# Generators for Topological Entropy and Expansiveness

by

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## 1. Introduction

The notion of topological entropy was introduced by Adler, Konheim, and McAndrew in [1], with some indications of the analogy to measure theoretic entropy. In this paper, we investigate the topological analogue of generators of measure-preserving transformations. The structure of flows with generators is examined, with the main result (Theorem (2.7)) being that every such flow is a transformation group homomorphic image of a subflow of a symbolic flow.

The notion of expansiveness in topological dynamics has been extensively investigated. In Section 3, we show that the discrete flows with generators are precisely the expansive flows. This connection between topological entropy and topological dynamics enables us to obtain significant information about each concept. On the one hand, the large class of flows which are known to be expansive yields flows in which the existence of a generator is not apparent. Moreover, we show that distal flows do not support generators since they are not expansive. On the other hand, we use the results from studying generators to obtain a new result about expansive flows as well as an alternate proof of a result due to W. Gottschalk [4, p. 345]. Moreover, we note some additional properties of flows with generators. Finally, in Section 4, we generalize the definition of a generator to an arbitrary transformation group. Although the notion of topological entropy is no longer applicable, we use the same techniques to provide results on expansive transformation groups analogous to those of expansive discrete flows.

Throughout this paper, $X$ will denote a compact Hausdorff space and $\varphi$ a homeomorphism of $X$ onto $X$. We introduce some modifications in the notation of [1]. Since the finite open covers are cofinal, it is no restriction to consider only finite open covers, and we shall do so. Moreover, we use the notion of refinement (i.e., $\mathscr{U}$ is refined by $\mathscr{V}$ ($\mathscr{U} \prec \mathscr{V}$) if each member of $\mathscr{V}$ is contained in some member of $\mathscr{U}$) and joins for any cover (i.e., whether it is an open cover or not). We let $\mathscr{A}$ denote the collection of finite open covers of $X$. Finally if $\mathscr{U} \in \mathscr{A}$ and $m$, $n$ are integers such that $m \leq n$, then $\mathscr{U}_{m,n} = \bigvee_{j=m}^{n} \varphi^{-j}\mathscr{U}$. All other notation follows [1].



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Let $\mathscr{U}_X$ be the compatible uniformity of $X$. Then $(X, \varphi)$ is *expansive* if there exists $\alpha \in \mathscr{U}_X$ (an *expansive index*) such that if $x, y \in X$ and $x \neq y$, then $(\varphi^n x, \varphi^n y) \notin \alpha$ for some integer $n$. If $(Y, \psi)$ is another discrete flow, we write $(X, \varphi) \precsim (Y, \psi)$ if there exists a continuous map $\rho$ from $X$ onto $Y$ such that $\rho\varphi = \psi\rho$. Such a $\rho$ is called a *transformation group homomorphism*.

## 2. Generators for the Topological Entropy

We now establish conditions on an open cover which imply that the entropy of $\varphi$ is given by the entropy of $\varphi$ with respect to this cover.

**LEMMA 2.1.** *If $\mathscr{U}, \mathscr{V} \in \mathscr{A}$ and $\mathscr{V} \prec \mathscr{U}_{-n,\,n}$ for some $n$, then $h(\varphi, \mathscr{V}) \leq h(\varphi, \mathscr{U})$.*

*Proof.* If $\mathscr{V} \prec \mathscr{U}_{-n,\,n}$, it follows that $\mathscr{V}_{0,\,k} \prec \mathscr{U}_{-n,\,n+k}$ for every positive $k$. Hence $H(\mathscr{V}_{0,\,k}) \leq H(\mathscr{U}_{-n,\,n+k}) \leq H(\mathscr{U}_{-n,\,0}) + H(\mathscr{U}_{0,\,n+k})$ and

$$\frac{1}{k+1} H(\mathscr{V}_{0,\,k}) \leq \frac{1}{k+1} H(\mathscr{U}_{-n,\,0}) + \frac{n+k+1}{k+1}\left(\frac{1}{n+k+1} H(\mathscr{U}_{0,\,n+k})\right),$$

since $k$ is positive. If we take limits, the result follows.

