Contributions to the Theory of Generalized Differential Equations. I.*

by

T. F. BRIDGLAND, JR.

Florida State University

Introduction

The existence of an intimate relationship between the theory of ordinary differential equations $\dot{x} = f(t, x, u(t))$ involving more or less arbitrary "control" functions u and the theory of contingent equations initiated by Marchaud [1, 2] and Zaremba [3] over thirty years ago is by now well recognized. An examination of this relationship and the further development of the Marchaud-Zaremba Theory has been the object of several studies in the recent past [4, 5, 6]. Implicit, and in some cases explicit, motivation for these studies resides in the fundamental work of Filippov [7] and Roxin [8] on optimal control theory.

The original object of the research which led to the preparation of this paper was to generalize the results of [9] on the minimum miss distance problem and of [10], [11] on the problem of approximation of optimal trajectories. The decision to enlarge the scope of this article beyond the treatment of the aforementioned generalizations was predicated upon the following factors:

(i) the lack of a readily accessible, English language treatment of the Marchaud-Zaremba Theory;

(ii) the author's opinion that the family of solutions of a contingent equation should be the principal object of study in the theory rather than the "funnel" ("zone of emission") emphasized by some writers including Marchaud and Zaremba or the "attainable set" emphasized in recent work.

Thus the purposes of this paper are: (a) to present a generalization of the Marchaud-Zaremba Theory from the point of view enunciated in (ii) above; (b) to develop this generalized theory to an extent sufficient to permit establishment of generalized forms of the aforementioned results of [9, 10, 11]. The remainder of this introduction will be devoted to a discussion of the manner in which these purposes are accomplished in the sequel.

* This research was carried out while the author was associated with the University of Alabama Research Institute. It was supported in part by the U.S. Army Missile Command under contract DA-AH01-67-C1630 and in part by a stipendiary grant from Brown Engineering Company.

MATHEMATICAL SYSTEMS THEORY, Vol. 3, No. 1.
Published by Springer-Verlag New York Inc.

In Section 1 fundamental properties are obtained for spaces of nonvoid compact subsets of several different Banach spaces. Two results of particular significance for the Marchaud-Zaremba Theory should be mentioned here. The first (Theorem 1.1) cites the existence of an isometry between the space Γ^n (metrized by Hausdorff distance) of nonvoid, compact, convex subsets of Euclidean n-space E^n and the space G^n, suitably metrized, of positively homogeneous, subadditive functions on E^n. This isometry permits replacing the study of functions with range Γ^n by a study of functions with range in G^n and, in particular, permits a new proof of Zaremba's approximation theorem [3, II.8.] which is analytic in character as opposed to the geometric character of Zaremba's proof. The second result to which we alluded above is actually the body of results comprehended by Theorems 1.4, 1.5, Lemmas 1.9, 1.11 and Corollary 1.3. This body of results deals with the properties of the space $\mathscr{H}^n(I)$ (metrized by Hausdorff distance) of nonvoid compact subsets of the Banach space $\mathscr{C}^n(I)$ (supremum norm) of continuous functions on I into E^n, where I is a compact interval of E^1. In Section 2 it is shown (Theorems 2.3 and 2.5) that for appropriate choice of I, the solution family of the generalized differential equation we treat is an element of $\mathscr{H}^n(I)$. All of the known results[1] concerning the "funnel" and the "attainable set" then follow as corollaries from the results of Section 1 concerning the sets $F(H)$ and $G(t;H)$ respectively.

In Section 2 the Cauchy problem for generalized differential equations is stated; the main existence theorems (Theorems 2.3, 2.5), as well as some elementary approximation theorems, are proved. Our proof of Theorems 2.3 and 2.5 follows the general lines laid down by Zaremba; however, there are important points of difference. Zaremba defines a "contingent derivative" of a vector-valued function and then requires that the contingent derivative of a continuous function satisfy certain conditions in order that the function be called a solution of the contingent equation. Prompted by the concerns of modern control theory, we consider as solutions of a generalized differential equation only *absolutely* continuous functions whose ordinary derivatives satisfy specified conditions. Hence, the core of our proof of Theorems 2.3 and 2.5 rests on the arguments used by Filippov in [7]. Ważewski [5] has written on the equivalence of contingent equations and generalized differential equations.

In Section 3, the elementary approximation theorems of Section 2 are utilized to obtain properties of solution families of generalized differential equations. In particular, Theorems 3.3 and 3.4 relate to the important question of continuous dependence on initial data. In Section 4 further approximation theorems are proved, the results of this section providing the basis for the generalization of the approximation theory developed in [10, 11].

In Section 5 an optimization theory is developed which is abstractly equivalent to that of [9]. This theory is based on the elementary proposition that a real-valued, lower semicontinuous function defined on a metric space attains a minimum on each compact subset of the space (cf. [13, p. 944]). The context

[1] I.e. all those results involving only compactness. We leave out of account here generalizations of the classical theorems of Kneser and Hukuhara [12, pp. 15–18] which involve connectivity.

in which the theory is developed is that of a class $\mathscr{E}$ of functions having range in $\mathscr{H}^n(I)$. The defining properties of $\mathscr{E}$ (see (18)) are quite simple; the nonemptiness of $\mathscr{E}$ is guaranteed by the existence theory of Section 2. This optimization theory is thus free of any direct reference to generalized differential equations; this freedom permits a new perspective on the relationship between optimality and controllability.

In Sections 6 and 7 approximation theories for optimization are established, that of Section 6 being abstractly equivalent to that stated in [10, Theorem 5] and that of Section 7 being abstractly equivalent to that stated in [11, Theorems 2, 3]. These developments being carried out in the context described above for Section 5, again there is freedom from direct reference to generalized differential equations. In Section 8, the results of Sections 5, 6, and 7 are interpreted for generalized differential equations, Theorems 8.1, 8.2 and 8.3 being the explicit statements of the aforementioned extensions of the results of [9, 10, 11].

1. Metric Spaces of Compact Sets

Let E^n denote Euclidean n-space, the inner product of a, $b \in E^n$ being designated by $a \circ b$; the norm of $a \in E^n$ is defined as usual by $(a \circ a)^{1/2}$ and denoted by $\|a\|$. With n fixed we shall use $|t| + \|x\|$ as a norm for $(t, x) \in E^1 \times E^n$ and designate this norm by $\|(t, x)\|$. Given a compact interval $I \subset E^1$, we shall denote by $\mathscr{C}^n(I)$ the Banach space of all continuous functions $\varphi: I \to E^n$ normed by

$$\langle \varphi \rangle = \max \{\|\varphi(t)\|: t \in I\}.$$

In this paper we shall be primarily concerned with four spaces, denoted by Ω^n, $\Psi^n(I)$, $\mathscr{H}^n(I)$ and Γ^n. The first three are the spaces of nonvoid compact subsets of E^n, $I \times E^n$ and $\mathscr{C}^n(I)$, respectively, whereas Γ^n is the space of nonvoid, compact, convex subsets of E^n. Each of these spaces may be metrized by the Hausdorff distance, the notation for which is introduced next.

For A, $B \in \Omega^n$ the Hausdorff distance is denoted by $\rho(A, B)$, where

$$\begin{aligned} \rho(A, B) &= \max \{\bar{\rho}(A, B), \bar{\rho}(B, A)\}, \\ \bar{\rho}(A, B) &= \max \{\alpha(a, B): a \in A\}, \\ \alpha(a, B) &= \min \{\|x-a\|: x \in B\}. \end{aligned}$$

We assume for Γ^n the metric of Ω^n.

For A, $B \in \mathscr{H}^n(I)$ the Hausdorff distance is denoted by $\sigma(A, B)$, where

$$\begin{aligned} \sigma(A, B) &= \max \{\bar{\sigma}(A, B), \bar{\sigma}(B, A)\}, \\ \bar{\sigma}(A, B) &= \max \{\beta(a, B): a \in A\}, \\ \beta(a, B) &= \min \{\langle x-a \rangle: x \in B\}. \end{aligned}$$

For A, $B \in \Psi^n(I)$ the Hausdorff distance is denoted by $\nu(A, B)$, where

$$\begin{aligned} \nu(A, B) &= \max \{\bar{\nu}(A, B), \bar{\nu}(B, A)\}, \\ \bar{\nu}(A, B) &= \max \{\omega((t, x), B): (t, x) \in A\}, \\ \omega((t, x), B) &= \min \{\|(t-\tau, x-\xi)\|: (\tau, \xi) \in B\}. \end{aligned}$$

The remainder of this section will be devoted to an examination of properties of these spaces which will be useful in the sequel. We shall investigate as well some properties of functions having domain and/or range in these spaces. Our first result is actually a step in the proof of the triangle law for the Hausdorff distance.

LEMMA 1.1. *For A, B, $C \in \Omega^n$, $\bar{\rho}(A, B) \leqq \bar{\rho}(A, C)+\bar{\rho}(C, B)$. Analogous results hold for Γ^n, $\Psi^n(I)$ and $\mathscr{H}^n(I)$.*

Proof. Let a, c denote points in A, C respectively; then $\alpha(a, B) \leqq \|a-c\|+\alpha(c, B)$ for all $c \in C$. Hence $\alpha(a, B) \leqq \alpha(a, C)+\alpha(c, B)$ for some $c \in C$ and then $\bar{\rho}(A, B) \leqq \bar{\rho}(A, C)+\alpha(c, B)$ for some $c \in C$. From this the assertion follows.

LEMMA 1.2. *For A, $B \in \Omega^n$ define the gap, $\gamma(A, B)$, between A, B by*

$$\gamma(A, B) = \min \{\|a-b\|: a \in A; b \in B\};$$

then $\gamma(\cdot, \cdot)$ is a uniformly Lipschitz continuous function on $\Omega^n \times \Omega^n$. Again, analogous results hold for Γ^n, $\Psi^n(I)$ and $\mathscr{H}^n(I)$.

Proof. Let A_0, B_0, A, $B \in \Omega^n$ and let a_0, b_0, a, b be points of the respective sets which satisfy $\|a_0-b_0\| = \gamma(A_0, B_0)$, $\|a-b\| = \gamma(A, B)$. Let $a^* \in A$, $b^* \in B$ be points nearest a_0, b_0 respectively. Then

$$\gamma(A, B) = \|a-b\| \leqq \|a^*-b^*\| \leqq \|a^*-a_0\|+\|b^*-b_0\|+\|a_0-b_0\|,$$

and this in turn implies

$$\gamma(A, B)-\gamma(A_0, B_0) \leqq \alpha(a_0, A)+\alpha(b_0, B) \leqq \bar{\rho}(A_0, A)+\bar{\rho}(B_0, B).$$

From this estimate and symmetry there follows

$$|\gamma(A, B)-\gamma(A_0, B_0)| \leqq \rho(A, A_0)+\rho(B, B_0).$$

Since $\alpha(a, B) = \gamma(\{a\}, B)$, the next result is an immediate consequence of Lemma 1.2.

COROLLARY 1.1. *The functions $\alpha(\cdot, \cdot)$, $\beta(\cdot, \cdot)$, $\omega(\cdot, \cdot)$ are uniformly Lipschitz continuous.*

From the proof of Lemma 1.2 it is evident that a general notion of semicontinuity is desirable for functions having domain and/or range in one of the Hausdorff metric spaces introduced earlier. Hence, let $\mathscr{M}_1$, $\mathscr{M}_2$, $\mathscr{M}_3$, be generic symbols for any of these spaces with the corresponding Hausdorff distances denoted by μ_1, μ_2, μ_3. Thus, for example, if $\mathscr{M}_1$ be taken as Ω^n then $\bar{\mu}_1$ is identified with $\bar{\rho}$.

Definition 1.1. *A function H: $\mathscr{M}_1 \times \mathscr{M}_2 \to \mathscr{M}_3$ is said to be upper semicontinuous at a point $(a_0, b_0) \in \mathscr{M}_1 \times \mathscr{M}_2$ if and only if for $\epsilon > 0$ there exists $\delta = \delta(\epsilon, a_0, b_0) > 0$ such that $\bar{\mu}_1(a, a_0) < \delta$ and $\bar{\mu}_2(b, b_0) < \delta$ imply $\bar{\mu}_3(H(a, b), H(a_0, b_0)) < \epsilon$.*

Since each of E^n, $I \times E^n$ and $\mathscr{C}^n(I)$ is isometrically imbedded in Ω^n, $\Psi^n(I)$ and $\mathscr{H}^n(I)$ (we say that E^n *generates* Ω^n, etc.), the corresponding definition of upper semicontinuity, when any of $\mathscr{M}_1$, $\mathscr{M}_2$ is replaced by its generating space, is obtained from Definition 1.1 by replacing the $\bar{\mu}$ in question by the metric of the corresponding generating space. The definition of lower semicontinuity is dual to that of upper semicontinuity in the sense that $\bar{\mu}_1(a, a_0)$, $\bar{\mu}_2(b, b_0)$, $\bar{\mu}_3(H(a, b), H(a_0, b_0))$ are replaced respectively by $\bar{\mu}_1(a_0, a)$, $\bar{\mu}_2(b_0, b)$, $\bar{\mu}_3(H(a_0, b_0), H(a, b))$. It is easy to see that a function which is both upper and lower semicontinuous at a point is continuous at the point.

Let us confine our attention briefly to the space Γ^n. For $A \in \Gamma^n$ we define the *support function*, $g(A, \cdot)$, of A by

$$g(A, p) = \max\{p \circ \sigma \colon \sigma \in A\}.$$

The fundamental properties of support functions of compact convex sets are detailed in [14, 15, 16]. In particular, for fixed $A \in \Gamma^n$, $g(A, \cdot)$ is positively homogeneous and subadditive on E^n—hence convex there—and by [15, Theorem 24] $g(A, \cdot)$ is thus continuous on E^n. Hence for $A, B \in \Gamma^n$ we may define $\bar{\Delta}(A, B)$ by

$$\begin{aligned} \bar{\Delta}(A, B) &= \max\{g(A, p) - g(B, p) \colon \|p\| = 1\}, \quad A \cap B' \neq \varnothing \\ \bar{\Delta}(A, B) &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad , A \cap B' = \varnothing, \end{aligned}$$

where the prime denotes complement with respect to E^n and $\varnothing$ denotes the null set. We define $\Delta(A, B)$ by

$$\Delta(A, B) = \max\{\bar{\Delta}(A, B), \bar{\Delta}(B, A)\}.$$

The easy proof of the next result is omitted (cf. [14, p. 35]).

LEMMA 1.3. *For $A, B \in \Gamma^n$, $\bar{\rho}(A, B) = \bar{\Delta}(A, B)$ so that $\rho(A, B) = \Delta(A, B)$; moreover,*

$$\Delta(A, B) = \max\{|g(A, p) - g(B, p)| \colon \|p\| = 1\}.$$

Somewhat more fundamental is the next lemma.

LEMMA 1.4. *If $A \in \Gamma^n$ has a nonvoid interior, then the interior of A has the representation*

$$\text{int } A = \{y \in E^n \colon y \circ p < g(A, p) \text{ for all } p \neq 0\}.$$

Proof. From [16, Theorem 5.3] we have

$$A = \{y \in E^n \colon y \circ p \leqq g(A, p) \text{ for all } p \neq 0\}.$$

Suppose that $y \in A$ satisfies $y \circ p_0 = g(A, p_0)$ for some $p_0 \neq 0$; then $(y + \eta p_0) \circ p_0 > g(A, p_0)$ for all $\eta > 0$ and y is thus a boundary point. On the other hand, if y is a boundary point of A, there exists a support hyperplane to A at y with normal $p_0 \neq 0$. This hyperplane has the expression $\{z \colon z \circ p_0 = g(A, p_0)\}$, so that one must have $y \circ p_0 = g(A, p_0)$.

A result closely related to Lemma 1.4 is Fenchel's theorem:

THEOREM 1.1. [16, pp. 62, 63] *If g is a positively homogeneous, subadditive function on E^n into E^1, then the set $A(g)$, defined by*

$$A(g) = \{z \in E^n \colon z \circ p \leqq g(p) \text{ for all } p \neq 0\},$$

is in Γ^n and g is its support function.

Now let us denote by G^n the space of all positively homogeneous, subadditive functions on E^n into E^1; for $g_1, g_2 \in G^n$ we define:

$$\bar{\nabla}(g_1, g_2) = \max\{g_1(p)-g_2(p): \|p\| = 1\} \text{ if there exists } p_0 \neq 0 \text{ such that } g_1(p_0) > g_2(p_0),$$

$$\bar{\nabla}(g_1, g_2) = 0 \text{ if } g_1(p) \leqq g_2(p) \text{ for all } p \neq 0;$$

$$\nabla(g_1, g_2) = \max\{\bar{\nabla}(g_1, g_2), \bar{\nabla}(g_2, g_1)\}.$$

It is easy to see that

$$\nabla(g_1, g_2) = \max\{|g_1(p)-g_2(p)|: \|p\| = 1\}$$

and that ∇ is a metric for G^n. The next result then follows from Lemma 1.3, Theorem 1.1 and [16, Theorem 5.3].

