in L^2(0, T)$,

$$|f| \leq \delta \tag{4.5}$$

such that the solution of (2.1) satisfying boundary condition (I) and initial conditions

$$u(\cdot, 0) = u_t(\cdot, 0) = 0 \tag{4.6}$$

satisfies

$$u(\cdot, T) = u_T(\cdot), \qquad u_t(\cdot, T) = u'_T(\cdot), \tag{4.7}$$

(II) Assume that (a) For some $\nu < \pi/2L$ *(4.1) holds; (b) (4.2), (4.3) hold, where* $\{\psi_n\}$ *are the eigenfunctions of* $\mathbf{B}_{II}$. *Then if* $T \geq 2L$, *the conclusion of Part (I) holds as well. Moreover, if* $T = 2L$, *the map*

$$f(\cdot) \to (u_T(\cdot), u'_T(\cdot)) \tag{4.8}$$

is a homeomorphism between the δ*-sphere in* $L^2(0, T)$ *and the* ϵ*-sphere in* $\mathbf{H}\times\mathbf{K}$.

(III) Assume that (a) For some $\nu < 0$ *(4.1) holds; (b) (4.2), (4.3) hold, where* $\{\psi_n\}$ *are the eigenfunctions of* $\mathbf{B}_{III}$. *Then if* $T > 2L$, *the conclusion of Part (I) holds.*

Most of the work necessary for the proof of Theorem 4.1 has been already carried out. We begin by observing that the second condition in (4.3) is similar to the one required in Theorem 2.1 and subsequent results in Section 2, but it is formulated in terms of the eigenfunctions of $\mathbf{B}$ rather than those of $\mathbf{A}$ as in Section 2. However, both formulations are equivalent, as $\varphi_n(x) = \psi_n(x)+O(1/n)$ $(n \to \infty)$ uniformly with respect to x, $0 \leq x \leq L$ (see [14], Chapter IV). Regrettably, the first condition in (4.3) cannot be formulated in terms of the eigenfunctions of $\mathbf{A}$, which are explicitly known.

As for the proof of Theorem 4.1, it has been established in Section 2 that the map Φ, which is none other than (4.8), is continuously differentiable everywhere in the sense of Fréchet as a map from $L^2(0, T)$ into $\mathbf{H}\times\mathbf{K}$, and it was proved in Section 3 that $d\Phi(0)$ is *onto* under the hypotheses written down in Theorem 4.1 in each of the three cases. Then the conclusion follows from the inverse function theorem in the form stated in [4] (Chapter X, p. 263, Exercise 8). This result can be stated (in a slightly less general form, but amply sufficient for our purposes) as follows.

LEMMA 4.1. *Let E, F be two Banach spaces, Φ a continuously differentiable map from a neighborhood of zero in E into F such that $\Phi(0) = 0$. Assume $d\Phi(0)$ is onto. Then the range of Φ contains a neighborhood of zero in F.*

The additional information contained in Part (II) of Theorem 4.1 about the map (4.8) can be deduced from the standard inverse function theorem ([4], Theorem 10.2.5).

There remains the question of relating the results just obtained for the equation (2.1) with the original equation (1.1). We note that the map (1.8)–(1.9) is a homeomorphism between the space $\mathbf{H}(0, X; A_0, A_1, B_0, B_1)$ (defined as $\mathbf{H}$ but with reference to the interval $[0, X]$ and the general boundary conditions (1.2)) and the space $\mathbf{H}(0, L, \tilde{A}_0, \tilde{A}_1, \tilde{B}_0, \tilde{B}_1)$ and it is also a homeomorphism between $\mathbf{K}(0, X, A_0, A_1, B_0, B_1)$ and $\mathbf{K}(0, L, \tilde{A}_0, \tilde{A}_1, \tilde{B}_0, \tilde{B}_1)$ (these spaces are defined modifying the definition of $\mathbf{K}$ in a way similar to the one above for $\mathbf{H}$). Accordingly, our results can be applied *as long as the transformed boundary conditions (1.11) have one of the three forms (I), (II) or (III) of Section 2.* This is a serious restriction, as only the form (I) is kept through the transformation (1.8)–(1.9) save when rather fortuitous relations between ρ and p hold.

We note, finally, that as the map $u(\cdot, t) \to u(\cdot, T-t)$ transforms solutions of (2.1)—or of (1.1)—again into solutions and preserves the boundary conditions as well, the conclusion of Theorem 4.1 can be re-interpreted by interchanging the role of the initial condition (4.6) and the final condition (4.7); that is, Theorem 4.1 can be thought of as a result assuring that "every sufficiently small state can be brought to equilibrium" under the conditions set forth in the theorem.

We end by illustrating the results obtained with an example where all the necessary assumptions can be instantly verified.

COROLLARY 4.1. *The conclusions of Theorem 4.1 hold for the equation $u_{tt} = u_{xx} - u^3 + xf(t)$ $(0 \le x \le L)$ for any $L > 0$ and any of the boundary conditions (I) or (II).*

REFERENCES

[1] W. F. AMES, Discontinuity formation in solutions of homogenous non-linear hyperbolic equations possessing smooth initial data, *Internat. J. Non-Linear Mech.* **5** (1970), 413–428.

[2] R. BELLMAN, *Stability Theory of Differential Equations*, McGraw-Hill, New York, 1953.

[3] M. CIRINÀ, Boundary controllability of nonlinear hyperbolic systems, *J. SIAM Control* **7** (1969), 198–212.

[4] J. DIEUDONNÉ, *Foundations of Modern Analysis*, Academic Press, New York, 1960.

[5] F. A. FICKEN and B. A. FLEISHMAN, Initial-value problems and time-periodic solutions for a nonlinear wave equation, *Comm. Pure Appl. Math.* **10** (1957), 331–356.
[6] E. B. LEE and L. MARKUS, Optimal control for nonlinear processes, *Arch. Rat. Mech. Anal.* **8** (1961), 36–58.
[7] E. B. LEE and L. MARKUS, *Foundations of Optimal Control Theory*, Wiley, New York, 1967.
[8] J. L. LIONS and E. MAGENES, *Problèmes aux Limites non homogènes et Applications*, Vol. I, Dunod, Paris, 1968.
[9] J. L. LIONS and E. MAGENES, *Problèmes aux Limites non homogènes et Applications*, Vol. II, Dunod, Paris, 1968.
[10] J. L. LIONS, *Quelques Methodes de Résolution des Problèmes aux Limites non linéaires*, Dunod-Gauthier-Villars, Paris, 1969.
[11] F. RIESZ and B. SZ.-NAGY, *Leçons d'Analyse Fonctionnelle*, 3ème éd., Akadémiai Kiadó, Budapest, 1955.
[12] D. L. RUSSELL, Nonharmonic Fourier series in the control theory of distributed parameter systems, *J. Math. Anal. Appl.* **18** (1967), 542–560.
[13] L. SCHWARZ, *Etude des Sommes d'Exponentielles*, 2ème éd., Hermann, Paris, 1959.
[14] F. TRICOMI, *Equazioni Differenziali*, Einaudi, Torino, 1953.

(*Received 29 January 1973*).