**Definition 2.2.** Let $\mathscr{U}, \mathscr{V} \in \mathscr{A}$. Then $\mathscr{V}$ is *$\varphi$-refined* by $\mathscr{U}$ (write $\mathscr{V} \prec_\varphi \mathscr{U}$) if for every bisequence $(A_i)$ of members of $\mathscr{U}$, there exists an integer $n$ and $B \in \mathscr{V}$ such that $\bigcap_{i=-n}^{n} \varphi^{-i}(A_i) \subset B$.

**LEMMA 2.3.** *Let $\mathscr{U}, \mathscr{V} \in \mathscr{A}$. Then $\mathscr{V} \prec_\varphi \mathscr{U}$ if and only if $\mathscr{V} \prec \mathscr{U}_{-n,\,n}$ for some $n$.*

*Proof.* Suppose that $\mathscr{V} \not\prec \mathscr{U}_{-n,\,n}$ for any $n$. Then for every $n$, there exists a sequence $(A_{in})_{-n}^{n}$ such that $\bigcap_{-n}^{n} \varphi^{-i}(A_{in}) \not\subset B$ for any $B \in \mathscr{V}$. Since $\mathscr{U}$ is finite, it follows easily by induction that there exists a bisequence $(A_i)$ such that $A_i = A_{im}$ for infinitely many $m$, say $I_i$, and $|i| \leq |j|$ implies $I_i \supset I_j$. Let $n$ be a positive integer. If $m \in I_n$ such that $m > n$, then

$$\bigcap_{-n}^{n} \varphi^{-i}(A_i) = \bigcap_{-n}^{n} \varphi^{-i}(A_{im}) \supseteq \bigcap_{-m}^{m} \varphi^{-i}(A_{im}).$$

Since $\bigcap_{-m}^{m} \varphi^{-i}(A_{im}) \not\subset B$ for any $B \in \mathscr{V}$, the same statement holds for $\bigcap_{-n}^{n} \varphi^{-i}(A_i)$. Thus, $\mathscr{V} \not\prec_\varphi \mathscr{U}$. Since the converse is clear, the lemma is established.

**Definition 2.4.** Let $\mathscr{U} \in \mathscr{A}$. Then $\mathscr{U}$ is a *generator* for $(X, \varphi)$ if for every bisequence $(A_i)$ of elements of $\mathscr{U}$, $\bigcap_{-\infty}^{\infty} \varphi^{-i}(A_i^-)$ is at most one point.

The analogy between Definition 2.4 and the corresponding measure-theoretic concept [9, p. 1] is quite clear. Moreover, it is easily seen that if $\mathscr{U}^*$ is the cover of all sets of the form $\bigcap_{-\infty}^{\infty} \varphi^{-i}(A_i^-)$, where $(A_i)$ is a bisequence in $\mathscr{U}$, then $\mathscr{U}$ is a generator if and only if $\mathscr{V} \prec \mathscr{U}^*$ for every $\mathscr{V} \in \mathscr{A}$.

**LEMMA 2.5.** *Let $\mathscr{U}$ be a generator for $(X, \varphi)$. Then $\mathscr{V} \prec_\varphi \mathscr{U}$ for every $\mathscr{V} \in \mathscr{A}$.*

*Proof.* Choose $\mathscr{V} \in \mathscr{A}$. Consider any bisequence $(A_i)$ in $\mathscr{U}$. Letting $C_n = \bigcap_{-n}^{n} \varphi^{-i}(A_i^-)$, it follows that $(C_n)$ is a nested sequence of closed sets. Since $\bigcap_0^\infty C_n$ is at most a point, $\bigcap_0^\infty C_n \subset B$ for some $B \in \mathscr{V}$. By compactness, $C_n \subset B$ for some $n$. Thus, $\bigcap_{-n}^{n} \varphi^{-i}(A_i) \subset B$. The result follows.

In general, the converse of Lemma 2.5 is undoubtedly false. However, if $X$ is a metric space and $\mathscr{U}$ is a closed-open cover (every set in $\mathscr{U}$ is both closed and open), the converse holds. This is easily seen by choosing for each $\epsilon > 0$ a cover $\mathscr{V}$ of $\epsilon$-neighborhoods and noting that every bisequence $(A_i)$ in $\mathscr{U}$ has diam $(\bigcap_{-\infty}^{\infty} \varphi^{-i}(A_i^-)) < 2\epsilon$.

The following result justifies the terminology of Definition 2.4.