THEOREM 1.2. *If $A(g)$ is defined as in Theorem 1.1, then the mapping defined by $g \to A(g)$ is an isometry of G^n onto Γ^n with inverse given by $A \to g(A, \cdot)$. In fact we have for $g_1, g_2 \in G^n$*

$$\bar{\nabla}(g_1, g_2) = \bar{\Delta}(A(g_1), A(g_2)).$$

Definition 1.2. A function g: $I \times E^n \to G^n$ is said to be upper semicontinuous at a point $(t_0, x_0) \in I \times E^n$ if and only if for $\epsilon > 0$ there exists $\delta = \delta(\epsilon, t_0, x_0) > 0$ such that $\|(t-t_0, x-x_0)\| < \delta$ implies $\bar{\nabla}(g(t, x), g(t_0, x_0)) < \epsilon$. Lower semicontinuity is defined dually.

The next result follows readily from Theorem 1.2 and Lemma 1.3.

COROLLARY 1.2. *A function R: $I \times E^n \to \Gamma^n$ is continuous [upper semicontinuous], (lower semicontinuous) if and only if the function g_R: $I \times E^n \to G^n$ defined by $g_R(t, x) = g(R(t, x), \cdot)$ is continuous [upper semicontinuous], (lower semicontinuous).*

Definition 1.3. *For $A \in \Omega^n$ the set $\{x \in E^n: \alpha(x, A) \leqq \eta\}$ is called the η-neighborhood of A and is denoted by A^η. Corresponding definitions for $\Psi^n(I)$ and $\mathscr{C}^n(I)$ may be stated in a similar way.*

LEMMA 1.5. *If $A \in \Omega^n[\Gamma^n, \Psi^n(I)]$, then for each $\eta > 0$, $A^\eta \in \Omega^n[\Gamma^n, \Psi^n(I)]$.*

Proof. In every case closure and boundedness follow from the continuity of α or ω (cf. proof of Lemma 1.10). If $A \in \Gamma^n$ the convexity of A^η is proved, for example, in [16, Theorem 3.5].

LEMMA 1.6. *If R: $I \times E^n \to \Gamma^n$ is upper semicontinuous and $D \subset I \times E^n$ is compact, then*

$$\sup\{|g(R(t, x), p)|: \|p\| = 1; (t, x) \in D\} < \infty.$$

Proof. Let $\epsilon = 1$; then there exists $\delta = \delta(t_0, x_0) > 0$ such that $\|(t-t_0, x-x_0)\| < \delta$ implies $\bar{\Delta}(R(t, x), R(t_0, x_0)) < 1$ and the sets

$$S(t_0, x_0) = \{(t, x) \in I \times E^n: \|(t-t_0, x-x_0)\| < \delta(t_0, x_0)\}$$

form an open cover of D from which one may extract a finite subcover $\bigcup_{i=1}^{k} S(t_i, x_i)$. Then $(t, x) \in D$ implies that

$$R(t, x) \subset \bigcup_{i=1}^{k} (R(t^i, x^i))^1,$$

this union being compact. But then so is the convex hull of this union and, denoting by g^* the support function of this hull, we find that for all $(t, x) \in D$

$$-g^*(-p) \leqq -g(R(t, x), -p) \leqq g(R(t, x), p) \leqq g^*(p).$$

Since g^* is continuous, we obtain

$$\sup \{|g(R(t, x), p)|: \|p\| = 1\} \leqq \max \{|g^*(p)|: \|p\| = 1\}.$$

Our next result was first proved by Zaremba [3, II.8]; the proof we shall give is motivated by a proof due to Hobson [17, pp. 151-152] of Baire's theorem on the monotone approximation to a semicontinuous function by continuous functions. Our proof also illustrates the usefulness of the support function and of Fenchel's theorem in dealing with functions having range in Γ^n. An earlier illustration of this usefulness appears in the proof of [11, Theorem 1].

THEOREM 1.3. *Let R: $D \to \Gamma^n$ be upper semicontinuous, where D is the sphere $\{(t, x) \in E^1 \times E^n: \|(t-t_0, x-x_0)\| \leqq r\}$; then there exists a sequence $\{R_m\}$ of functions R_m: $E^1 \times E^n \to \Gamma^n$ possessing the following properties:*

(i) *for each m, R_m is continuous on D;*
(ii) *for each m and all $(t, x) \in D$, $\bar{\Delta}(R_{m+1}(t, x), R_m(t, x)) = 0$;*
(iii) *for each m and all $(t, x) \in D$, $R(t, x) \subset \operatorname{int} R_m(t, x)$;*
(iv) *for each $(t, x) \in D$, $\lim_{m\to\infty}\bar{\Delta}(R_m(t, x), R(t, x)) = 0$.*

Proof. In the interest of brevity, in this proof we denote $E^1 \times E^n$ by M and a generic point of M by a single symbol such as y. We may extend g_R (*vide* Corollary 1.2) as an upper semicontinuous function to all of M by requiring

$$g_R(y) = g_R(r\|y-y_0\|^{-1}(y-y_0)), \ \|y-y_0\|>r;$$

a corresponding extension of R ensues by virtue of Theorem 1.1. Let K be a simplex in M containing D; by repeated barycentric subdivision one may obtain, corresponding to each value of m, a simplicial partition of K for which the fundamental subsimplices have diameter less than 2^{-m}. For each m, a point $y \in K$ has a unique representation of the form

$$y = \sum_{i=1}^{n+2} \alpha_i^m(y)\cdot y_i,$$

where the y_i, $i = 1, \cdots, n+2$, are the vertices of the fundamental subsimplex $\sigma_m(y)$ to which y belongs and where $\alpha_i^m(y) \geqq 0$, $\sum_{i=1}^{n+2}\alpha_i^m(y) = 1$ and the functions α_i^m are continuous on K. By virtue of Lemma 1.6 we may define

$$g^m(y_i, \nu) = \max \{\sup \{g(R(\bar{y}), \nu)|\bar{y} \in \sigma_m\}|\sigma_m \in \mathscr{F}_m(y_i)\}+2^{-m},$$

where $\|\nu\| = 1$ and $\mathscr{F}_m(y_i)$ is the (finite) family of fundamental subsimplices σ_m of which y_i is a vertex. Then for a point $y \in K$ we may define

$$g^m(y, \nu) = \sum_{i=1}^{n+2} \alpha_i^m(y)g^m(y_i, \nu).$$

Now we have $g(y, \nu)<g^m(y_i, \nu)$ for each vertex y_i of $\sigma_m(y)$ so that

(a) $$g(y, \nu) = \sum_{i=1}^{n+2} \alpha_i^m(y) g(y, \nu) < \sum_{i=1}^{n+2} \alpha_i^m(y) g^m(y_i, \nu) = g^m(y, \nu).$$

We may extend $g^m(y, \cdot)$ to all of E^n by defining

$$g^m(y, p) = \|p\| g^m(y, p/\|p\|), \; p \in E^n - \{0\},$$
$$g^m(y, 0) = 0,$$

and then the fact that $g^m(y, \cdot) \in G^n$ for each $y \in K$ follows from $g(y, \cdot) \in G^n$ and the construction. The continuity of the function g_m: $y \to g^m(y, \cdot)$ is a consequence of the continuity of the functions α_i^m together with the fact that

$$\max \{g^m(y_i, p) : y_i \in K; \; \|p\| = 1\} < \infty.$$

That

(b) $$g^m(y, p) \geqq g^{m+1}(y, p)$$

for all $y \in K$, $p \in E^n$ and all m is an easy consequence of the construction.

By virtue of the upper semicontinuity of g_R it follows that given $\eta \in (0, 1)$ there exists $\delta = \delta(\eta, y_0) \in (0, 1)$ such that $\|y - y_0\| < \delta$ implies $\bar{\nabla}(g_R(y), g_R(y_0)) < \eta/2$. If ζ is a point belonging to a fundamental subsimplex of order m which shares a vertex y_i with $\sigma_m(y_0)$, then

$$\|\zeta - y_0\| \leqq \|\zeta - y_i\| + \|y_i - y_0\| < 2^{-m} + 2^{-m} = 2^{-(m-1)}.$$

Hence, if we let $\mu = \min \{\eta, \delta\}$, it follows from our construction that

(c) $$\bar{\nabla}(g_m(y_0), g_R(y_0)) < \eta, \qquad m > 1 - (\ln \mu / \ln 2).$$

If we define

$$R_m(y) = \{z \in E^n : z \circ p \leqq g^m(y, p) \text{ for all } p \neq 0\},$$

then Theorem 1.1 implies that $R_m(y) \in \Gamma^n$ and that $g_m(y)$ is its support function. Now (i) follows from Corollary 1.2 and the continuity of g_m, whereas (ii) is implied by (b). Condition (iii) is a consequence of (a) and Lemma 1.4. Finally (iv) is implied by (c) and Theorem 1.2.

The next lemma seems to be due to Ważewski [5, Lemme 1]; it is a trivial consequence of Theorem 1.1 and [16, Theorem 5.3].

LEMMA 1.7. *If $A \in \Gamma^n$ and if x: $[a, b] \to E^n$ is absolutely continuous and its derivative $\dot{x}$ satisfies $\alpha(\dot{x}(t), A) = 0$ almost everywhere on $[a, b]$, then $\alpha((b-a)^{-1} (x(b) - x(a)), A) = 0$.*

Our final result in connection with Γ^n is

LEMMA 1.8. *Let R: $I \times E^n \to \Gamma^n$ be continuous and let $R(t, x)$ have a nonvoid interior for each $(t, x) \in I \times E^n$; then for each $(t_0, x_0) \in I \times E^n$ there exists $\delta = \delta(t_0, x_0) > 0$ such that*

$$\bigcap \{R(t, x) : \|(t - t_0, x - x_0)\| \leqq \delta\} \neq \varnothing.$$

Proof. Let ζ be an arbitrary point of the interior of $R(t_0, x_0)$; we demonstrate the existence of a δ for which ζ is in the given intersection. Suppose no such δ exists; then there exists a sequence $\{(t_m, x_m)\} \subset I \times E^n$ such that $(t_m, x_m) \neq (t_0, x_0)$, $\lim_{m \to \infty} \|(t_m - t_0, x_m - x_0)\| = 0$, and $\alpha(\zeta, R(t_m, x_m)) > 0$. Hence, by [16, Theorem 5.3], there exists a sequence $\{\nu_m\} \subset E^n$, $\|\nu_m\| = 1$, for which

$$\zeta \circ \nu_m > g(R(t_m, x_m), \nu_m);$$

let $\bar{\nu}$ be an accumulation point of $\{\nu_m\}$ and denote a subsequence converging to $\bar{\nu}$ by the same indices. From the continuity of R together with Corollary 1.2 there follows

$$\zeta \circ \bar{\nu} \geqq g(R(t_0, x_0), \bar{\nu});$$

by virtue of Lemma 1.4 this contradicts the status of ζ as an interior point of $R(t_0, x_0)$.

We now turn our attention to the space $\mathscr{H}^n(I)$. For a nonvoid subset H of $\mathscr{C}^n(I)$ we denote by $G(t; H)$ the set $\{\varphi(t) \colon \varphi \in H\}$.

Remark 1.1. It is elementary that if $H \subset \mathscr{C}^n(I)$, then $H \in \mathscr{H}^n(I)$ if and only if H is closed and conditionally compact. The Arzela-Ascoli theorem asserts that a set $H \subset \mathscr{C}^n(I)$ is conditionally compact if and only if it is both bounded and equicontinuous [18, pp. 166-167].

THEOREM 1.4. *If $H \in \mathscr{H}^n(I)$ then $G(t; H) \in \Omega^n$ for each $t \in I$.*

Proof. By Remark 1.1, together with [18, Theorems 3-15 IV, 3-8II], we know that for each $t \in I$, $G(t; H)$ is conditionally compact. There only remains to show that $G(t; H)$ is closed and this is trivial if H is finite. For the infinite case, let ξ be a point of the closure of $G(t; H)$ and let $\{\xi_m\} \subset G(t; H)$ satisfy $\lim_{m\to\infty}\xi_m = \xi$. Let $\{h_m\} \subset H$ satisfy $h_m(t) = \xi_m$; then there is a subsequence of $\{h_m\}$ which converges to a limit $\bar{h} \in H$. Hence $\bar{h}(t) = \xi$ so that $\xi \in G(t; H)$.

THEOREM 1.5. *$G(\cdot\,; \cdot)$ is continuous on $I \times \mathscr{H}^n(I)$.*

Proof. We make the following assertions: (a) for each $H \in \mathscr{H}^n(I)$, $G(\cdot\,; H)$ is continuous on I; (b) the family $\{G(t; \cdot) \colon t \in I\}$ is equicontinuous on $\mathscr{H}^n(I)$. The theorem then follows from these assertions *via* the triangle law. We shall prove (a) here; (b) is an immediate consequence of Lemma 1.9 below. By the definition and continuity of α, there exists $a \in G(t; H)$ such that

$$\alpha(a, G(t_0; H)) = \bar{\rho}(G(t; H), G(t_0, H));$$

let $b \in G(t_0; H)$ be a point nearest a. Now there exists $\varphi \in H$ such that $\varphi(t) = a$ and evidently $\|a - \varphi(t_0)\| \geqq \|a - b\|$; but, given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $|t - t_0| < \delta$ and $\varphi \in H$ imply $\|\varphi(t) - \varphi(t_0)\| < \epsilon$. Hence, $|t - t_0| < \delta$ implies $\bar{\rho}(G(t; H), G(t_0; H)) = \|a - b\| < \epsilon$, which yields upper semicontinuity. Lower semicontinuity follows by symmetry.

LEMMA 1.9. *If $H_1, H_2 \in \mathscr{H}^n(I)$, then*

$$\sup\{\bar{\rho}(G(t; H_2), G(t; H_1)) \colon t \in I\} \leqq \bar{\sigma}(H_2, H_1).$$

Proof. There exists $\varphi \in H_2$ such that:

$$\begin{aligned}\bar{\rho}(G(t; H_2), G(t; H_1)) &= \alpha(\varphi(t), G(t; H_1)) \\ &= \min\{\|\varphi(t) - \gamma\| \colon \gamma \in G(t; H_1)\} \\ &= \min\{\|\varphi(t) - \psi(t)\| \colon \psi \in H_1\} \\ &\leqq \min\{\langle \varphi - \psi \rangle \colon \psi \in H_1\} \\ &\leqq \beta(\varphi, H_1) \leqq \bar{\sigma}(H_2, H_1).\end{aligned}$$

Now let $\Pi: I \to \Omega^n$ and define

$$S_\Pi = \{(t, x) \in I \times E^n : \alpha(x, \Pi(t)) = 0\};$$

we have

LEMMA 1.10. *If* Π *is continuous, then* $S_\Pi \in \Psi^n(I)$.

Proof. This is a direct consequence of the compactness of $\{0\}$, the continuity of $\alpha(\cdot, \Pi(\cdot))$ and the easily verified fact that there exists a number $q \geqq 0$ such that

$$\alpha(x, \Pi(t)) + q \geqq \|x\|, \qquad (t, x) \in I \times E^n.$$

By virtue of Theorem 1.5 and Lemma 1.10 we may define $F: \mathscr{H}^n(I) \to \Psi^n(I)$ by

$$F(H) = \{(t, x) \in I \times E^n : \alpha(x, G(t; H)) = 0\};$$

it is easy to see that this is equivalent to

$$F(H) = \{(t, \varphi(t)) : t \in I;\ \varphi \in H\}.$$

We prove

LEMMA 1.11. *If* $H_1, H_2 \in \mathscr{H}^n(I)$, *then* $\bar{\nu}(F(H_2), F(H_1)) \leqq \bar{\sigma}(H_2, H_1)$.

Proof. There exists $(t, x) \in F(H_2)$ such that

$$\begin{aligned}\bar{\nu}(F(H_2), F(H_1)) &= \omega((t, x), F(H_1)) \\ &= \min\{\|(t-\tau, x-\xi)\| : (\tau, \xi) \in F(H_1)\} \\ &\leqq \alpha(x, G(t; H_1)) \leqq \bar{\rho}(G(t; H_2), G(t; H_1))\end{aligned}$$

and the assertion follows from this estimate and Lemma 1.9. An immediate consequence of Lemma 1.11 is

COROLLARY 1.3. $F(\cdot)$ *is uniformly Lipschitz continuous on* $\mathscr{H}^n(I)$.