**THEOREM 2.6.** *Let $\mathscr{U}$ be a generator for $(X, \varphi)$. Then $h(\varphi) = h(\varphi, \mathscr{U})$.*

*Proof.* By Lemma 2.5, $\mathscr{V} \prec_{\varphi} \mathscr{U}$ for every $\mathscr{V} \in \mathscr{A}$. The result follows by Lemmas 2.3 and 2.1.

The converse of Theorem 2.6 obviously fails by considering $\varphi$ to be the identity.

The next few results show that the flows with generators are intimately related to subflows of the symbolic flows on finite sets, i.e., the shift on bisequences on a finite space [5, Chapter 12].

**THEOREM 2.7.** *Suppose $(X, \varphi)$ has a generator. Then there exists a $d$-ary symbolic flow $(Y, \sigma)$ and a closed invariant subset $Z$ of $Y$ such that $(Z, \sigma) \rightrightarrows (X, \varphi)$.*

*Proof.* Suppose that $\mathscr{U} = \{A_0, \cdots, A_{d-1}\}$ is a generator for $(X, \varphi)$. Let $(Y, \sigma)$ be the $d$-ary symbolic flow and consider $Z = \{m \mid m = (m_i) \in Y$ and $\bigcap_{-\infty}^{\infty} \varphi^{-i}(A_{m_i}^-) \neq \varnothing\}$. Suppose that $(m^j)$ is a sequence in $Z$ such that $m^j \to m$. Set $\{y_j\} = \bigcap_{-\infty}^{\infty} \varphi^{-i}(A_{m_i^j}^-)$. By passing to a subsequence if necessary, assume that $y_j \to y$. Since for all $i$, $m_i^j \to m_i$, we have that $m_i^j = m_i$ for $j \geq j_i$. Thus $y_j \in \varphi^{-i}(A_{m_i}^-)$ for all $j \geq j_i$ and hence $y \in \varphi^{-i}(A_{m_i}^-)$. Hence $\bigcap_{-\infty}^{\infty} \varphi^{-i}(A_{m_i}^-) \neq \varnothing$ and $m \in Z$. Thus $Z$ is closed. Clearly $Z$ is invariant, since if $\bigcap_{-\infty}^{\infty} \varphi^{-i}(A_{m_i}^-) \neq \varnothing$, then

$$\bigcap_{-\infty}^{\infty} \varphi^{-i+1}(A_{m_i}^-) = \bigcap_{-\infty}^{\infty} \varphi^{-i}(A_{m_{i+1}}^-) = \bigcap_{-\infty}^{\infty} \varphi^{-i}(A_{\sigma(m)_i}^-) \neq \varnothing,$$

and $\sigma(m) \in Z$.

Define $\psi$: $Z \to X$ by $\psi(m) = \bigcap_{-\infty}^{\infty} \varphi^{-i}(A_{m_i}^-)$. Then

$$\psi\sigma(m) = \bigcap_{-\infty}^{\infty} \varphi^{-i}(A_{\sigma(m)_i}^-) = \bigcap_{-\infty}^{\infty} \varphi^{-i}(A_{m_{i+1}}^-) = \varphi\left(\bigcap_{-\infty}^{\infty} \varphi^{-i}(A_{m_i}^-)\right) = \varphi\psi(m)$$

for $m \in Z$. Since $\mathscr{U}$ is a cover, $\psi$ is clearly onto. Suppose $m^j \to m$. By the first paragraph, every convergent subsequence of $(\psi(m^j))$ converges to $\psi(m)$. Since $X$ is compact, it follows that $\psi(m^j) \to \psi(m)$. Thus $\psi$ is continuous. The result follows.

As a consequence of Theorem 2.7, we have that $(X, \varphi)$ is isomorphic to $(Z/R, \sigma)$, where $R$ is the compatible equivalence relation induced by $\psi$.

**COROLLARY 2.8.** *Suppose $(X, \varphi)$ has a generator. Then $X$ is metrizable.*

*Proof.* By Theorem 2.7, $(Z, \sigma) \rightrightarrows (X, \varphi)$, and $Z$ is metrizable. The proof is completed by the following well-known topological result: If $\rho$ is a continuous map from a second-countable compact space $W$ onto a Hausdorff space $V$, then $V$ is second-countable. For if $\mathscr{B}$ is a countable base for $W$, let $\mathscr{B}_1$ be the collection of finite unions of members of $\mathscr{B}$, and let $\mathscr{C}$ be the complements in $V$ of images of complements of members of $\mathscr{B}_1$. Since $\rho$ is closed, each $C \in \mathscr{C}$ is open. It is direct to verify that $\mathscr{C}$ is a base for $V$.