Comment. It is easy to deduce that both $\nu(F(H_2), F(H_1))$ and max $\{\rho(G(t; H_2), G(t; H_1)) : t \in I\}$ are metrics for $\mathscr{H}^n(I)$. However, each is definitely a weaker metric than $\sigma(H_2, H_1)$, as the following simple example in $\mathscr{H}^1(I)$ shows. Letting $I = [-1, 1]$, define $y: I \times I \to I$ and $x: I \times I \to I$ by

$$\begin{aligned} y(t, \theta) &= \sin(t + \theta\pi/2), \\ x(t, \theta) &= \theta. \end{aligned}$$

If we define $H_1, H_2 \in \mathscr{H}^1(I)$ by

$$\begin{aligned} H_1 &= \{y(\cdot, \theta) : \theta \in I\}, \\ H_2 &= \{x(\cdot, \theta) : \theta \in I\}, \end{aligned}$$

it is easy to see that

$$\nu(F(H_2), F(H_1)) = \max\{\rho(G(t; H_2), G(t; H_1)) : t \in I\} = 0$$

but that $\sigma(H_2, H_1) > 0$.

The following convergence principle will prove to be of fundamental importance in the sequel.

THEOREM 1.6. *Let the sequence* $\{H_m\} \subset \mathscr{H}^n(I)$ *satisfy* $\bar{\sigma}(H_{m+1}, H_m) = 0$ *for all m and denote by* H^* *the set* $\bigcap H_m$; *then* $H^* \in \mathscr{H}^n(I)$ *and* $\lim_{m\to\infty} \bar{\sigma}(H_m, H^*) = 0$.

Proof. By Cantor's theorem $H^* \neq \varnothing$; it is certainly closed and then, as a subset of the compact space H_m, H^* is compact. Hence $H^* \in \mathscr{H}^n(I)$. There exists $h_m \in H_m$ such that $\bar{\sigma}(H_m, H^*) = \beta(h_m, H^*)$; let $\bar{\beta}$ be an accumulation point of $\{\beta(h_m, H^*)\}$ and denote by the same indices a subsequence of $\{h_m\}$ for which $\lim_{m\to\infty} \beta(h_m, H^*) = \bar{\beta}$. If $\bar{h}$ is an accumulation point of $\{h_m\}$, then $\beta(\bar{h}, H^*) = 0$; but then by continuity of β we have

$$\lim_{m\to\infty} \bar{\sigma}(H_m, H^*) = \lim_{m\to\infty} \beta(h_m, H^*) = \bar{\beta} = 0.$$

We close this section with the statement of a well-known lemma of Filippov [7].

LEMMA 1.12. *Let f: $I \times E^m \to E^n$ be continuous, let Q: $I \to \Omega^m$ be upper semicontinuous and define R: $I \to \Omega^n$ by*

$$R(t) = \{f(t, u)\colon u \in Q(t)\};$$

if z: $I \to E^n$ has Lebesgue measurable components and $\alpha(z(t), R(t)) = 0$ almost everywhere on I, then there exists u: $I \to E^m$ having Lebesgue measurable components which satisfies $\alpha(u(t), Q(t)) = 0$ on I and $z(t) = f(t, u(t))$ almost everywhere on I.

2. Generalized Differential Equations: Existence Theory

We start by formulating precisely the Cauchy problem for generalized differential equations.

PROBLEM. Let D be an open subset of $E^1 \times E^n$, (t_0, x_0) a fixed point in D and R: $D \to \Omega^n$ a given function. We wish to determine conditions on R sufficient to ensure the existence of an open interval $J \subset E^1$ and an absolutely continuous function x: $E^1 \to E^n$ satisfying:

(i) $t_0 \in J$ and $x(t_0) = x_0$;
(ii) for each $t \in J$, $(t, x(t)) \in D$;
(iii) $\dot{x}(t) \in R(t, x(t))$ almost everywhere in J.

We denote this problem by

$$(1) \qquad \dot{x} \in R(t, x), \qquad x(t_0) = x_0;$$

an absolutely continuous function x: $E^1 \to E^n$ satisfying (i), (ii), (iii) will be called a solution of (1). The existence theory we develop for (1) is parallel to that developed by Zaremba [3] for contingent equations; even our proofs are modelled largely on those given by Zaremba.

THEOREM 2.1. *Let R: $D \to \Gamma^n$ be continuous and satisfy* int $R(t, x) \neq \varnothing$ *on D; then for each $(t_0, x_0) \in D$ a solution of* (1) *exists.*

Proof. Since D is open there exists $\delta = \delta(t_0, x_0) > 0$ such that $S(t_0, x_0) \equiv \{t, x)\colon \|(t-t_0, x-x_0)\| \leqq \delta\} \subset D$ and, by taking δ sufficiently small we may have, by virtue of Lemma 1.8, $\bigcap \{R(t, x)\colon (t, x) \in S(t_0, x_0)\} \neq \varnothing$. Let ζ be an arbitrary element of this intersection. We define x: $E^1 \to E^n$ by $x(t) = x_0 + (t-t_0)\zeta$ and let

$$M = \max\{\sigma(t, x) : (t, x) \in S(t_0, x_0)\},$$

where

$$\sigma(t, x) = \max\{\|q\| : q \in R(t, x)\}.$$

The continuity of σ on D follows readily from that of R, so that M is well-defined. Let $\alpha = \delta(M+1)^{-1}$; then for $|t-t_0| \leqq \alpha$ we have $\|(t-t_0, x(t)-x_0)\| \leqq \delta$, so that $(t, x(t)) \in S(t_0, x_0)$. Moreover, on this same interval $\dot{x}(t) = \zeta \in R(t, x(t))$. Certainly $x(t_0) = x_0$ and this completes the proof.

COROLLARY 2.1. *Under the hypotheses of Theorem 2.1 every solution of* (1) *may be continued over the interval* $[t_0-\alpha, t_0+\alpha]$.

The proof is like that for ordinary differential equations.

COROLLARY 2.2. *Let* $Q: D \to \Gamma^n$ *satisfy the hypotheses of Theorem 2.1 as well as the condition* $\bar{\Delta}(Q(t, x), R(t, x)) = 0$ *on* $S(t_0, x_0)$*; then every solution of*

$$(2) \qquad \dot{x} \in Q(t, x), \qquad x(t_0) = x_0$$

may be continued over the interval $[t_0-\alpha, t_0+\alpha]$, *where* α *is that quantity determined relative to* R *in Theorem 2.1.*

Proof: Obviously every solution of (2) is a solution of (1) and then the assertion follows from Corollary 2.1.

Now let I be a common interval of existence for the solutions of (1) and denote by $\bar{H}_I(t_0, x_0)$ the family of restrictions to I of all solutions of (1); evidently $\bar{H}_I(t_0, x_0) \subset \mathscr{C}^n(I)$.

THEOREM 2.2. *Let* $R: D \to \Gamma^n$ *satisfy the hypotheses of Theorem 2.1 and let* $I = [t_0-\alpha, t_0+\alpha]$, *where* α *is determined as in the proof of Theorem 2.1; then* $\bar{H}_I(t_0, x_0) \in \mathscr{H}^n(I)$.

Proof. The case in which $\bar{H}(t_0, x_0)$ is finite being trivial, we consider $\bar{H}(t_0, x_0)$ to be infinite. From $x(t) = x_0 + \int_{t_0}^{t} \dot{x}(\tau)d\tau$ we have readily $\|x(t_2) - x(t_1)\| \leqq M|t_2-t_1|$ for $t_1, t_2 \in I$; thus $\bar{H}_I(t_0, x_0)$ is conditionally compact by Ascoli's theorem. Let $\{x^m\} \subset \bar{H}_I(t_0, x_0)$ be a sequence converging to a point φ in the closure of $\bar{H}_I(t_0, x_0)$; there follows easily $\|\varphi(t_2)-\varphi(t_1)\| \leqq M|t_2-t_1|$ for $t_1, t_2 \in I$, so that φ is in fact absolutely continuous and $\varphi(t_0) = x_0$. Now let R^η denote the function whose value on D is given by $(R(t, x))^\eta$; by Lemma 1.5 $R^\eta(t, x) \in \Gamma^n$ and it is easy to see that R^η is continuous. Since $\dot{x}^m(t) \in R(t, x^m(t))$ almost everywhere on I, equicontinuity implies that for sufficiently small $\delta = \delta(\eta) > 0$,

$$\dot{x}^m(t) \in R^{\eta/2}(t^*, x^m(t^*))$$

for almost all $t \in [t^*-\delta, t^*+\delta]$. If t^* is a point at which $\dot{\varphi}$ exists, then for all large m,

$$\dot{x}^m(t) \in R^\eta(t^*, \varphi(t^*))$$

almost everywhere on $[t^*-\delta, t^*+\delta]$. Hence it follows from Lemma 1.7 that

$$(t-t^*)^{-1}[x^m(t)-x^m(t^*)] \in R^\eta(t^*, \varphi(t^*))$$

for all $|t-t^*| \leqq \delta$. But then by the compactness of $R^\eta(t^*, \varphi(t^*))$ we find that

$$(t-t^*)^{-1}[\varphi(t)-\varphi(t^*)] = \lim_{m\to\infty} (t-t^*)^{-1}[x^m(t)-x^m(t^*)] \in R^\eta(t^*, \varphi(t^*)), \ |t-t^*| \leqq \delta.$$

By the same token $\alpha(\dot{\varphi}(t^*), R^\eta(t^*, \varphi(t^*))) = 0$, and if we let $\eta \to 0$, the continuity of α yields $\dot{\varphi}(t^*) \in R(t^*, \varphi(t^*))$. Thus $\bar{H}_I(t_0, x_0)$ is closed and then by Remark 1.1 the assertion of the theorem follows.

Comment. Of course the above proof is almost precisely that used by Filippov [7, Theorem 1]; we have reproduced it here partly for completeness and partly for later reference.

We are now in a position to prove our main existence theorem.

THEOREM 2.3. *Let R: $D \to \Gamma^n$ be upper semicontinuous; then for each $(t_0, x_0) \in D$ there exists a compact interval I having t_0 as midpoint for which $\bar{H}_I(t_0, x_0) \in \mathscr{H}^n(I)$.*

Proof. On a compact neighborhood of (t_0, x_0) contained in D one may approximate R by a sequence $\{R_m\}$ having the properties (i), $\cdots$, (iv) of Theorem 1.3. In a perhaps smaller neighborhood of (t_0, x_0) there follows from Corollaries 2.1 and 2.2 the fact that there exists I having midpoint t_0 on which the sets $\bar{H}_I^m(t_0, x_0)$, corresponding to

$$\dot{x} \in R_m(t, x), \qquad x(t_0) = x_0, \qquad m = 1, 2, 3, \cdots,$$

satisfy $\bar{H}_I^m(t_0, x_0) \in \mathscr{H}^n(I)$. Since $\bar{\Delta}(R_{m+1}(t, x), R_m(t, x)) = 0$ for all m, there follows easily $\bar{\sigma}(\bar{H}_I^{m+1}(t_0, x_0), \bar{H}_I^m(t_0, x_0)) = 0$ and now Theorem 1.6 applies to yield the following facts:

$$H^*(t_0, x_0) = \bigcap \bar{H}_I^m(t_0, x_0) \in \mathscr{H}^n(I)$$

and

$$\lim_{m\to\infty} \bar{\sigma}(\bar{H}_I^m(t_0, x_0), H^*(t_0, x_0)) = 0.$$

Let $\varphi \in H^*(t_0, x_0)$; then $\varphi \in \bar{H}_I^m(t_0, x_0)$ for all m. This is equivalent to $\alpha(\dot{\varphi}(t), R_m(t, \varphi(t))) = 0$ for all m and almost all $t \in I$. Hence the continuity of α together with Theorem 1.3 (iv) permits the conclusion that φ is a solution of (1). Thus $H^*(t_0, x_0) \subset \bar{H}_I(t_0, x_0)$ and, the reverse inclusion being obvious, the proof is complete.

In applications of the theory of generalized differential equations, more interest attaches to the case of nonlocal existence than to the purely local theory we have developed up to this point. The next theorem contains two conditions guaranteeing such nonlocal existence for the solutions of (1); the proof is omitted since it is easy and readily accessible [8], [9].[2]

THEOREM 2.4. *If R: $I \times E^n \to \Gamma^n$ is upper semicontinuous, where $I \subset E^1$ is a compact interval and if R satisfies either (i) or (ii) below then for each $(t_0, x_0) \in I \times E^n$, $\bar{H}_I(t_0, x_0) \in \mathscr{H}^n(I)$:*

(i) *there exists $P \in \Omega^n$ such that $R(t, x) \subset P$ on $I \times E^n$;*

(ii) $\max\{g(R(t, x), x), g(R(t, x), -x)\} \leqq C(\|x\|^2+1)$ *on $I \times E^n$ for some $C>0$.*

With R satisfying the conditions of Theorem 2.4 we may extend the Cauchy

[2] For this proof, reference must also be made to Bebernes, *et al*, *J. Differential Equations* **2** (1966), 102–106.

problem stated at the beginning of this section in the following way. We replace conditions (i), (ii) by the following ones:

(i′) $(t_0, G_0) \in I \times \Omega^n$ and $x(t_0) \in G_0$;
(ii′) for each $t \in I$, $(t, x(t)) \in I \times E^n$.

We denote this problem by

$$\dot{x} \in R(t, x), \qquad x(t_0) \in G_0, \tag{3}$$

and we denote the family of all solutions of (3) by $\bar{H}_I(t_0, G_0)$.

THEOREM 2.5. *If* R: $I \times E^n \to \Gamma^n$ *is upper semicontinuous and satisfies either* (*i*) *or* (*ii*) *of Theorem 2.4 then for each* $(t_0, G_0) \in I \times \Omega^n$, $\bar{H}_I(t_0, G_0) \in \mathscr{H}^n(I)$.

Proof. Let φ be a point in the closure of $\bar{H}_I(t_0, G_0)$—that $\bar{H}_I(t_0, G_0)$ is nonempty is a consequence of Theorem 2.3—and let $\{x^m\} \subset \bar{H}_I(t_0, G_0)$ be a sequence converging to φ. Letting $\bar{x}$ be an accumulation point of $\{x^m(t_0)\}$, we denote by the same indices a subsequence of $\{x^m\}$ for which $\{x^m(t_0)\}$ converges to $\bar{x}$; evidently $\bar{x} \in G_0$ and $\varphi(t_0) = \bar{x}$. The argument that φ satisfies (1) with x_0 replaced by $\bar{x}$ is a repetition with minor changes of that of Theorem 2.2 as is the demonstration that $\bar{H}_I(t_0, G_0)$ is conditionally compact. The assertion of the theorem then follows from Remark 1.1.

In the sequel we shall be concerned exclusively with questions associated with nonlocal existence; since in this case the interval I is stipulated *a priori* we shall henceforth omit this symbol as a subscript in $\bar{H}_I(t_0, G_0)$. Strictly speaking, when $G_0 = \{x_0\}$ we should write $\bar{H}(t_0, \{x_0\})$ but for convenience we choose to retain the earlier notation $\bar{H}(t_0, x_0)$. In view of the fact that $\bar{H}(t_0, G_0) = \bigcup_{x_0 \in G_0} \bar{H}(t_0, x_0)$ the choice is justified.

As is the case with ordinary differential equations, approximation theorems are of fundamental importance in the theory of generalized differential equations. In the remainder of this section we shall devote our efforts to proving several such theorems.

THEOREM 2.6. *Let* R: $I \times E^n \to \Gamma^n$ *satisfy the conditions of Theorem 2.5; then for each* $(t_0, G_0) \in I \times \Omega^n$, *the function* $\eta \to \bar{H}(t_0, G_0^\eta)$ *is continuous at* $\eta = 0$.

Proof. Evidently $\bar{\sigma}(\bar{H}(t_0, G_0), \bar{H}(t_0, G_0^\eta)) = 0$ for all $\eta > 0$; hence we need prove only that $\lim_{\eta \to 0} \bar{\sigma}(\bar{H}(t_0, G_0^\eta), \bar{H}(t_0, G_0)) = 0$. Now there exists $y^\eta \in \bar{H}(t_0, G_0^\eta)$ such that

$$\bar{\sigma}(\bar{H}(t_0, G_0^\eta), \bar{H}(t_0, G_0)) = \beta(y^\eta, \bar{H}(t_0, G_0)) \equiv \beta^\eta.$$

Let $\{\eta_m\}$ be a positive null sequence; then $\{\beta^{\eta_m}\}$ has an accumulation point $\bar{\beta}$ as does the sequence $\{y^{\eta_m}\}$—call it $\bar{y}$—and then by the continuity of $\beta(\cdot, \cdot)$ we find that $\bar{\beta} = \beta(\bar{y}, \bar{H}(t_0, G_0))$. Since $\bar{\sigma}(\bar{H}(t_0, G_0^{\eta_{m+1}}), \bar{H}(t_0, G_0^{\eta_m})) = 0$ for all m we have from Theorem 1.6 that $\bar{y} \in H^*(t_0, G_0)$, and then an argument like that for Theorem 2.3 permits the conclusion that $\beta(\bar{y}, \bar{H}(t_0, G_0)) = 0$ so that $\bar{\beta} = 0$, whence follows the assertion of the theorem.