It is obvious that if $\mathscr{U}$ is a generator for $(X, \varphi)$ and $\mathscr{U} \prec \mathscr{V}$, then $\mathscr{V}$ is a generator for $(X, \varphi)$. We use this in the following corollary.

**COROLLARY 2.9.** *Suppose $X$ is zero-dimensional and $(X, \varphi)$ has a generator. Then there exists a d-ary symbolic flow $(Y, \sigma)$ such that $(X, \varphi)$ is imbedded in $(Y, \sigma)$.*

*Proof.* Since closed-open covers are cofinal, we can choose a closed-open generator $\mathscr{U}$ for $(X, \varphi)$. Let $\mathscr{V}$ be the partition generated by $\mathscr{U}$. Then $\mathscr{V}$ is a closed-open generator for $(X, \varphi)$, and if $(Z, \sigma)$ is the flow obtained from applying the construction of Theorem 2.7 to $\mathscr{V}$, it follows that $\psi$ is one-to-one. The proof is completed.

It is known in measure theory [8] that a generator yields the measure algebra underlying the space. We examine the corresponding topological notion.

*Remark 2.10. Let $\mathscr{U}$ be a generator for $(X, \varphi)$. Then $\bigcup_{-\infty}^{\infty} \mathscr{U}_{-n,\, n}$ is a base for the topology of $X$.*

*Proof.* Since each $\varphi^{-i}\mathscr{U}$ is a cover, it follows that every $x \in X$ has a bisequence $(A_{ix})$ in $\mathscr{U}$ such that $x \in \bigcap_{-\infty}^{\infty} \varphi^{-i}(A_{ix})$. Then

$$\bigcap_{-\infty}^{\infty} \varphi^{-i}(A_{ix}^{-}) = \bigcap_{-\infty}^{\infty} \varphi^{-i}(A_{ix}) = \{x\}.$$

Choose $B$ open and $x \in B$. Since $\bigcap_{-\infty}^{\infty} \varphi^{-i}(A_{ix}^{-}) \subset B$, it follows as in Lemma 2.5 that $\bigcap_{-n}^{n} \varphi^{-i}(A_{ix}) \subset B$ for some $n$. Since $x \in \bigcap_{-n}^{n} \varphi^{-i}(A_{ix})$, the result follows.

Note that we have actually shown that if $\mathscr{U} \in \mathscr{A}$ satisfies the property that each $x \in X$ has a bisequence $(A_{ix})$ in $\mathscr{U}$ with $\bigcap_{-\infty}^{\infty} \varphi^{-i}(A_{ix}^{-}) = \bigcap_{-\infty}^{\infty} \varphi^{-i}(A_{ix}) = \{x\}$, then $\bigcup_{-\infty}^{\infty} \mathscr{U}_{-n,\, n}$ is a base. In general, the converse of Remark 2.10 fails. However, we do have the following.

**COROLLARY 2.11.** *If $X$ is first countable and $\mathscr{U}$ is an open partition of $X$, then $\mathscr{U}$ is a generator for $(X, \varphi)$ if and only if $\bigcup_{-\infty}^{\infty} \mathscr{U}_{-n,\, n}$ is a base for the topology of $X$.*

*Proof.* Let $x \in X$ and $\{U_n \mid n \geq 1\}$ be a decreasing neighborhood base of $x$. Then for each $n$, there exists $B_n = \bigcap_{-j_n}^{j_n} \varphi^{-i}(A_i^n) \subset U_n$. Suppose $j_n = j$ for an infinite set of $n$'s, say $N_0$. Then there exists an infinite subset $N_1 \subset N_0$ and a sequence $(A_i)_{-j}^{j}$ such that $A_i^n = A_i$ for every $n \in N_1$. It follows that $\{x\} = \bigcap_{-j}^{j} \varphi^{-i}(A_i)$, and there exists a bisequence $(C_i)$ in $\mathscr{U}$ such that $\{x\} = \bigcap_{-\infty}^{\infty} \varphi^{-i}(C_i)$.

Now suppose that $j_n \to \infty$. We can assume that $(j_n)$ is increasing. Suppose that $n \leq m$. Since $x \in B_n \cap B_m$ implies that

$$B_n \cap \bigcap_{-j_n}^{j_n} \varphi^{-i}(A_i^m) \neq \varnothing,$$

and that $\mathscr{U}$ is a partition, it follows that $A_i^n = A_i^m = A_i$ if $|i| \leq j_n$. Thus, there exists a bisequence $(A_i)$ in $\mathscr{U}$ for which $A_i^n = A_i$ if $|i| \leq j_n$. It follows again that $\{x\} = \bigcap_{-\infty}^{\infty} \varphi^{-i}(A_i)$. Since $\mathscr{U}_{-\infty,\infty}$ is a partition, the proof is easily completed.

As a corollary, we note that if $\mathscr{U}$ is a closed-open cover for which $\bigcup_{-\infty}^{\infty} \mathscr{U}_{-n,n}$ is a base, then $(X, \varphi)$ has a generator. For, using the partition $\mathscr{V}$ generated by $\mathscr{U}$, it follows easily that $\mathscr{V}$ has the same property. The result follows by applying Corollary 2.11.

## 3. Expansive Flows and the Existence of Generators

We now show that the flows with generators are precisely the expansive flows.

**LEMMA 3.1.** *$(X, \varphi)$ is expansive if and only if there exists $\mathscr{U} \in \mathscr{A}$ such that for every bisequence $(A_i)$ in $\mathscr{U}$, $\bigcap_{-\infty}^{\infty} \varphi^{-i}(A_i)$ is at most one point.*

*Proof.* If $\alpha$ is an expansive index for $(X, \varphi)$, choose a symmetric open $\beta \in \mathscr{U}_X$ such that $\beta^2 \subset \alpha$. Let $\mathscr{U}$ be a finite covering of $\beta$-neighborhoods. Suppose that some bisequence $(A_i)$ in $\mathscr{U}$ has $\bigcap_{-\infty}^{\infty} \varphi^{-i}(A_i)$ containing two distinct points $x, y$. If $A_i = x_i\beta$, then for every $i$, $(\varphi^i x, \varphi^i y) \in (x_i\beta) \times (x_i\beta) \subset \beta^2 \subset \alpha$. But this contradicts the assumption that $\alpha$ is an expansive index.

Suppose that $\mathscr{U}$ satisfies the condition of the lemma on bisequences. Let $\alpha \in \mathscr{U}_X$ be a Lebesgue index for $\mathscr{U}$. It is easily seen that $\alpha$ is an expansive index for $(X, \varphi)$. The result follows.

We call a cover satisfying the condition of Lemma 3.1 a *weak generator.*

**THEOREM 3.2.** *$(X, \varphi)$ has a generator if and only if $(X, \varphi)$ is expansive.*

*Proof.* Suppose that $\mathscr{U}$ is a weak generator. For every $x \in X$, choose an open neighborhood $V_x$ and $A_x \in \mathscr{U}$ such that $V_x^- \subset A_x$. Then $X = \bigcup_{x \in F} V_x$ for some finite $F \subset X$. Letting $\mathscr{V} = \{V_x | x \in F\}$, it follows that $\mathscr{V}$ is a generator for $(X, \varphi)$. The converse being obvious, the proof is completed.

Note that, in general, a weak generator is probably not a generator.

Theorem 3.2 yields that a large class of flows have generators. Many toral flows are expansive (see [3, Theorem 4]) and the symbolic flows are expansive. It also shows that many minimal flows fail to satisfy the converse of Theorem 2.6 (e.g., the equicontinuous flows, cf. [1, Example 1a]). Moreover, the following result is immediate from Theorem 2.7 and Corollary 2.9.

**COROLLARY 3.3.** *Let $(X, \varphi)$ be expansive. Then* (1) *There exists a d-ary symbolic flow $(Y, \sigma)$ and a closed invariant subset $Z$ such that $(Z, \sigma) \precsim (X, \varphi)$;* (2) *If $X$ is zero-dimensional, then $(X, \varphi)$ is imbedded in some d-ary symbolic flow.*

The second statement of Corollary 3.3 is originally due to Gottschalk.