Let us consider now the autonomous equation

$$\dot{x} \in S(x), \qquad x(t_0) \in G_0; \tag{4}$$

for such equations we may state the following lemma whose proof is trivial.

LEMMA 2.1. *Let $S: E^n \to \Gamma^n$ be upper semicontinuous and have the property that all its solutions are continuable to E^1; then y is a solution of* (4) *if and only if the function z defined by $z(t) = y(t+t_0)$ is a solution of* (4) *with t_0 replaced by* 0.

COROLLARY 2.3. *If $N \in \Omega^n$ and if S satisfies the hypotheses of Theorem 2.5, then for* (4) *the family $\{H(\cdot, G): \bar{\rho}(G, N) = 0\}$ is equicontinuous on I.*

Proof. If $y_0 \in \bar{H}(\tau, G)$ then by Lemma 2.1 there exists $z \in \bar{H}(0, G)$ for which $y_0(t_0) = z(t_0-\tau)$; by the same token the function y_1 defined by $y_1(t) = z(t-t_0)\cdot$ is in $\bar{H}(t_0, G)$. Now for any G satisfying $\bar{\rho}(G, N) = 0$ a solution $z \in \bar{H}(0, G)$ is lipschitzian with a Lipschitz constant K that depends only on N; hence $\langle y_0 - y_1 \rangle \leqq K|\tau - t_0|$. Hence $\bar{\sigma}(\bar{H}(\tau, G), \bar{H}(t_0, G)) \leqq K|\tau - t_0|$; the assertion of the corollary follows by symmetry.

Corollary 2.3 may be improved considerably; in fact we will prove

THEOREM 2.7. *If $R: I \times E^n \to \Gamma^n$ satisfies the hypotheses of Theorem 2.5, then for each $N \in \Omega^n$ the family $\{\bar{H}(\cdot, G): \bar{\rho}(G, N) = 0\}$ is equicontinuous on I.*

Proof. We may extend g_R as an upper semicontinuous function to all of $E^1 \times E^n$ by defining, when $I = [a, b]$,

$$g_R(t, x) = g_R(a, x), \qquad t<a,$$
$$g_R(t, x) = g_R(b, x), \qquad t>b;$$

a corresponding extension of R ensues by virtue of Theorem 1.1. It then follows that whichever of the conditions (i), (ii) of Theorem 2.5 is satisfied by the given R is also satisfied by the extended R. Mapping $E^1 \times E^n$ onto E^{n+1} by

$$(t, x) \to y = (y^0, y^1, \cdots, y^n)^T \equiv (y^0, y^1)^T,$$

where the superscript ^{T} denotes the transpose, we define $R^*: E^{n+1} \to \Gamma^n$ by $R^*(y) = R(t, x)$; then we may define $R^*: E^{n+1} \to \Gamma^{n+1}$ by

$$R'(y) = \{\xi \in E^{n+1}: \xi^0 = 1; \xi' \in R^*(y)\}.$$

Further we define $G_0' \in \Omega^{n+1}$ by

$$G_0' = \{\xi \in E^{n+1}: \xi^0 = 0; \xi' \in G_0\}.$$

It is easy to verify that R' is upper semicontinuous so that the solutions of

$$\dot{y} = R'(y), \qquad y(t_0) \in G_0', \tag{5}$$

exist locally and every solution of (5) satisfies

$$y(t) = (t-t_0, x(t))^T, \tag{6}$$

where x is a solution of (3). Conversely, if x is a solution of (3), then the function y defined by (6) is a solution of (5). The continuability to all of E^1 of the solutions of (5) is thus the consequence of the continuability of the solutions of (3). If we let $\bar{J}(t_0, G_0')$ denote the family of solutions, restricted to I, of (5), a straightforward computation yields both $\bar{\rho}(G', G_0') = \bar{\rho}(G, G_0)$ and

$$\bar{\sigma}^2(\bar{J}(\tau, G_0'), \bar{J}(t_0, G_0')) = (\tau - t_0)^2 + \bar{\sigma}^2(\bar{H}(\tau, G_0), \bar{H}(t_0, G_0)).$$

With this result, the assertion of the theorem follows by virtue of Corollary 2.3 and the autonomy of R'.

LEMMA 2.2. *For $\eta \geqq 0$ let R^η denote the function defined in the proof of Theorem 2.2. If $R: I\times E^n \to \Gamma^n$ is upper semicontinuous and satisfies (i) [(ii)] of Theorem 2.4, then R^η is upper semicontinuous for each $\eta > 0$ and satisfies (i) [(ii)] of Theorem 2.4.*

Proof. If (i) is satisfied by R then it is satisfied by R^η with P replaced by P^η. If (ii) is satisfied by R then, since

$$g(R^\eta(t, x), p) = g(R(t, x), p) + \eta \|p\|,$$

(ii) is satisfied by R^η with C replaced by $C+\eta$.

In view of Lemma 2.2, if $R: I\times E^n \to \Gamma^n$ satisfies the conditions of Theorem 2.5, then so does R^η for all $\eta > 0$, and then we may denote by $\overline{H}_\eta(t_0, G_0)$ the family of restrictions to I of the solutions of

$$\dot{x} \in R^\eta(t, x), \qquad x(t_0) \in G_0, \tag{7}$$

so that $\overline{H}_\eta(t_0, G_0) \in \mathscr{H}^n(I)$. Only minor modifications to the proof of Theorem 2.6 permit the establishment of the following results.

THEOREM 2.8. *If $R: I\times E^n \to \Gamma^n$ satisfies the conditions of Theorem 2.5, then for each $(t_0, G_0) \in I\times\Omega^n$ the function $\eta \to \overline{H}_\eta(t_0, G_0)$ is continuous at $\eta = 0$.*

Let us now define a function $\mathscr{A}: I\times I\times\Omega^n \to \Omega^n$ by

$$\mathscr{A}(t, t_0, G_0) = G(t; \overline{H}(t_0, G_0));$$

then $\mathscr{A}(\cdot, t_0, G_0)$ is the "attainability function" which has been of such great interest in recent work (e.g., [6]). The following theorems are direct consequences of Theorem 1.5 together with, respectively, Theorems 2.6, 2.7, 2.8.

THEOREM 2.9. *Let $R: I\times E^n \to \Gamma^n$ satisfy the conditions of Theorem 2.5; then for each $(t_0, G_0) \in I\times\Omega^n$ the family $\{\eta \to \mathscr{A}(t, t_0, G_0^\eta): t \in I\}$ is equicontinuous at $\eta = 0$.*

THEOREM 2.10. *If $R: I\times E^n \to \Gamma^n$ satisfies the hypotheses of Theorem 2.5, then for each $N \in \Omega^n$ the family $\{\mathscr{A}(t, \cdot, G): t \in I;\ \bar{\rho}(G, N) = 0\}$ is equicontinuous on I.*

THEOREM 2.11. *If $R: I\times E^n \to \Gamma^n$ satisfies the hypotheses of Theorem 2.5, then for each $(t_0, G_0) \in I\times\Omega^n$ the family $\{\eta \to \mathscr{A}_\eta(t, t_0, G_0): t \in I\}$ is equicontinuous at $\eta = 0$, where $\mathscr{A}_\eta(t, t_0, G_0) \equiv G(t; \overline{H}_\eta(t_0, G_0))$.*

Remark 2.1. Scrutinized in the light provided by the comment following Corollary 1.3, Theorems 2.9, 2.10, 2.11 provide cogent evidence for the value of making the solution family of (3) the fundamental object of analysis rather than, say, the attainability function. For whereas these theorems may be (and Theorem 2.9 has been) proved independently, it is not apparent that Theorems 2.6, 2.7, 2.8 may be obtained as corollaries to Theorems 2.9, 2.10, 2.11 respectively; indeed in view of the comment following Corollary 1.3, one is tempted to conjecture that such an implication is not valid.

Remark 2.2. Defining a function $\mathscr{F}: I\times\Omega^n \to \Psi^n(I)$ by $\mathscr{F}(t_0, G_0) = F(\overline{H}(t_0, G_0))$, one obtains the "funnel" or "zone of emission" [1, 2]. Theorems similar to Theorems 2.9, 2.10, 2.11 may be stated for $\mathscr{F}$; the statements and proofs (which depend on Corollary 1.3) of these theorems are left to the reader. Being forewarned, the reader will observe that at several points in the remainder of this paper theorems which are stated for $\bar{H}$ have easy (unstated) corollaries concerning $\mathscr{A}$ and $\mathscr{F}$ which stem from Theorem 1.5 and Corollary 1.3.

3. Generalized Differential Equations: Properties of $\bar{H}(\cdot, \cdot)$

One may observe that in the results of the preceding section no essential loss occurs through replacing the hypotheses of Theorem 2.4 by the following condition:

(*) R: $E^1 \times E^n \to \Gamma^n$ is upper semicontinuous, all solutions of (1) may be continued to E^1, and for each compact $I \subset E^1$ and each $(t, G) \in I \times \Omega^n$, $\bar{H}(t, G)$ is a bounded subset of $\mathscr{C}^n(I)$.

Hence the hypotheses of the remaining theorems of this paper will include (*) explicitly or implicitly and then the choice of the compact interval I is independent of R.

The theorems of this section are concerned with the continuity of the function $\bar{H}(\cdot, \cdot)$; aside from their intrinsic interest these theorems will prove to be of fundamental importance in the theory developed in later sections. In Theorem 2.7 we have already proved in effect the following theorem.

THEOREM 3.1. *If R: $E^1 \times E^n \to \Gamma^n$ satisfies (*), then for each $N \in \Omega^n$ the family $\{\bar{H}(\cdot, G)$: $\bar{\rho}(G, N) = 0\}$ is equicontinuous on I.*
For $\bar{H}(\cdot, \cdot)$ we have

THEOREM 3.2. *If R: $E^1 \times E^n \to \Gamma^n$ satisfies (*), then $\bar{H}(\cdot, \cdot)$ is upper semicontinuous on $I \times \Omega^n$.*

Proof. From Lemma 1.1 we obtain

$$\bar{\sigma}(\bar{H}(\tau, G), \bar{H}(t_0, G_0)) \leqq \bar{\sigma}(\bar{H}(\tau, G), \bar{H}(t_0, G)) + \bar{\sigma}(\bar{H}(t_0, G), \bar{H}(t_0, G_0)),$$

so that if $\bar{\rho}(G, G_0) < \eta$, then

$$\bar{\sigma}(\bar{H}(\tau, G), \bar{H}(t_0, G_0)) \leqq \bar{\sigma}(\bar{H}(\tau, G), \bar{H}(t_0, G)) + \bar{\sigma}(\bar{H}(t_0, G_0^\eta), \bar{H}(t_0, G_0)).$$

The assertion then follows from this inequality together with Theorems 2.6 and 3.1.

The following well-known lemma provides the bond—together with Lemma 1.12—between modern control theory and the theory of generalized differential equations. The easy proof is omitted.

LEMMA 3.1. *Let Φ: $E^1 \times E^n \to \Omega^m$ be upper semicontinuous* [*continuous*], *let f: $E^1 \times E^n \times E^m \to E^n$ be continuous, and define R: $E^1 \times E^n \to \Omega^n$ by*

$$R(t, x) = \{f(t, x, \varphi) \colon \varphi \in \Phi(t, x)\}.$$

Then R is upper semicontinuous [*continuous*] *on $E^1 \times E^n$.*

Definition 3.1. If for R: $E^1 \times E^n \to \Gamma^n$ there exists a continuous f : $E^1 \times E^n \times E^m \to E^n$ and a Φ: $E^1 \times E^n \to \Omega^m$ which is upper semicontinuous [continuous] (constant) which satisfy the remaining condition of Lemma 3.1 then R will be said to have an upper semicontinuous [continuous] (constant) representation.

We shall denote by $U(t_0, x_0)$ the set of all functions $u\colon E^1 \to E^m$ satisfying:

(a) the components of u restricted to I are Lebesgue measurable and bounded;

(b) for $t \in I$, $u(t) \in \Phi(t, x(t))$, where x is a solution of

$$\dot{x} = f(t, x, u(t)), \qquad x(t_0) = x_0, \qquad (t_0, x_0) \in I \times E^n. \tag{8}$$

THEOREM 3.3. *If $R\colon E^1 \times E^n \to \Gamma^n$ satisfies (*) and has a constant representation and if, for each $(t_0, x_0) \in I \times E^n$ and all $u \in U(t_0, x_0)$, (8) has a unique solution, then $\bar{H}(\cdot, \cdot)$ is continuous on $I \times \Omega^n$.*

Proof. By virtue of Theorems 3.1, 3.2 and Lemma 1.1 it is sufficient to prove that for fixed $t_0 \in I$, $\bar{H}(t_0, \cdot)$ is lower semicontinuous on Ω^n. In fact it is sufficient to prove that $\lim_{m\to\infty} \bar{\sigma}(\bar{H}(t_0, G_0), \bar{H}(t_0, G_m)) = 0$ for every sequence $\{G_m\}$ satisfying $0 < \rho(G_m, G_0)$, $\lim_{m\to\infty} \rho(G_m, G_0) = 0$; it is this last assertion which we shall prove. To this end let $\{G_m\}$ be a sequence satisfying the last stated conditions; there exists $y(G_m) \in \bar{H}(t_0, G_0)$ such that

$$\bar{\sigma}(\bar{H}(t_0, G_0), \bar{H}(t_0, G_m)) = \beta(y(G_m), \bar{H}(t_0, G_m)) \equiv \beta(G_m).$$

It is easy to find that the sequence $\{\beta(G_m)\}$ has an accumulation point $\bar{\beta}$. There is thus a subsequence $\{G_{m_k}\}$ for which $\{\beta(G_{m_k})\}$ converges to $\bar{\beta}$. Now $\{y(G_{m_k})\}$ has an accumulation point $\bar{y}$ and a subsequence of $\{G_{m_k}\}$ may be selected for which the corresponding subsequence of $\{y(G_{m_k})\}$ converges to $\bar{y}$. For brevity we denote our final selections simply by $\{G_m\}$, $\{\beta_m\}$ and $\{y_m\}$, where $\beta_m \to \bar{\beta}$ and $y_m \to \bar{y}$ as $m \to \infty$. Since $\bar{H}(t_0, G_0)$ is compact, $\bar{y} \in \bar{H}(t_0, G_0)$ and by Lemma 1.12 and the hypotheses there exists $\bar{u} \in U(t_0, x_0)$ such that $\bar{y}$ is a solution of (8) with u, x_0 replaced by $\bar{u}$, $\bar{y}(t_0)$. Let z_m be the solution of

$$\dot{x} = f(t, x, \bar{u}(t)), \ x(t_0) = \xi_m,$$

where $\xi_m \in G_m$ is a point nearest $y_m(t_0)$. Then $z_m \in \bar{H}(t_0, G_m)$ so that $\beta_m \leqq \langle y_m - z_m \rangle$. Since $\lim_{m\to\infty} \|y_m(t_0) - \xi_m\| = 0$ and $\lim_{m\to\infty} \|y_m(t_0) - \bar{y}(t_0)\| = 0$, we have $\lim_{m\to\infty} \|\xi_m - \bar{y}(t_0)\| = 0$. But then by uniqueness we must have $\lim_{m\to\infty} \langle z_m - \bar{y} \rangle = 0$; hence $\lim_{m\to\infty} \langle y_m - z_m \rangle = 0$. Consequently $\bar{\beta} = 0$, from which the assertion of the theorem follows since $\{G_m\}$ was arbitrary.

Let us now introduce a strengthened form of condition (*).

(**) (a) $R\colon E^1 \times E^n \to \Gamma^n$ is continuous, all solutions of (1) are continuable to E^1 and, for each compact $I \subset E^1$ and each $(t, G) \in I \times \Omega^n$, $\bar{H}(t,G)$ is a bounded subset; moreover,

(b) there exists $\bar{\eta} > 0$ and a function $\rho\colon E^1 \times E^1 \to E^1$ which is continuous, non-negative and satisfies $\rho(t, 0) = 0$ on E^1 and is such that (i) $\dot{u} = \pm\, \rho(t, u)$, $u(t_0) = u_0$, $|u_0| \leqq \bar{\eta}$ has a unique solution for each $t_0 \in E^1$, and (ii) for all $(t, x_i) \in E^1 \times E^n$, $i = 1, 2$,

$$\Delta(R(t, x_2), R(t, x_1)) \leqq \rho(t, \|x_2 - x_1\|).$$

Certainly (**) implies (*). We may now prove

THEOREM 3.4. *If R: $E^1 \times E^n \to \Gamma^n$ satisfies* (**), *then* $\bar{H}(\cdot, \cdot)$ *is continuous on* $I \times \Omega^n$.