If $(X, \varphi)$ is a point-transitive or minimal, then $(Z, \sigma)$ can be chosen with the same property by choosing the orbit closure of a point in $Z$ which maps onto the transitive point in the first case and choosing an almost periodic point in $Z$ in the second case.

*Remark 3.4. Suppose $X$ is an infinite metric space and $(X, \varphi)$ is distal. Then $(X, \varphi)$ does not have a generator.*

*Proof.* By Theorem 3.2, we need only show that $(X, \varphi)$ is not expansive. But if $(X, \varphi)$ is expansive, then there exist distinct points $a, b$ such that $a$ and $b$

are positively asymptotic [2, Theorem 2]. This contradicts the assumption that $(X, \varphi)$ is distal. The result follows.

As a corollary, we have that the only closed distal subflows of a $d$-ary symbolic flow are the finite subflows.

If $X$ is metrizable, the weakly mixing flows [7] are "almost" expansive, since almost all points in the product have dense orbit. However, in general, they are not expansive and we cannot assert the existence of a generator. For example, consider the flow on the 4-torus given by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & -3 & 3 \end{bmatrix}.$$

By examining the eigenvalues, it is seen that the flow is weakly mixing (in fact, weakly mixing with respect to Lebesgue measure [6, p. 55]) but not expansive [3, Theorem 4].

*Remark 3.5. Suppose that $X$ is a connected finite-dimensional topological group and $\varphi$ is a group automorphism of $X$. If $(X, \varphi)$ has a generator, then $X$ is abelian.*

*Proof.* This follows from [10].

It is easy to see that if each of the transformation groups $(X_i, \varphi_i)$ $(i = 1, \cdots, n)$, where $X_i$ is compact Hausdorff, has a generator $\mathscr{U}_i$, then $\times\, \mathscr{U}_i$ is a generator for $\left(\bigtimes_{i=1}^{n} X_i, \bigtimes_{i=1}^{n} \varphi_i\right)$. The infinite analogue fails, since if $((X_i, \varphi_i) \mid i \in I)$ is an infinite collection, the only way for which $(\times X_i, \times \varphi_i)$ can be expansive is if $X_i$ is trivial for all but a finite number of $i$. Also, if $(X, \varphi)$ has a generator, then $h(\varphi) < \infty$. Thus, any flow with infinite entropy is not expansive. In particular, the shift flow (or symbolic flow) on an infinite compact metric space is not expansive, cf. [1, p. 315]. Finally, it follows that if $(X, \varphi)$ has a generator and if $n \neq 0$, then $(X, \varphi^n)$ has a generator, since the corresponding result holds for the expansive case. However, this is immediate by noting that if $\mathscr{U}$ is a generator for $(X, \varphi)$, then $\mathscr{U}_{0,\,|n|-1}$ is a generator for $(X, \varphi^n)$.

## 4. Generalizations

Instead of considering the homeomorphism $\varphi$, we now assume that $T$ is a discrete group such that $(X, T)$ is a transformation group. Motivated by Definition 2.4, we have the following definition.

**Definition 4.1.** Let $\mathscr{U} \in \mathscr{A}$. Then $\mathscr{U}$ is an *(abstract) generator* for $(X, T)$ if for every $T$-family $(A_t \mid t \in T)$ in $\mathscr{U}$, $\bigcap_{t \in T} (A_t^-) t^{-1}$ is at most a point.

If $T$ is any discrete group and $Z$ any finite set, then the left symbolic transformation group over $T$ to $Z$ is the transformation group $(Z^T, T)$ with action $(z_t \mid t \in T)t_0 = (z_{t_0 t} \mid t \in T)$, i.e., $T$ acts on the index set $T$ by group left multiplication. The following result generalizes Theorem 2.7.