Proof. We may consider the first seven sentences of the proof of Theorem 3.3 to have been restated here. Again $\bar{y} \in \bar{H}(t_0, G_0)$ and we denote by ξ_m a point of G_m nearest $y_m(t_0)$. Let $q_R\ (t, x, y)$ denote the unique point of $R(t, x)$ nearest $y \in E^n$; by virtue of [11, Lemma 3] $q_R(\cdot, \cdot, \cdot)$ is continuous on $E^1 \times E^n \times E^n$. It is readily verified that the function whose value is $q_R(t, x, \dot{y}_m(t))$ satisfies Carathéodory's conditions for the local existence of solutions of the differential equation

$$\dot{x} = q_R(t, x, \dot{y}_m(t)), \; x(t_0) = \xi_m. \tag{9}$$

The continuability to E^1 of the solutions of (9) is a consequence of (**). Hence denoting by z_m a solution of (9) we have $z_m \in \bar{H}(t_0, G_m)$ so that $\beta_m \leqq \langle y_m - z_m \rangle$. Let ζ_m denote the function whose value is given by $\zeta_m(t) = \|y_m(t) - z_m(t)\|$. From its definition and the properties of y_m, z_m it follows readily that ζ_m is lipschitzian, hence absolutely continuous on I. An easy estimate then shows that almost everywhere on $I \cap [t_0, \infty)$

$$\dot{\zeta}_m(t) \leqq \|\dot{y}_m(t) - q_R(t, z_m(t), \dot{y}_m(t))\| = \alpha(\dot{y}_m(t), R(t, z_m(t))),$$

so that

$$\dot{\zeta}_m(t) \leqq \Delta(R(t, y_m(t)), R(t, z_m(t))) \leqq \rho(t, \zeta_m(t)).$$

Hence, setting $\eta_m = \rho(G_m, G_0)$ and denoting by $u(\cdot\,; t_0, \eta_m)$ the solution of $\dot{u} = \rho(t, u), u(t_0) = \eta_m$, we have from [19, Theorem 16.2] that $\zeta_m(t) \leqq u(t; t_0, \eta_m)$. Uniqueness then implies that $\lim_{m\to\infty} \zeta_m(t) = 0$ uniformly on $I \cap [t_0, \infty)$. A similar argument holds to the left of $t_0 \in I$ and we conclude that $\bar{\beta} = 0$. Since the original $\{G_m\}$ was arbitrary the assertion of the theorem follows.

Remark 3.1. The proof of the following assertion is trivial. *If R: $E^1 \times E^n \to \Gamma^n$ has a constant representation, then there exists $\varphi = \varphi(x, y) \in \bar{\Phi}$, where $\bar{\Phi}$ denotes the constant value of Φ, such that*

$$\bar{\Delta}(R(t, x), R(t, y)) \leqq \|f(t, x, \varphi) - f(t, y, \varphi)\|.$$

4. Generalized Differential Equations: Approximation Theory

For the type of approximation with which we shall deal in this section we have already obtained a first result in the form of Theorem 2.8. If the hypothesis of that theorem—which is, in effect, (*)—is replaced by (**), the behavior of $\eta \to \bar{H}_\eta(t_0, G_0)$ may be further restricted as shown in the next theorem.

THEOREM 4.1. *If R: $E^1 \times E^n \to \Gamma^n$ satisfies* (**) *with $\rho(t, u) \equiv Ku$, then the family $\{\eta \to \bar{H}_\eta(t, G)$: $(t, G) \in I \times \Omega^n\}$ is equicontinuous at $\eta = 0$.*

Proof. Evidently $\bar{\sigma}(\bar{H}(t_0, G_0), \bar{H}_\eta(t_0, G_0)) = 0$ for all $\eta > 0$, we need to prove that $\lim_{\eta\to 0} \bar{\sigma}(\bar{H}_\eta(t_0, G_0), \bar{H}(t_0, G_0)) = 0$ uniformly for $(t_0, G_0) \in I \times \Omega^n$. Now for each $\eta > 0$ there exists $y^\eta \in \bar{H}_\eta(t_0, G_0)$ such that

$$\bar{\sigma}(\bar{H}_\eta(t_0, G_0), \bar{H}(t_0, G_0)) = \beta(y^\eta, \bar{H}(t_0, G_0)) \equiv \beta^\eta.$$

Let $q_R(\cdot, \cdot, \cdot)$ be the function defined in the proof of Theorem 3.4. We denote by z^η a solution of

$$\dot{x} = q_R(t, x, \dot{y}^\eta(t)), \; x(t_0) = y^\eta(t_0);$$

then $z^\eta \in \bar{H}(t_0, G_0)$ and $\beta^\eta \leqq \langle y^\eta - z^\eta \rangle$. But we find that

$$\begin{aligned} \|y^\eta(t)-z^\eta(t)\| &= \left\|\int_{t_0}^t [\dot{y}^\eta(\tau)-q_R(\tau, z^\eta(\tau), \dot{y}^\eta(\tau))]\, d\tau\right\| \\ &\leqq \left|\int_{t_0}^t \alpha(\dot{y}^\eta(\tau), R(\tau, z^\eta(\tau)))\, d\tau\right| \\ &\leqq \left|\int_{t_0}^t \Delta(R^\eta(\tau, y^\eta(\tau)), R(\tau, z^\eta(\tau)))\, d\tau\right|, \end{aligned}$$

and from the hypotheses and the last inequality we find, for $t \in I$, that

$$\|y^\eta(t)-z^\eta(t)\| \leqq \eta\|I\| + \left|\int_{t_0}^t K\|y^\eta(\tau)-z^\eta(\tau)\|\, d\tau\right|$$

where $\|I\|$ is the length of I. Hence, by the Bellman-Gronwall inequality, we find that

$$\beta^\eta \leqq \langle y^\eta - z^\eta \rangle \leqq \eta\|I\| \exp(K\|I\|),$$

from which we conclude that $\lim_{\eta\to 0} \beta^\eta = 0$ uniformly on $I \times \Omega^n$. This completes the proof.

In the same vein we may state the following result.

THEOREM 4.2. *If $R: E^1 \times E^n \to \Gamma^n$ satisfies (*) and has a constant representation, and if there exists $K \geqq 0$ for which $(t, x_i, \varphi) \in E_1 \times E^n \times E^m$, $i = 1, 2$, implies $\|f(t, x_1, \varphi) - f(t, x_2, \varphi)\| \leqq K\|x_1 - x_2\|$ for the function f of the representation, then the family $\{\eta \to \bar{H}_\eta(t, G) \colon (t, G) \in I \times \Omega^n\}$ is equicontinuous at $\eta = 0$.*

Proof. As in the proof of Theorem 4.1 we let $y^\eta \in \bar{H}_\eta(t_0, G_0)$ satisfy $\bar{\sigma}(\bar{H}_\eta(t_0, G_0), \bar{H}(t_0, G_0)) = \beta(y^\eta, \bar{H}(t_0, G_0)) \equiv \beta^\eta$. Now since $\dot{y}^\eta(t) \in R^\eta(t, y^\eta(t))$ almost everywhere on I, it follows that

$$\dot{y}^\eta(t) = q_R(t, y^\eta(t), \dot{y}^\eta(t)) + \eta\nu(t) \tag{10}$$

where $q_R(\cdot, \cdot, \cdot)$ is the function defined in the proof of Theorem 3.4. As in the proof of that theorem we assert that $q_R(\cdot, y^\eta(\cdot), \dot{y}^\eta(\cdot))$ is measurable on I; hence the function $\nu(\cdot)$, whose value is defined by (10), is also measurable on I. Moreover, $\|\nu(t)\| \leqq 1$. By Lemma 1.12 we conclude that there exists $\bar{u} \in U(t_0, y^\eta(t_0))$ such that y^η is the solution on I of

$$\dot{x} = f(t, x, \bar{u}(t)) + \eta\nu(t), \qquad x(t_0) = y^\eta(t_0).$$

Now let z^η be the solution of $\dot{x} = f(t, x, \bar{u}(t))$, $x(t_0) = y^\eta(t_0)$; then $z^\eta \in \bar{H}(t_0, G_0)$ and we find, with $\|I\|$ denoting the length of I, that

$$\|y^\eta(t)-z^\eta(t)\| \leqq \eta\|I\| + \left|\int_{t_0}^t \|f(\tau, y^\eta(\tau), \bar{u}(\tau)) - f(\tau, z^\eta(\tau), \bar{u}(\tau))\|\, d\tau\right|, \quad t \in I.$$

From the Lipschitz condition satisfied by f together with the Bellman-Gronwall inequality we find that

$$\beta^\eta \leqq \langle y^\eta - z^\eta \rangle \leqq \eta\|I\| \exp(K\|I\|),$$

from which the assertion of the theorem follows.

Comment. A simpler proof, displaying Theorem 4.2 as a corollary to Theorem 4.1, may be based on Remark 3.1.

Theorems 2.8, 4.1 and 4.2 carry important implications for the approximation theory developed in [10, 11]. To substantiate this assertion let us first introduce the concept of a *uniform approximation*.

Definition 4.1. A family $\{S_\eta\colon 0 \leqq \eta \leqq \bar{\eta}\}$ of functions $S_\eta\colon E^1 \times E^n \to \Gamma^n$ is said to be a uniform approximation to a function $R\colon E^1 \times E^n \to \Gamma^n$ if and only if the following conditions are satisfied:

(i) if $\eta \in (0, \bar{\eta}]$ then $\bar{\Delta}(R(t, x), S_\eta(t, x)) = 0$ on $E^1 \times E^n$;

(ii) $R = S_0$ and the family $\{\eta \to S_\eta(t, x)\colon (t, x) \in E^1 \times E^n\}$ is equicontinuous at $\eta = 0$.

LEMMA 4.1. *If $\{S_\eta\colon 0 \leqq \eta \leqq \bar{\eta}\}$ is a uniform approximation to $R\colon E^1 \times E^n \to \Gamma^n$ and if R satisfies either of the conditions (i), (ii) of Theorem 2.4, then there exists $\eta^* \in (0, \bar{\eta}]$ such that every member of the family $\{S_\eta\colon 0 \leqq \eta \leqq \eta^*\}$ satisfies the same condition.*

Proof. Given $\epsilon > 0$ there follows from Definition 4.1 (ii) the existence of an $\eta^* = \eta^*(\epsilon) \in (0, \bar{\eta}]$ such that $\bar{\Delta}(S_\eta(t, x), R(t, x)) \leqq \epsilon$ for all $\eta \in [0, \eta^*]$ and all $(t, x) \in I \times E^n$. But this implies that $\bar{\Delta}(S_\eta(t, x), R^\epsilon(t, x)) = 0$. The assertion then follows from Lemma 2.2 and the fact, stemming from the last equation, that

$$-g(R^\epsilon(t, x), -p) \leqq -g(S_\eta(t, x), -p) \leqq g(S_\eta(t, x), p) \leqq g(R^\epsilon(t, x), p).$$

The implications to which we referred previously are embodied in the following result.

THEOREM 4.3. *Let $\{S_\eta\colon 0 \leqq \eta \leqq \bar{\eta}\}$ be a uniform approximation to $R\colon E^1 \times E^n \to \Gamma^n$; then for the function $\eta \to \bar{J}_\eta(t, G)$, where $\bar{J}_\eta(t_0, G_0)$ is the family of solutions restricted to I of $\dot{x} \in S_\eta(t, x)$, $x(t_0) \in G_0$, the following properties obtain.*

(i) *If for each $\eta \in [0, \bar{\eta}]$, S_η satisfies (*), then for each $(t, G) \in I \times \Omega^n$ the function $\eta \to \bar{J}_\eta(t, G)$ is continuous at $\eta = 0$.*

(ii) *If for each $\eta \in (0, \bar{\eta}]$, S_η satisfies (*) and if R satisfies (**), then the family $\{\eta \to \bar{J}_\eta(t, G)\colon (t, G) \in I \times \Omega^n\}$ is equicontinuous at $\eta = 0$.*

(iii) *If for each $\eta \in (0, \bar{\eta}]$, S_η satisfies (*) and if R satisfies the hypotheses of Theorem 4.2, then the family $\{\eta \to \bar{J}_\eta(t, G)\colon (t, G) \in I \times \Omega^n\}$ is equicontinuous at $\eta = 0$.*

Proof. Define, for $\eta \in (0, \bar{\eta}]$,

$$\epsilon^*(\eta, t, x) = \inf\{\epsilon > 0\colon \bar{\sigma}(S_\eta(t, x), R^\epsilon(t, x)) = 0\};$$

then for all sufficiently small $\eta > 0$, $\epsilon^*(\eta, t, x)$ is independent of $(t, x) \in E^1 \times E^n$ by virtue of Definition 4.1 (ii) so that we may denote this number by $\epsilon^*(\eta)$. We then define $\epsilon(\eta)$ as $\epsilon(\eta) = \max\{\eta, \epsilon^*(\eta)\}$ and, again by Definition 4.1 (ii), it follows that $\lim_{\eta \to 0^+} \epsilon(\eta) = 0$. Moreover, $\epsilon(\eta) > 0$ for $\eta > 0$. Since $\bar{\sigma}(\bar{J}_\eta(t_0, G_0), \bar{H}_{\epsilon(\eta)}(t_0, G_0)) = 0$, it follows from Lemma 1.1 that

$$\bar{\sigma}(\bar{J}_\eta(t_0, G_0), \bar{H}(t_0, G_0)) \leqq \bar{\sigma}(\bar{H}_{\epsilon(\eta)}(t_0, G_0), \bar{H}(t_0, G_0)).$$

The assertions (i), (ii), (iii) of the theorem are then consequences of this estimate together with, respectively, Theorems 2.8, 4.1, and 4.2 for we already have from Definition 4.1 (i) that

$$\bar{\sigma}(\bar{H}(t_0, G_0), \bar{J}_\eta(t_0, G_0)) = 0 \text{ for } \eta \in [0, \bar{\eta}], (t_0, G_0) \in I \times \Omega^n.$$

5. Optimal Control and Domains of Controllability

The basis for the optimization theory developed in this section is the classical theorem which states that a real-valued, lower semicontinuous function defined on a metric space has a minimum on each compact subset of that space. The particular form of this theorem which we shall need is the following.

THEOREM 5.1. *Let* $\lambda: \mathscr{C}^n(I) \to E^1$ *be lower semicontinuous; then on* $\mathscr{H}^n(I)$, $\min \{\lambda(\varphi): \beta(\varphi, H) = 0\}$ *exists.*

Now let $\Pi: E^1 \to \Omega^n$ be continuous and define $\delta_\Pi: I \times \mathscr{C}^n(I) \to E^1$ by

(11) $$\delta_\Pi(t, \varphi) = \min \{\alpha(\varphi(\tau), \Pi(\tau)): \tau \in I[t]\},$$

where $I[t]$ denotes $I \cap [t, \infty)$. We shall prove a series of lemmas.

LEMMA 5.1. $\delta_\Pi(\cdot, \cdot)$ *is continuous on* $I \times \mathscr{C}^n(I)$.

Proof. From Corollary 1.1 it follows that for all $t \in I$,

$$|\alpha(\varphi_2(t), \Pi(t)) - \alpha(\varphi_1(t), \Pi(t))| \leqq \langle \varphi_2 - \varphi_1 \rangle.$$

This result, together with well-known estimates, then yields

$$\begin{aligned} &|\delta_\Pi(t_2, \varphi_2) - \delta_\Pi(t_1, \varphi_1)| \\ &\leqq \langle \varphi_2 - \varphi_1 \rangle + |\min \{\alpha(\varphi_1(\tau), \Pi(\tau)): \tau \in I[t_2]\} \\ &\quad - \min \{\alpha(\varphi_1(\tau), \Pi(\tau)): \tau \in I[t_1]\}|, \end{aligned}$$

from which the assertion follows by virtue of the continuity of $\alpha(\varphi_1(\cdot), \Pi(\cdot))$.

We may associate with δ_Π a function $t_\Pi: I \times \mathscr{C}^n(I) \to I$ defined by

(12) $$t_\Pi(t, \varphi) = \min \{\tau \in I[t]: \alpha(\varphi(\tau), \Pi(\tau)) = \delta_\Pi(t, \varphi)\}.$$

As the inverse image of a closed set under a continuous map the set appearing in the right-hand member of (12) is closed; it is obviously bounded so that t_Π is well defined by (12).

LEMMA 5.2. $t_\Pi(\cdot, \cdot)$ *is lower semicontinuous on* $I \times \mathscr{C}^n(I)$.

Proof. For fixed $(t_0, \varphi_0) \in I \times \mathscr{C}^n(I)$ let τ be the lim inf of $t_\Pi(t, \varphi)$ as $\varphi \to \varphi_0$, $t \to t_0$; then there exists $\{(t_m, \varphi_m)\} \subset I \times \mathscr{C}^n(I)$ satisfying $\lim_{m \to \infty} (t_m, \varphi_m) = (t_0, \varphi_0)$, $(t_m, \varphi_m) \neq (t_0, \varphi_0)$ and $\lim_{m \to \infty} t_\Pi(t_m, \varphi_m) = \tau$. Since we have, for all $(t, \varphi) \in I \times \mathscr{C}^n(I)$,

$$\alpha(\varphi(t_\Pi(t, \varphi)), \Pi(t_\Pi(t, \varphi))) = \delta_\Pi(t, \varphi),$$

it follows from the foregoing that $\alpha(\varphi_0(\tau), \Pi(\tau)) = \delta_\Pi(t_0, \varphi_0)$ by virtue of the continuity of the functions involved. From this and (12) we conclude that $\tau \geqq t_\Pi(t_0, \varphi_0)$ since certainly $\tau \in I[t_0]$.