**THEOREM 4.2.** *Suppose $(X, T)$ has a generator. Then there exists a d-ary left symbolic transformation group $(Z^T, T)$ and a closed T-invariant subset $Z_0$ of $Z^T$ such that $(Z_0, T) \precsim (X, T)$.*

*Proof.* The proof follows that of Theorem 2.7 with only minor modifications. If $\mathscr{U} = \{A_0, \cdots, A_{d-1}\}$ is a generator for $(X, T)$, let $Z = \{0, \cdots, d-1\}$ and consider $(Z^T, T)$. Let $Z_0 = \{n | n = (n_t) \in Z^T$ and $\bigcap_T(A^-_{n_t}t^{-1}) \neq \varnothing\}$. Let $n \in Z_0$. Then $\bigcap (A^-_{n_t}t^{-1}) \neq \varnothing$. If $s \in T$, then $\bigcap (A^-_{n_t}t^{-1}s) \neq \varnothing$. Letting $w^{-1} = t^{-1}s$, we have that $t = sw$ and $\bigcap (A_{n_{sw}}w^{-1}) \neq \varnothing$. Hence $ns \in Z_0$, and $Z_0$ is invariant. Using nets instead of sequences, we can show that $Z_0$ is closed.

Define $\psi$: $Z_0 \to X$ by $\psi(n) = \bigcap (A^-_{n_t}t^{-1})$. It easily follows that $\psi\pi^t = \pi^t\psi$ for all $t$, and that $\psi$ is continuous onto. The proof is completed.

The analogue of Corollary 2.9 follows in a similar fashion.

In precisely the same way as in Section 3, we have that $(X, T)$ is expansive if and only if there exists $\mathscr{U} \in \mathscr{A}$ with the property that for every $T$-family $(A_t | t \in T)$ in $\mathscr{U}$, $\cap A_{n_t}t^{-1}$ is at most one point, and that $(X, T)$ is expansive if and only if $(X, T)$ has a generator. Thus, we have the following result.

**THEOREM 4.3.** *Let $(X, T)$ be expansive. Then* (1) *There exists a d-ary left symbolic flow $(Z^T, T)$ and a closed T-invariant subset $Z_0$ of $Z^T$ such that $(Z_0, T) \precsim (X, T)$;* (2) *If $X$ is zero-dimensional, then $(X, T)$ can be imbedded in some d-ary symbolic flow $(Z^T, T)$.*

Theorem 4.3 (2) is originally due to Gottschalk (unpublished).

A large class of expansive flows which are not necessarily discrete flows is contained in the index-permuting transformation groups [7, Definition (2.2)]. For consider $(Y^I, G)$, with action $(yg)_i = y_{g(i)}$ and where $Y$ is finite, and suppose that $(I, G)$ is transitive, i.e., for all $i, j \in I$, there exists $g \in G$ such that $g(i) = j$. Let $i \in I$, and consider the index $\alpha_i = \{(y, z) | y, z \in Y^I$ and $y_i = z_i\}$. Suppose that $x, y \in Y^I$ and $x \neq y$. Then for some $j$, $x_j \neq y_j$. If $g \in G$ is such that $g(i) = j$, then $(xg)_i = x_{g(i)} = x_j \neq y_j = y_{g(i)} = (yg)_i$. Thus, $(xg, yg) \notin \alpha_i$, and $(Y^I, G)$ is expansive.

**THEOREM 4.4.** *Let $X$ be a zero-dimensional topological group. Let $T$ be a discrete group of automorphisms of $X$. Let $(X, T)$ be expansive. Then there exists a finite group $Z$ such that $(X, T)$ is imbedded in $(Z^T, T)$ both as a transformation group and as a subgroup.*

*Proof.* Note that $T$ acts on $Z^T$ as a group of automorphisms. By compactness, we can assume that a generator $\mathscr{U}$ is refined by $\{xU | x \in X\}$, where $U$ is a neighborhood of the identity of $X$. Since $X$ is zero-dimensional, $U$ contains some closed-open normal subgroup $N$. Then for some finite $F \subset X$, the finite closed-open partition $\mathscr{V} = \{xN | x \in F\}$ is a generator. The set $Z$ in Theorem 4.2 can be identified with the finite group $X/N$, and

$$Z_0 = \{n | n = (n_t) \in (X/N)^T \text{ and } \bigcap ((x_{n_t}N)^- t^{-1}) = \bigcap (x_{n_t}Nt^{-1}) \neq \varnothing\}.$$