By virtue of Theorem 5.1 and Lemma 5.1 a function $V^\Pi: I \times \mathscr{H}^n(I) \to E^1$ is well defined by

(13) $$V^{\Pi}(t, H) = \min\{\delta_{\Pi}(t, \varphi): \beta(\varphi, H) = 0\}.$$

Associated with $V^{\Pi}(t, H)$ is the following subset of H:

$$L^{\Pi}(t, H) = \{\varphi \in H: \delta_{\Pi}(t, \varphi) = V^{\Pi}(t, H)\},$$

for which we have the following result.

LEMMA 5.3. *For each $(t, H) \in I \times \mathscr{H}^n(I)$, $L^{\Pi}(t, H) \in \mathscr{H}^n(I)$.*

Proof. By definition $L^{\Pi}(t, H)$ is nonvoid and the lemma is trivially true when $L^{\Pi}(t, H)$ has only finitely many elements. Suppose then that $L^{\Pi}(t, H)$ has infinitely many elements and let $\bar{\varphi}$ be a point of the closure of $L^{\Pi}(t, H)$; then $\bar{\varphi} \in H$. Let $\{\varphi_m\} \subset L^{\Pi}(t, H)$ satisfy $\lim_{m\to\infty} \varphi_m = \bar{\varphi}$; then by the continuity of δ_{Π} and the definition of $L^{\Pi}(t, H)$ we have $\delta_{\Pi}(t, \bar{\varphi}) = V^{\Pi}(t, H)$ so that $\bar{\varphi} \in L^{\Pi}(t, H)$. Thus as a closed subset of a compact set $L^{\Pi}(t, H)$ is compact.

As a consequence of Theorem 5.1 and Lemmas 5.2 and 5.3 we may define a function $T^{\Pi}: I \times \mathscr{H}^n(I) \to I$ by

(14) $$T^{\Pi}(t, H) = \min\{t_{\Pi}(t, \varphi): \varphi \in L^{\Pi}(t, H)\}.$$

Associated with $T^{\Pi}(t, H)$ is the following subset of $L^{\Pi}(t, H)$:

$${}^{\Pi}L(t, H) = \{\varphi \in L^{\Pi}(t, H): t_{\Pi}(t, \varphi) = T^{\Pi}(t, H)\};$$

an obvious variation of the proof of Lemma 5.3, based on Lemma 5.2, permits the following assertion.

LEMMA 5.4. *For each $(t, H) \in I \times \mathscr{H}^n(I)$, ${}^{\Pi}L(t, H) \in \mathscr{H}^n(I)$.*

Having defined V^{Π} and T^{Π}, we next proceed to obtain useful characterizations of these functions alternative to (13) and (14). We may define a function $\Delta^{\Pi}: I \times \mathscr{H}^n(I) \to E^1$ by

(15) $$\Delta^{\Pi}(t, H) = \gamma(G(t; H), \Pi(t)),$$

where $\gamma(\cdot, \cdot)$ is the gap function defined in Lemma 1.2.

LEMMA 5.5. *(i) Given $H_1, H_2 \in \mathscr{H}^n(I)$, $\Delta^{\Pi}(t, H_1) - \Delta^{\Pi}(t, H_2) \leqq \bar{\sigma}(H_2, H_1)$ for all $t \in I$. (ii) $\Delta^{\Pi}(\cdot, \cdot)$ is continuous on $I \times \mathscr{H}^n(I)$.*

Proof. (i) is a direct consequence of Lemma 1.9, (15) and the proof of Lemma 1.2. To show (ii), we observe that by virtue of Lemma 1.2 and Theorem 1.5, $\Delta^{\Pi}(\cdot, H)$ is continuous on I for each $H \in \mathscr{H}^n(I)$. Since by (i) the family $\{\Delta^{\Pi}(t, \cdot): t \in I\}$ is equicontinuous, the assertion follows *via* the triangle law.

THEOREM 5.2. *For all $(t, H) \in I \times \mathscr{H}^n(I)$,*

(16) $$V^{\Pi}(t, H) = \min\{\Delta^{\Pi}(\lambda, H): \lambda \in I[t]\};$$

(17) $$T^{\Pi}(t, H) = \min\{\lambda \in I[t]: \Delta^{\Pi}(\lambda, H) = V^{\Pi}(t, H)\}.$$

Proof. That the right-hand sides of (16) and (17), to be denoted respectively as $V^*(t, H)$ and $T^*(t, H)$, are well defined is a consequence of Lemma 5.5. Now there exists $t_1 \in I[t]$ such that $V^*(t, H) = \gamma(G(t_1; H), \Pi(t_1))$ and then there exists $\varphi_0 \in H$ such that $V^*(t, H) = \alpha(\varphi_0(t_1), \Pi(t_1))$. Clearly $\alpha(\varphi_0(\tau), \Pi(\tau)) \geqq \alpha(\varphi_0(t_1), \Pi(t_1))$ for $\tau \in I[t]$ so that $V^*(t, H) = \delta_{\Pi}(t, \varphi_0) \geqq V^{\Pi}(t, H)$.

On the other hand there exists $\varphi_1 \in H$ such that $V^\Pi(t, H) = \delta_\Pi(\varphi_1)$ and then there exists $t_0 \in I[t]$ such that

$$\delta_\Pi(t, \varphi_1) = \alpha(\varphi_1(t_0), \Pi(t_0)) \geqq \Delta^\Pi(t_0, H) \geqq V^*(t, H).$$

Hence $V^\Pi(t, H) = V^*(t, H)$ and this proves (16).

If one interprets t_1 above as $T^*(t, H)$, then by virtue of (16) it follows that $T^*(t, H) \geqq T^\Pi(t, H)$ since $\varphi_0 \in L^\Pi(t, H)$. On the other hand we may replace t_0 above by $T^\Pi(t_0, H)$, with $\varphi_1 \in L^\Pi(t, H)$, and then

$$\Delta^\Pi(T^\Pi(t, H), H) = \alpha(\varphi_1(T^\Pi(t, H)), \Pi(T^\Pi(t, H))) = V^\Pi(t, H),$$

so that $T^\Pi(t, H) \geqq T^*(t, H)$. This proves (17).

THEOREM 5.3. (*i*) *Given* $H_1, H_2 \in \mathscr{H}^n(I)$; then $V^\Pi(t, H_1) - V^\Pi(t, H_2) \leqq \bar{\sigma}(H_2, H_1)$ *for all* $t \in I$. (*ii*) $V^\Pi(\cdot, \cdot)$ *is continuous on* $I \times \mathscr{H}^n(I)$.

Proof. (i) Well known estimates based on (16) together with Lemma 5.5 (i) suffice to establish this result. (ii) follows from (i) and (16) by means of an argument like that for Lemma 5.1.

Making use of (17), the proof of the following corollary of Theorem 5.3 is like that of Lemma 5.2.

COROLLARY 5.1. $T^\Pi(\cdot, \cdot)$ *is lower semicontinuous on* $I \times \mathscr{H}^n(I)$.

Comment. In Corollary 5.1 lower semicontinuity is to be construed not in the sense of Definition 1.1 but in the usual sense.

Henceforth we concern ourselves with the class $\mathscr{E}$ of functions $\bar{H}: I \times \Omega^n \to \mathscr{H}^n(I)$ having the following properties:

(18a) $$G(t_0;\ \bar{H}(t_0, G_0)) = G_0 \text{ for all } (t_0, G_0) \in I \times \Omega^n;$$

(18b) $$\bar{\sigma}(\bar{H}(t_0, G_0), \bar{H}(t, G(t; \bar{H}(t_0, G_0)))) = 0 \text{ for all } (t, t_0, G_0) \in I \times I \times \Omega^n;$$

(18c)
$$\bar{\rho}(G, G_0) = 0 \text{ implies } \bar{\sigma}(\bar{H}(t_0, G), \bar{H}(t_0, G_0)) = 0 \text{ for all } (t_0, G_0, G) \in I \times \Omega^n \times \Omega^n.$$

To see that $\mathscr{E}$ is non-empty, observe that the existence of $\bar{H}: I \times \Omega^n \to \mathscr{H}^n(I)$ satisfying (18a) is guaranteed by Theorem 2.5, and that the $\bar{H}$ of that theorem satisfies (18b); (18c) is subject to trivial verification. Indeed, we have used (18c) repeatedly in previous sections.

For fixed $\bar{H} \in \mathscr{E}$, we define functions $V_\Pi: I \times \Omega^n \to E^1$ and $T_\Pi: I \times \Omega^n \to I$ by means of

(19) $$V_\Pi(t, G) = V^\Pi(t, \bar{H}(t, G));$$

(20) $$T_\Pi(t, G) = T^\Pi(t, \bar{H}(t, G)).$$

The next theorem is a direct consequence of (19), (20), Theorem 5.3 and its corollary.

THEOREM 5.4. (*i*) *If for* $N \in \Omega^n$ *the family* $\{\bar{H}(\cdot, G): \bar{\rho}(G, N) = 0\}$ *is equicontinuous, then the family* $\{V_\Pi(\cdot, G): \bar{\rho}(G, N) = 0\}$ *is equicontinuous.* (*ii*) *If* $\bar{H}(\cdot, \cdot)$ *is continuous on* $I \times \Omega^n$, *then* $V_\Pi(\cdot, \cdot)$ *is continuous on* $I \times \Omega^n$. (*iii*) *If* $\bar{H}(\cdot, \cdot)$ *is upper semicontinuous on* $I \times E^n$, *then* $V_\Pi(\cdot, \cdot)$ *is lower semicontinuous on* $I \times E^n$. (*iv*) *If* $\bar{H}(\cdot, \cdot)$ *is continuous on* $I \times E^n$, *then* $T_\Pi(\cdot, \cdot)$ *is lower semicontinuous on* $I \times E^n$.

It is easy to see that the restrictions to $\bar{H}(t, G)$ of δ_Π and t_Π are generalizations of the concepts "miss distance" and "first time of closest approach", respectively, defined in [9]. With this interpretation, $V_\Pi(t, G)$ and $T_\Pi(t, G)$ are respectively the minimum miss distance and minimum time to attain minimum miss distance.

Our next result is a trivial consequence of (19), (20) and Theorem 5.2.

THEOREM 5.5. *For all* $(t, G) \in I \times \Omega^n$, $V_\Pi(t, G) \leqq \gamma(G, \Pi(t))$ *with equality holding if and only if* $T_\Pi(t, G) = t$.

Put in other words, Theorem 5.5 states that the problem of determining $\varphi \in \bar{H}(t, G)$ which minimizes $\delta_\Pi(t, \varphi)$ is trivial if $T_\Pi(t, G) = t$ on $I \times \Omega^n$ since then any φ will do. I.e., in this case $L^\Pi(t, \bar{H}(t, G)) = \bar{H}(t, G)$. This problem only becomes nontrivial when the set

(21a) $$C_\infty(\Pi) = \{(t, G) \in I \times \Omega^n\colon\ T_\Pi(t, G) > t\}$$

is non-empty.

The problem of determining conditions sufficient to ensure the non-emptiness of $C_\infty(\Pi)$ is called the *problem of weak controllability* (cf.[20]). Shortly we shall confine our attention to the problem of weak controllability as restricted to the image of E^n imbedded in Ω^n by the map $x \to \{x\}$. In this case we prefer to study the following set rather than $C_\infty(\Pi)$:

(21b) $$B_\infty(\Pi) = \{(t, x) \in I \times E^n\colon\ T_\Pi(t, x) > t\}.$$

Associated with $B_\infty(\Pi)$ are the sets

(21c) $$\begin{aligned} B_\eta(\Pi) &= \{(t, x) \in B_\infty(\Pi)\colon\ V_\Pi(t, x) < \eta\};\\ B^0(\Pi) &= \bigcap \{B_\eta(\Pi)\colon\ \eta > 0\};\\ B_0(\Pi) &= \{(t, x) \in B_\infty(\Pi)\colon\ V_\Pi(t, x) = 0\}. \end{aligned}$$

Remark 5.1. Sets $C_\eta(\Pi)$, $C^0(\Pi)$, $C_0(\Pi)$ associated with $C_\infty(\Pi)$ may be defined in a manner like that of (21c). In the remaining theorems of this section, results which are valid when B is replaced by C and E by Ω are designated by an asterisk. In each such case, only minor modifications are required in the proofs.

***LEMMA 5.6.** (*i*) $B_\infty(\Pi) = \bigcup \{B_\eta(\Pi)\colon\ \eta > 0\}$; (*ii*) $B_0(\Pi) = B^0(\Pi)$.

Proof. Only the proof that $B^0(\Pi) \subset B_0(\Pi)$ is not obvious. Let $(t, x) \in B^0(\Pi)$; then there exists $\{\varphi_m\} \subset L^\Pi(t, \bar{H}(t, x))$ satisfying $\delta_\Pi(t, \varphi_m) < m^{-1}$. Any accumulation point $\bar{\varphi}$ of $\{\varphi_m\}$ satisfies $\bar{\varphi} \in L^\Pi(t, \bar{H}(t, x))$ by virtue of Lemma 5.3. With this, continuity of $\delta_\Pi(t, \cdot)$ yields $\delta_\Pi(t, \bar{\varphi}) = 0$, so that $(t, x) \in B_0(\Pi)$.

We next state useful representations of $B_\infty(\Pi)$, $B_\eta(\Pi)$ and $B_0(\Pi)$ which are equivalent to those of (21); we omit the easy proof. In the sequel, $I(t]$ denotes $I \cap (t, \infty)$.

***LEMMA 5.7.**

(a) $B_\infty(\Pi) = \{(t_0, x_0) \in I \times E^n\colon\ \exists t_1 \in I(t_0] \ni \Delta^\Pi(t_1, \bar{H}(t_0, x_0)) < \alpha(x_0, \Pi(t_0))\}$;

(a′) $B_\infty(\Pi) = \{(t_0, x_0) \in I \times E^n\colon\ \exists \varphi_0 \in \bar{H}(t_0, x_0) \ni \delta_\Pi(t_0, \varphi_0) < \alpha(x_0, \Pi(t_0))\}$;

(b) $B_\eta(\Pi) = \{(t_0, x_0) \in B_\infty(\Pi)\colon \exists t_1 \in I(t_0] \ni \Delta^\Pi(t_1, \bar{H}(t_0, x_0)) < \eta\}$;

(b′) $B_\eta(\Pi) = \{(t_0, x_0) \in B_\infty(\Pi): \exists\varphi_0 \in \bar{H}(t_0, x_0) \ni \delta_\Pi(t_0, \varphi_0) < \eta\}$;

(c) $B_0(\Pi) = \{(t_0, x_0) \in B_\infty(\Pi): \exists t_1 \in I(t_0] \ni \Delta^\Pi(t_1, \bar{H}(t_0, x_0)) = 0\}$;

(c′) $B_0(\Pi) = \{(t_0, x_0) \in B_\infty(\Pi): \exists\varphi_0 \in \bar{H}(t_0, x_0) \ni \delta_\Pi(t_0, \varphi_0) = 0\}$.

In view of the representations (a′) and (c′) of Lemma 5.7 we are justified in referring to $B_\infty(\Pi)$, and $B_0(\Pi)$ as the "domain of weak controllability" and the "domain of controllability", respectively.

We are now in a position to prove a result which is essential to the approximation methods developed in [11] and in Section 7.

***THEOREM 5.6.** *If $\bar{H}(\cdot, \cdot)$ is continuous on $I \times E^n$, then $B_\eta(\Pi)$, $\eta \in (0, \infty]$, is relatively open in $I \times E^n$.*

Proof. The theorem is trivial if $B_\infty(\Pi)$ is empty. For the rest we have

$$\alpha(x, \Pi(t)) - V_\Pi(t, x) = [\alpha(x, \Pi(t)) - \alpha(x_0, \Pi(t_0))] + [\alpha(x_0, \Pi(t_0)) - V_\Pi(t_0, x_0)] + [V_\Pi(t_0, x_0) - V_\Pi(t, x)].$$

If $(t_0, x_0) \in B_\infty(\Pi)$, then the second term on the right-hand side of this equation is a positive number; by virtue of the continuity of $\alpha(\cdot, \Pi(\cdot))$ and of $V_\Pi(\cdot, \cdot)$ (Theorem 5.4 (ii)), the entire right-hand side is positive provided $|t-t_0|$ and $\|x-x_0\|$ are sufficiently small. This proves that $B_\infty(\Pi)$ is relatively open in $I \times E^n$. If $(t_0, x_0) \in B_\eta(\Pi)$ for some $\eta \in (0, \infty)$, then the first part of the proof shows that $(t, x) \in B_\infty(\Pi)$ for sufficiently small $|t-t_0|$, $\|x-x_0\|$. Taking these quantities still smaller if necessary yields $(t, x) \in B_\eta(\Pi)$ by virtue of (21) and the continuity of $V_\Pi(\cdot, \cdot)$.

THEOREM 5.7. *If $\bar{H}(\cdot, \cdot)$ is upper semicontinuous on $I \times E^n$, then $B_0(\Pi)$ is relatively closed in $I \times E^n$.*

Proof. This is trivial if $B_0(\Pi)$ is empty. Otherwise the assertion is a consequence of Theorem 5.4 (iii) and the fact that the set of zeros of a non-negative, lower semicontinuous function is closed (cf. [11, Remark 5]).

THEOREM 5.8. *If for each $G \in \Omega^n$, $\bar{H}(\cdot, G)$ is continuous on I, then $B_\eta(\Pi)$ is bounded for each $\eta \in [0, \infty)$.*

Proof. If the assertion is true for $\eta \in (0, \infty)$, it is true for $\eta = 0$ since $B_0(\Pi) \subset B_\eta(\Pi)$. Hence, let $\eta \in (0, \infty)$ be fixed and let $(\tau, \xi) \in B_\eta(\Pi)$. Then there exist $\varphi \in \bar{H}(\tau, \xi)$ and $t^* \in I(\tau]$ such that $\alpha(\varphi(t^*), \Pi(t^*)) < \eta$. Since Π is continuous, there exists $G^* \in \Omega^n$ such that $(\Pi(t))^\eta \subset G^*$ for all $t \in I$. By hypothesis and by Theorem 1.5, $G(\cdot; \bar{H}(\cdot, G^*))$ is continuous on $I \times I$ so that there exists $\Psi \in \Omega^n$ such that $G(t; \bar{H}(t_0, G^*)) \subset \Psi$ for all $(t, t_0) \in I \times I$. Now by (18), $\xi \in G(\tau; \bar{H}(t^*, G^*))$ so that $\xi \in \Psi$. Since (τ, ξ) was arbitrary, the assertion follows.

Now let us define, for $(t, x) \in I \times E^n$, a set

$$K(t, x) = \{\varphi \in \bar{H}(t, x): (t_\Pi(t, \varphi), \varphi(t_\Pi(t, \varphi))) \notin B_\infty(\Pi)\}; \tag{25}$$

for this set we have the following theorem and corollary.

THEOREM 5.9. *(i) On $I \times E^n$, $L_\Pi(t, x) \subset K(t, x)$, where $L_\Pi(t, x) \equiv L^\Pi(t, \bar{H}(t, x))$. (ii) If $\bar{H}(\cdot, \cdot)$ is continuous on $I \times E^n$ and if for each $(t, x) \in B_\infty(\Pi)$, $t_\Pi(t, \cdot)$ is continuous on $K(t, x)$, then for all $(t, x) \in I \times E^n$, $K(t, x) \in \mathscr{H}^n(I)$.*

COROLLARY 5.2. *Under the hypothesis of Theorem 5.9 (ii),*

$$V_\Pi(t, x) = \min\{\delta_\Pi(t, \varphi): \beta(\varphi, K(t, x)) = 0\}, \qquad (t, x) \in I \times E^n. \tag{26}$$

Proof. A simple proof by contradiction suffices to establish (i). For (ii), we observe first of all that by (i), $K(t, x)$ is non-empty on $I \times E^n$. Since $\bar{H}(\cdot, \cdot)$ is continuous, it follows from Theorem 5.6 that the complement of $B_\infty(\Pi)$ is relatively closed in $I \times E^n$. For $(t, x) \in B_\infty(\Pi)$ the assertion of (ii) then follows from this closure property together with (25) and the continuity of $t_\Pi(t, \cdot)$. For $(t, x) \notin B_\infty(\Pi)$ the assertion of (ii) is trivially true. Corollary 5.2 is an obvious consequence of Theorem 5.9.

6. Optimal Control: Approximation Theory. I

In this section we assume $\Pi: I \to \Omega^n$ and $\bar{H}: I \times \Omega^n \to \mathscr{H}^n(I)$ given, the former being continuous and the latter being an element of $\mathscr{E}$. We also assume the existence of a one-parameter family $\{\bar{J}_\eta: 0 \leqq \eta \leqq 1\}$ of functions in $\mathscr{E}$ having the property that $\bar{H} = \bar{J}_0$. The sets $C_\infty(\Pi)$, etc., of Section 5 are those defined relative to the given $\bar{H}$. For each $\eta \in [0, 1]$ we define

$$V_\Pi^\eta(t, G) = V^\Pi(t, \bar{J}_\eta(t, G));$$
$$T_\Pi^\eta(t, G) = T^\Pi(t, \bar{J}_\eta(t, G)).$$

Our first result is an immediate consequence of this definition together with Theorem 5.3 and its corollary.

THEOREM 6.1. *(i) If $\eta \to \bar{J}_\eta(t, G)$ is continuous at $\eta = 0$ for fixed $(t, G) \in I \times \Omega^n$, then $\eta \to V_\Pi^\eta(t, G)$ is continuous at $\eta = 0$* and *$\eta \to T_\Pi^\eta(t, G)$ is lower semicontinuous at $\eta = 0$. (ii) If the family $\{\eta \to \bar{J}_\eta(t, G): (t, G) \in I \times \Omega^n\}$ is equicontinuous at $\eta = 0$, then the family $\{\eta \to V_\Pi^\eta(t, G): (t, G) \in I \times \Omega^n\}$ is equicontinuous at $\eta = 0$.*

Now for each $\eta \in [0, 1]$ let us define

$$\bar{J}_\eta^\Pi(t, G) = \{\varphi \in \bar{J}_\eta(t, G): \delta_\Pi(\varphi) = V_\Pi^\eta(t, G)\}; \tag{27}$$

by virtue of Lemma 5.3, $\bar{J}_\eta^\Pi(t, G) \in \mathscr{H}^n(I)$ for all $\eta \in [0, 1]$ and all $(t, G) \in I \times \Omega^n$. We shall prove

COROLLARY 6.1. *If for fixed $(t, G) \in I \times \Omega^n$, $\eta \to \bar{J}_\eta(t, G)$ is continuous at $\eta = 0$, and if for sufficiently small $\eta \in (0, 1]$, $\bar{\sigma}(\bar{H}(t, G), \bar{J}_\eta(t, G)) = 0$, then the mapping $\eta \to \bar{J}_\eta^\Pi(t, G)$ is upper semicontinuous at $\eta = 0$.*

Proof. Now there exists $y^\eta \in \bar{J}_\eta^\Pi(t, G)$ such that, with $L_\Pi(t, G) \equiv L^\Pi(t, \bar{H}(t, G))$,

$$\bar{\sigma}(\bar{J}_\eta^\Pi(t, G), L_\Pi(t, G)) = \beta(y^\eta, L_\Pi(t, G)) \equiv \beta^\eta,$$

and it is easy to see that $\{\beta^\eta: 0 \leqq \eta \leqq 1\}$ is bounded. Let $\{\eta_m\}$ be a positive null sequence and let $\bar{\beta}$ be an accumulation point of $\{\beta^{\eta_m}\}$; we denote by the same indices a subsequence of $\{\beta^{\eta_m}\}$ converging to $\bar{\beta}$. Corresponding to $\{\beta^{\eta_m}\}$ there is a sequence $\{y^{\eta_m}\}$ having an accumulation point $\bar{y}$ and again we denote by the same indices a subsequence of $\{y^{\eta_m}\}$ converging to $\bar{y}$. Selecting a further subsequence of $\{\beta^{\eta_m}\}$ if necessary we find by continuity of $\beta(\cdot, \cdot)$ that $\beta(\bar{y},$

$L_\Pi(t, G)) = \bar{\beta}$. But now $\bar{y} \in \bar{H}(t, G)$ and the continuity of $\eta \to \bar{J}_\eta(t, G)$ together with (27), Theorem 6.1 (i) and the continuity of δ_Π permits the conclusion that $\beta(\bar{y}, L_\Pi(t, G)) = 0$ so that $\bar{\beta} = 0$. This establishes the corollary.

Before proceeding further along these lines we introduce the concept of a *monotone* one-parameter family.

Definition 6.1. A one parameter family $\{M_\eta : 0 \leqq \eta \leqq 1\}$ of functions $M_\eta : I \times \Omega^n \to \mathscr{K}^n(I)$ is said to be *monotone* if and only if $\bar{\sigma}(M_{\eta_1}(t, G), M_{\eta_2}(t, G)) = 0$ on $I \times \Omega^n$ for all η_1, η_2 satisfying $0 \leqq \eta_1 \leqq \eta_2 \leqq 1$.

The following lemma is a trivial consequence of the definitions.

LEMMA 6.1. *If* $\{\bar{J}_\eta : 0 \leqq \eta \leqq 1\}$ *is a monotone family, then for each* $(t, G) \in I \times \Omega^n$ *the function* $\eta \to V_\Pi^\eta(t, G)$ *is nonincreasing on* $[0, 1]$.

The hypothesis of monotonicity permits us to define an approximation alternative to (27). This assertion is made precise in Theorem 6.2 below. We need

LEMMA 6.2. *If* $\{\bar{J}_\eta : 0 \leqq \eta \leqq 1\}$ *is a monotone family, then the family* $\{P_\eta^\Pi : 0 \leqq \eta \leqq 1\}$ *defined by*

$$P_\eta^\Pi(t, G) = \{\varphi \in \bar{J}_\eta(t, G) : V_\Pi^\eta(t, G) \leqq \delta_\Pi(t, \varphi) \leqq V_\Pi(t, G)\} \tag{28}$$

is a monotone family.

Proof. That $P_\eta^\Pi(t, G) \in \mathscr{K}^n(I)$ is the consequence of an argument like that for Lemma 5.3. Monotonicity follows easily from Lemma 6.1.

By means of a proof so like that of Corollary 6.1 that we omit it, one may establish

THEOREM 6.2. *If* $\{\bar{J}_\eta : 0 \leqq \eta \leqq 1\}$ *is a monotone family and if* $\eta \to \bar{J}_\eta(t, G)$ *is continuous at* $\eta = 0$, *then* $\eta \to P_\eta^\Pi(t, G)$ *is continuous at* $\eta = 0$.

The next result follows easily from Lemma 6.2.

LEMMA 6.3. *If* $\{\bar{J}_\eta : 0 \leqq \eta \leqq 1\}$ *is a monotone family, then for each* $(t, G) \in I \times \Omega^n$ *the function* $\eta \to \tau_\Pi^\eta(t, G)$ *defined by*

$$\tau_\Pi^\eta(t, G) = \min\{t_\Pi(t, \varphi) : \varphi \in P_\eta^\Pi(t, G)\} \tag{29}$$

is nonincreasing on $[0, 1]$.

THEOREM 6.3. *If* $\{\bar{J}_\eta : 0 \leqq \eta \leqq 1\}$ *is a monotone family and if, for fixed* $(t, G) \in I \times \Omega^n$, $\eta \to \bar{J}_\eta(t, G)$ *is continuous at* $\eta = 0$, *then the function* $\eta \to \tau_\Pi^\eta(t, G)$ *defined by* (29) *is continuous at* $\eta = 0$.

Proof. Lemma 6.3 implies that the function under discussion is upper semicontinuous at $\eta = 0$. By means of an argument which is by now standard, based on Lemma 5.2 and Theorem 6.2, it follows that the function is lower semicontinuous at $\eta = 0$.

Comment. The significance of Theorems 6.2 and 6.3 is apparent when one observes that $P_0^\Pi(t, G) = \bar{H}^\Pi(t, G)$ and $\tau_\Pi^0(t, G) = T_\Pi(t, G)$. Also, it is easily concluded from (28) and (29) that since $\bar{\sigma}(\bar{J}_\eta^\Pi(t, G), P_\eta^\Pi(t, G)) = 0$ we have $\tau_\Pi^\eta(t, G) \leqq T_\Pi^\eta(t, G)$ for $\eta \in [0, 1]$.

Now let us define

(30) $$Q_\eta^\Pi(t, G) = \{\varphi \in P_\eta^\Pi(t, G): \tau_\Pi^\eta(t, G) \leqq t_\Pi(t, \varphi) \leqq T_\Pi(t, G)\};$$

we may state

THEOREM 6.4. *If* $\{\bar{J}_\eta: 0 \leqq \eta \leqq 1\}$ *is a monotone family, then the family* $\{Q_\eta^\Pi: 0 \leqq \eta \leqq 1\}$ *defined by* (30) *is a monotone family. If, in addition,* $\eta \to \bar{J}_\eta(t, G)$ *is continuous at* $\eta = 0$ *for fixed* $(t, G) \in I \times \Omega^n$, *then* $\eta \to Q_\eta^\Pi(t, G)$ *is continuous at* $\eta = 0$.

Proof. The first assertion is a consequence of (30) and Lemma 6.2 together with an argument like that suggested for Lemma 5.4. The second assertion follows from Theorems 6.2, 6.3 and an argument differing only in minor details from that of Corollary 6.1.

Comment. The significance of Theorem 6.4 becomes apparent upon observing that $Q_0^\Pi(t, G) = {}^\Pi L(t, \bar{H}(t, G))$.

7. Optimal Control: Approximation Theory. II

For the theory to be developed in this section we assume, as in the preceding section, that we are given $\Pi: I \times \Omega^n$ and $\bar{H}: I \times \Omega \to \mathscr{H}^n(I)$, both being continuous and the latter being an element of $\mathscr{E}$. We also assume the existence of a *monotone* one-parameter family $\{\bar{J}_\eta: 0 \leqq \eta \leqq 1\}$ of functions in $\mathscr{E}$ having the properties that $\bar{H} = \bar{J}_0$ and $\eta \to \bar{J}_\eta(t_0, x_0)$ is continuous at $\eta = 0$ for each $(t_0, x_0) \in I \times E^n$. The set $B_\infty(\Pi)$ with which we shall be concerned is that defined relative to $\bar{H}$; throughout this section we shall assume that $B_\infty(\Pi)$ is nonvoid. It will prove to be convenient—and no loss of generality—to assume that $I = [0, 1]$.

By virtue of the assumptions made in the preceding paragraph, Theorem 5.6 implies that $B_\infty(\Pi)$ is relatively open in $I \times E^n$; hence, the complement of $B_\infty(\Pi)$, which will be denoted by b_∞, is relatively closed in $I \times E^n$. The set $b_\infty(\Pi)$ is certainly nonvoid since $\{(1, x): x \in E^n\} \subset b_\infty(\Pi)$. In view of these facts, the number

(31) $$\tau^*(t_0, x_0, \varphi) = \min \{\tau \in [t_0, 1]: (\tau, \varphi(\tau)) \in b_\infty(\Pi)\}$$

is well defined for each $(t_0, x_0) \in I \times E^n$ and all $\varphi \in \bar{J}_1(t_0, x_0)$. Moreover, $t_0 \leqq t^*(t_0, x_0, \varphi)$, with equality holding if and only if $(t_0, x_0) \in b_\infty(\Pi)$. It is not difficult to establish that $\varphi \to \tau^*(t_0, x_0, \varphi)$ is lower semicontinuous on $\bar{J}_1(t_0, x_0)$; however, we shall require the stronger condition:

(7.0) for each $(t_0, x_0) \in I \times E^n$, $\tau^*(t_0, x_0, \cdot)$ is continuous on $\bar{J}_1(t_0, x_0)$.

Throughout the remainder of this section (7.0) will be a standard hypothesis.

Let us now define, for each $(t_0, x_0) \in I \times E^n$ and all $\varphi \in \bar{J}_1(t_0, x_0)$,

(32) $$\mu(t_0, x_0, \varphi) = \alpha(\varphi(\tau^*(t_0, x_0, \varphi)), \Pi(\tau^*(t_0, x_0, \varphi)));$$

by virtue of (7.0), $\mu(t_0, x_0, \cdot)$ is continuous on $\bar{J}_1(t_0, x_0)$ for each $(t_0, x_0) \in I \times E^n$. Hence a number $W_\eta^\Pi(t_0, x_0)$ is well defined by

(33) $$W_\eta^\Pi(t_0, x_0) = \min \{\mu(t_0, x_0, \varphi): \beta(\varphi, \bar{J}_\eta(t_0, x_0)) = 0\}$$

for all $(t_0, x_0) \in I \times E^n$ and all $\eta \in [0, 1]$. From (33) and the monotonicity of

$\{J_\eta: 0 \leqq \eta \leqq 1\}$ it follows readily that the function $\eta \to W_\eta^\Pi(t_0, x_0)$ is nonincreasing on [0, 1]. An easy argument based on (33) and Corollary 5.2 supports the conclusion that $W_0^\Pi(t_0, x_0) = V_\Pi(t_0, x_0)$. One needs only to observe that by virtue of (11), (12), (25), (31), and (32) one may write (26) as

$$V_\Pi(t, x) = \min\{\mu(t, x, \varphi): \beta(\varphi, K(t, x)) = 0\}; \tag{34}$$

an assumption that the right-hand side of (34) is greater than the right-hand side of (33) then leads to a contradiction of Theorem 5.8 (i). A standard argument from (33) utilizing the continuity of $\mu(t_0, x_0, \cdot)$ and of $\eta \to \bar{J}_\eta(t_0, x_0)$ establishes the fact that $\eta \to W_\eta^\Pi(t_0, x_0)$ is lower semicontinuous at $\eta = 0$. Combined with the monotonicity cited in the preceding paragraph, this last result implies that $\eta \to W_\eta^\Pi(t_0, x_0)$ is continuous at $\eta = 0$.