Consider the inverse $\psi^{-1}$ of the homeomorphism $\psi$: $Z_0 \to X$. If $x = \bigcap x_{n_t}Nt^{-1}$, $y = \bigcap x_{m_t}Nt^{-1}$, then

$$xy = (\bigcap x_{n_t}Nt^{-1})(\bigcap x_{m_t}Nt^{-1}) = \bigcap (x_{n_t}x_{m_t})Nt^{-1},$$

since $xt \in x_{n_t}N$ and $yt \in x_{m_t}N$ implies that $(xy)t \in x_{n_t}x_{m_t}N$. Then

$$\psi^{-1}(x)\cdot\psi^{-1}(y) = (x_{n_t}N\,|\,t \in T)\cdot(x_{m_t}N\,|\,t \in T) = (x_{n_t}\, x_{m_t}N\,|\,t \in T) = \psi^{-1}(xy),$$

and $\psi^{-1}$ is a group homomorphism. The result follows.

Again, Theorem 4.4 is due to Gottschalk (unpublished).

The restriction that $T$ be discrete is not severe. Recalling that $(X, T)$ effective means $t = e$ if $\pi^t = id_X$, it is known that if $(X, T)$ is expansive effective, $X$ is compact Hausdorff, and $T$ is locally compact with equivalent left and right uniformities (in particular, $T$ can be locally compact abelian), then $T$ is discrete. A more general statement involving the period of a transformation group asserts that the non-trivial action of $T$ on $X$ is given by a discrete quotient group of $T$.

We also note that all the results, except Corollary 2.8, hold when $X$ is compact uniformizable.

Finally, there is a natural extension of Definition 2.4 in case that $\varphi$ is simply a continuous map. This is the notion of a one-sided generator. Namely, $\mathscr{U} \in \mathscr{A}$ is a *(one-sided) generator* if for every sequence $(A_i)$ of members of $\mathscr{U}$, $\bigcap_0^\infty \varphi^{-i}(A_i^-)$ is at most one point. With obvious modifications, everything in Section 2 through Theorem 2.6 is still valid.

The notion of a discrete semi-flow is known (see [3], for example). This corresponds to the notions in Section 1 when a continuous map acts on a space. Invariance in this case means that the image of a set is contained in the set. All the one-sided symbolic semi-flows have one-sided generators, and it is easy to generalize Theorem 2.7 to the corresponding statement with semi-flows and one-sided generators. Thus Corollaries 2.8 and 2.9 follow; moreover, it is easy to see that the invariant set $Z$ actually satisfies $\sigma Z = Z$. Using the notion of positively expansive introduced in [3] for semi-flows, we may generalize Lemma 3.1 and Theorem 3.2 immediately to obtain the result that $(X, \varphi)$ is positively expansive if and only if $(X, \varphi)$ has a one-sided generator. Thus, Corollary 3.3 holds for one-sided symbolic flows and positively expansive semi-flows.

We now note the surprising fact that the above one-sided notion, unlike many others in topological dynamics, does not generalize the two-sided notion. Indeed, there does not exist any homeomorphism on an infinite metric space satisfying this property. For suppose $X$ is an infinite metric space and $\mathscr{U}$ is a one-sided generator. Then it easily follows that diam $(\mathscr{U}_{0,\,n}) \to 0$ as $n \to \infty$. But it is shown in [1, Theorem, p. 316] that if $\mathscr{V} \in \mathscr{A}$, diam $(\mathscr{V}_{1,\,n})$ is bounded away from 0. Since $\varphi$ is a homeomorphism, by letting $\mathscr{V} = \varphi(\mathscr{U}) = \{\varphi(U)\,|\,U \in \mathscr{U}\}$, we obtain that diam $(\mathscr{U}_{0,\,n-1})$ is bounded away from 0, a contradiction. So homeomorphisms on infinite compact metric spaces never have one-sided generators or, in other words, a homeomorphism *never* yields a positively expansive action on an infinite compact metric space. This result provides a simple alternate proof of [5, Theorem 10.30] (the hypothesis that $X$ be self-dense in [5, Theorems 10.30 and 10.36] can be replaced by $X$ being infinite. Note that [3, Example, p. 319] shows that the above result fails in non-compact spaces, and it is easily seen that finite spaces can support positively expansive homeomorphisms. Moreover, the one-sided symbolic flows show that the result fails for continuous, open and closed maps. Since [1, Theorem, p. 316] can be generalized to an assertion about refinements of covers if $X$ is simply compact Hausdorff, we can obtain the same result for infinite compact Hausdorff spaces (cf. [2, Theorem 1]).

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