Developments thus far permit a nonvoid set $K_\eta^\Pi(t_0, x_0) \subset \bar{J}_\eta(t_0, x_0)$ to be defined by

$$K_\eta^\Pi(t_0, x_0) = \{\varphi \in \bar{J}_\eta(t_0, x_0): W_\eta^\Pi(t_0, x_0) \leqq \mu(t_0, x_0, \varphi) \leqq V_\Pi(t_0, x_0)\}. \tag{35}$$

A repetition, with minor variations, of the arguments of the preceding section completes the proof of the following central result of this section.

THEOREM 7.1. *The assumptions of the first paragraph of this section together with* (7.0) *imply:* (*i*) *the function* $\eta \to W_\eta^\Pi(t_0, x_0)$ *defined by* (33) *is nonincreasing on* [0, 1] *and continuous at* $\eta = 0$, *where* $W_0^\Pi(t_0, x_0) = V_\Pi(t_0, x_0)$; (*ii*) *the family* $\{K_\eta^\Pi: 0 \leqq \eta \leqq 1\}$ *defined by* (35) *is monotone and for each* $(t_0, x_0) \in I \times E^n$ *the function* $\eta \to K_\eta^\Pi(t_0, x_0)$ *is continuous at* $\eta = 0$, *where* $K_0^\Pi(t_0, x_0) = L_\Pi(t_0, x_0)$.

8. Generalized Differential Equations and Optimal Control

In this section we examine the ramifications of the preceding three sections in the case in which the members of $\mathscr{E}$ to be examined are solution families of (1) or (3) corresponding to particular choices of the right-hand members of these equations. Throughout this section we assume given a continuous (target) function $\Pi: I \to \Omega^n$ and a function $R: E^1 \times E^n \to \Gamma^n$.

THEOREM 8.1. *If* $R: E^1 \times E^n \to \Gamma^n$ *satisfies* (*) *and if* $\bar{H}: I \times \Omega^n \to \mathscr{H}^n(I)$ *is the restriction to* I *of the solution family of* (3), *then* (*i*) $B_0(\Pi)$ *is relatively closed in* $I \times E^n$; (*ii*) *for each* $\eta \in [0, \infty)$, $B_\eta(\Pi)$ *is bounded;* (*iii*) *the conclusions of* (*i*), (*iii*) *of Theorem 5.4 hold. If R satisfies* (**) *or if R satisfies the hypotheses of Theorem 3.3, then in addition to* (*i*), (*ii*), (*iii*) *there follow:* (*iv*) *the conclusions of* (*ii*), (*iv*) *of Theorem 5.4 hold;* (*v*) *for each* $\eta \in (0, \infty)$, $B_\eta(\Pi)$ *is relatively open in* $I \times E^n$, *as is* $B_\infty(\Pi)$.

Proof. (i) Theorems 3.2, 5.7; (ii) Theorems 3.1, 5.8; (iii) Theorems 3.1, 3.2, 5.4; (iv) Theorems 3.3, 3.4, 5.4; (v) Theorems 3.3, 3.4, 5.6.

Comment. Theorem 8.1, together with the remaining results of Section 5 which obtain by virtue of the fact that $\bar{H} \in \mathscr{E}$, is the promised generalization of all but the last section of [9]. Indeed, the hypotheses of that paper are precisely those of the present Theorem 3.3. A comparison of Theorem 8.1 with [9, Theorem 4] makes evident the advantages accruing to the formulation of optimal control theory given in Section 5. For whereas in the latter theorem an hypothesis is required to the effect that $B_\infty(\Pi)$ is nonvoid, no such "controllability" hypothesis appears in Theorem 8.1.

In order to investigate the implications of Sections 6, 7 for generalized differential equations we must introduce the following definition.

Definition 8.1. A family $\{S_\eta: 0 \leqq \eta \leqq 1\}$ of functions $S_\eta: E^1 \times E^n \to \Gamma^n$ is said to be a *monotone approximation* to a function $R: E^1 \times E^n \to \Gamma^n$ if and only if the following conditions are satisfied:

(i) for η_1, η_2 satisfying $0 \leqq \eta_1 \leqq \eta_2 \leqq 1$, $\bar{\Delta}(S_{\eta_1}(t, x), S_{\eta_2}(t, x)) = 0$ on $E^1 \times E^n$ and $R = S_0$;

(ii) the family $\{\eta \to S_\eta(t, x): (t, x) \in E^1 \times E^n\}$ is equicontinuous at $\eta = 0$.

From this and Definition 4.1, it follows that a monotone approximation is a uniform approximation. Moreover, it is easy to see that the family $\{\bar{J}_\eta: 0 \leqq \eta \leqq 1\}$ of solution families corresponding to the equations of Theorem 4.3 is monotone in the sense of Definition 6.1.

The following two theorems are immediate consequences of Definition 8.1, Theorem 4.3 and the corresponding results of Section 6 and 7.

THEOREM 8.2. *If* $\{S_\eta: 0 \leqq \eta \leqq 1\}$ *is a monotone approximation to* $R: E^1 \times E^n \to \Gamma^n$ *and if for each* $\eta \in [0, 1]$, S_η *satisfies* (*), *then for each* $(t, G) \in I \times \Omega^n$ *the following conclusions are valid: (i) the function* $\eta \to P_\eta^\Pi(t, G)$ *defined by* (28) *is continuous at* $\eta = 0$; *(ii) the function* $\eta \to Q_\eta^\Pi(t, G)$ *defined by* (30) *is continuous at* $\eta = 0$.

THEOREM 8.3. *If* $\{S_\eta: 0 \leqq \eta \leqq 1\}$ *is a monotone approximation to* $R: E^1 \times E^n \to \Gamma^n$, *if* S_η *satisfies* (*) *for* $\eta \in (0, 1]$, *if R satisfies either* (**) *or the hypotheses of Theorem 4.2 and if condition* (7.0) *is satisfied, then for each* $(t, x) \in I \times E^n$ *the function* $\eta \to K_\eta^\Pi(t_0, x_0)$ *is continuous at* $\eta = 0$.

Comment. Theorems 8.2 and 8.3 are the promised generalizations of [10, Theorem 5] and [11, Theorem 3], respectively.

Remark 8.1. The following device is, in essence, what is used in [10] and [11], and may be extended to apply in our present, more general, context. With R assumed to be continuous on $E^1 \times E^n$ an approximation $\{S_\eta: 0 \leqq \eta \leqq 1\}$ to R is determined on a compact set $\mathscr{D} \subset E^1 \times E^n$ in such a way that the following conditions are satisfied:

(a) condition (i) of Definition 8.1 holds;

(b) for each $\eta \in (0, 1]$, S_η is continuous on $\mathscr{D}$;

(c) for each $(t, x) \in \mathscr{D}$, $\eta \to S_\eta(t, x)$ is continuous at $\eta = 0$.

Conditions (a), (b), (c), together with Dini's theorem on monotone convergence, imply that

(d) the family $\{\eta \to S_\eta(t, x): (t, x) \in \mathscr{D}\}$ is equicontinuous at $\eta = 0$.

With $\mathscr{D}$ being either a cylinder (as in [10], [11]) or a sphere (as in Theorem 1.3), the functions of the family $\{S_\eta: 0 \leqq \eta \leqq 1\}$ may be extended, by means of the device used in the proof of Theorem 1.3, to $E^1 \times E^n$ in such a way that conditions (i) and (ii) of Definition 8.1 hold, and the extended S_η is continuous on $E^1 \times E^n$ for all $\eta \in [0, 1]$.

In connection with [11, Theorem 3] and Theorem 8.3 above it is of great importance to determine conditions under which the function $\tau^*(t_0, x_0, \cdot)$ defined by (31) satisfies (7.0). We state such a condition in Theorem 8.4 below; first we restrict the set $B_\infty(\Pi)$. Set $\mathscr{M} = \overline{B_\infty(\Pi)} \cap b_\infty(\Pi)$, where the superior bar denotes closure, and let $(\lambda_0, \xi_0) \in \mathscr{M}$. We assume without loss of generality that $I = [0, 1]$ and that

(8.0) there exists $\psi: E^n \to (0, 1]$ of class C^1 and a neighborhood $\mathscr{N}$ of (λ_0, ξ_0) such that

$$\mathscr{M} \cap \mathscr{N} = \{(t, x) \in \mathscr{N}: t = \psi(x)\}$$

and

$$\{(t, x) \in \mathscr{N}: t < \psi(x)\} \subset B_\infty(\Pi).$$

THEOREM 8.4. *Let $B_\infty(\Pi)$ satisfy* (8.0); *given $(t_0, x_0) \in I \times E^n$, if for all $(t, x) \in F(\overline{H}(t_0, x_0))$ and all $\sigma \in R(t, x)$,*

$$0 \leqq \psi_x(x) \circ \sigma \leqq \|\psi_x(x)\| \quad \|\sigma\| < 1, \tag{8.1}$$

where $\overline{H}$ is the restriction to I of the solution family of (1), *then $\tau^*(t_0, x_0, \cdot)$ is continuous on $\overline{H}(t_0, x_0)$.*

Proof. Let $\varphi_0 \in \overline{H}(t_0, x_0)$ be fixed and set $\lambda_0 = \tau^*(t_0, x_0, \varphi_0)$, $\xi_0 = \varphi_0(\tau^*(t_0, x_0, \varphi_0))$; then $(\lambda_0, \xi_0) \in \mathscr{M}$ and (8.0) applies. Define $\theta: [\lambda_0, 1] \to (0, 1]$ by $\theta(t) = \psi(\varphi_0(t))$; it follows readily that for $t_1, t_2 \in [\lambda_0, 1]$

$$\theta(t_2) - \theta(t_1) = \int_{t_1}^{t_2} \psi_x(\varphi_0(\tau)) \circ \dot{\varphi}_0(\tau)\, d\tau. \tag{36}$$

From (36) and (8.1) follow the easy consequences

$$|\theta(t_2) - \theta(t_1)| \leqq K|t_2 - t_1| \text{ for some } K \in [0, 1) \text{ and all } t_1, t_2 \in [\lambda_0, 1]$$

and

$$\lambda_0 \leqq \theta(t) \leqq t \text{ for all } t \in [\lambda_0, 1].$$

Hence θ is a contraction mapping on $[\lambda_0, 1]$ and from Banach's fixed-point theorem it follows that λ_0 is the unique fixed point of θ in $[\lambda_0, 1]$. Now let $\bar{\tau}$ be an arbitrary accumulation point of the sequence $\{\tau^*(t_0, x_0, \varphi_m)\}$, where $\lim_{m\to\infty} \varphi_m = \varphi_0$. Then by continuity and the definition of τ^*, $\bar{\tau}$ satisfies $\bar{\tau} = \theta(\bar{\tau})$, from which we conclude, by virtue of the fixed-point property and the definition of λ_0, that $\bar{\tau} = \lambda_0$.

Comment. It is noteworthy that (8.1) is an only slightly strengthened form of [11, (vii)].

9. Final Remarks

Recently, Castaing [21] has formulated an existence theory for a class of generalized differential equations in which the "contingent" $R: E^1 \times E^n \to \Gamma^n$ is required to satisfy conditions weaker than those assumed in the present article. Filippov [13] had already stated, without proof, results related to those of Castaing. The Castaing-Filippov theory bears a relationship to the existence theory expounded in Section 2 which is the analogue of the relationship, in ordinary differential equations, which the Carathéodory existence theory bears to the Cauchy-Peano theory. Castaing emphasizes, as Filippov does not, the properties of the solution family, so that his main theorem [21, Theorem 2] is very close to being a generalization of the content of the present Theorems 2.3 and 3.2. One thus necessarily anticipates that the bulk of the theory developed in this article may be generalized by adopting the Castaing-Filippov hypotheses.

A theory of generalized dynamical systems, initially formulated by Barbashin [22] and expanded by Roxin [23], has stemmed from an abstraction of the properties of the attainability function of a generalized differential equation (or contingent equation). With [23] as a starting point, Varaiya [24] (cf. [25]) has recently shown the value of examining the properties of the family of "motions" (Roxin [23] calls them "trajectories") corresponding to a given generalized dynamical system. In particular, Varaiya obtains a result [24, Theorem 3.1], similar to the present Theorem 2.3, which he applies to an optimization problem in a manner analogous to that of Section 5. In the light of these developments and of the theory expounded in this paper, one is led to conjecture that a fruitful theory of generalized dynamical systems—alternative to the Barbashin-Roxin theory—may be based on the postulated existence of a mapping $\bar{H}: I \times \Omega^n \to \mathscr{H}^n(I)$ possessing properties such as (18) together with other suitable abstractions of the properties of solution families of generalized differential equations.

REFERENCES

[1] A. MARCHAUD, Sur les champs des demi-droites et les équations différentielles du premier ordre, *Bull. Soc. Math. France* **63** (1934), 1–38.

[2] A. MARCHAUD, Sur les champs continus de demi-cones convexes et leurs intégrales, *Compositio Math.* **3** (1936), 89–127.

[3] S. C. ZAREMBA, Sur les équations au paratingent, *Bull. Sci. Math.* (2) **60** (1936), 139–160.

[4] T. WAŻEWSKI, Systèmes de commande et équations au contingent, *Bull. Acad. Polon. Sci.* **9** (1961), 151–155.

[5] T. WAŻEWSKI, Sur une condition équivalente à l'équation au contingent, *Bull. Acad. Polon. Sci.* **9** (1961), 865–867.

[6] E. ROXIN, On generalized dynamical systems defined by contingent equations, *J. Differential Equations* **1** (1965), 188–205.

[7] A. F. FILIPPOV, On certain questions in the theory of optimal control, *J. SIAM Control Ser. A* **1** (1962), 76–84.

[8] E. ROXIN, The existence of optimal controls, *Michigan Math. J.* **9** (1962), 109–119.

[9] T. F. BRIDGLAND, JR., and J. S. HINKEL, The minimum miss distance problem, *Proc. Amer. Math. Soc.* **18** (1967), 457–464.

[10] H. HERMES, The equivalence and approximation of optimal control problems, *J. Differential Equations* **1** (1965), 409–426.

[11] T. F. BRIDGLAND, JR., On the problem of approximate synthesis of optimal controls, *J. SIAM Control* **5** (1967), 326–344; *ibid.* **6** (1968), 58.

[12] P. HARTMAN, *Ordinary Differential Equations*, J. Wiley & Sons, New York, 1964.

[13] A. F. FILIPPOV, Differential equations with many-valued discontinuous right-hand side, *Soviet Math.* **4** (1963), 941–945.

[14] T. BONNESEN and W. FENCHEL, *Theorie der Konvexen Körper*, Chelsea Publ. Co., New York, 1948.

[15] H. G. EGGLESTON, *Convexity*, Cambridge University Press, Cambridge, 1958.

[16] F. A. VALENTINE, *Convex Sets*, McGraw-Hill, New York, 1964.

[17] E. W. HOBSON, *The Theory of Functions of a Real Variable and the Theory of Fourier Series*, Vol. II, Dover, New York, 1957.
[18] A. E. TAYLOR, *General Theory of Functions and Integration*, Blaisdell, New York, 1965.
[19] J. SZARSKI, *Differential Inequalities*, Polish Scientific Publishers, Warsaw, 1965.
[20] T. F. BRIDGLAND, JR., A criterion for weak controllability, *Math. Systems Theory* **2** (1968), 51–56.
[21] C. CASTAING, Sur les équations différentielles multivoques, *C. R. Acad. Sci. Paris* **263** (1966), A63–A66.
[22] E. A. BARBASHIN, On the theory of generalized dynamical systems, *Uchen. Zapiski Moskov. Gos. Univ.* No. 135 (1949), 110–133.
[23] E. ROXIN, Stability in general control systems, *J. Differential Equations* **1** (1965), 115–150.
[24] P. P. VARAIYA, On the existence of solutions to a differential game, *J. SIAM Control* **5** (1967), 153–162.
[25] E. O. ROXIN, On Varaiya's definition of a differential game, *Proc. U.S.–Japan Seminar on Diff. and Functional Equations* (ed. W. A. Harris, Jr., and Y. Sibuya), pp. 491–495, W. A. Benjamin Inc., New York, 1967.

(Received 8 December 1967)