Mathematical Systems Theory: Causality*

by

T. G. WINDEKNECHT

Case Institute of Technology
Cleveland, Ohio

The concept of state is studied in a new set-theoretic formalism for systems theory. Starting with the notion of a time system as a set of ordered pairs of abstract time functions, the concepts of (i) non-anticipation and (ii) causality are introduced. It is proved that the class of causal systems (those possessing a set of states and functional state transitions) is precisely the class of non-anticipatory systems. It is shown that every causal system has a series decomposition consisting of a transition system followed by a static system. It is proved that a state set for a causal system is always constructible using a class of "natural" partitions of the system input set. This latter construction generalizes the result known for certain functional discrete systems to a much more general situation.

1. Introduction. Recently, there have been significant efforts made to clarify the concept of state in systems theory. For example, Zadeh [1] has explicitly discussed this question for systems which are non-probabilistic and non-anticipatory. Nerode [2] has given a construction in automata theory which indicates that the states of a sequential machine are essentially a set of "natural" equivalences classes of input sequences. Kalman [3] made a similar construction for the case of linear discrete-time systems and has shown there are "natural" equivalence classes of input sequences to serve as states in this case. It is important in these latter two cases that a "system" is taken to be a certain kind of map on sequences.

Apparently, to define the concept of state for systems in a more general situation, one first has to arrive at an appropriately general concept of system. Mesarović [4] has dealt with the notion of a (general) system as an n-ary relation. In his development, the state of a system is identified with the "connecting set" which arises in the general decomposition (reticulation) of n-ary relations. This concept would appear to be unrelated to the others mentioned; however, the concept of system employed is very much more general than that referred to by the other investigators.

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The purpose of this paper is to consider again the concept of state and, in particular, to show the interrelation of the several previously mentioned points of view. To do this, we set up a more restrictive concept of system than that in [4]; however, it is consistent with that given and it includes the "systems" of [2] and [3] as special cases. Our formalism is similar to that proposed by Mesarović [5]. Our principal findings are: (i) a state set of a (time) system exists if and only if the system is non-anticipatory; (ii) a state set is constructible from "natural" partitions of the system input set (even in the non-functional, non-discrete-time case); (iii) every system with a state set may be decomposed into the series interconnection of a transition system followed by a static system; and (iv) (which follows from (iii)) the set of "state trajectories" associated with a state set for a system is in fact a "connecting set" for this basic decomposition of the system.

Our set-theoretic notation is relatively standard and by and large consistent with [6, 7].

2. Non-Anticipatory Time Systems. We begin by introducing a formalism for general time systems. The formal concept of system employed (Definition 2.2 below) is Mesarović's general system specialized in two ways: (i) the system is taken to be a 2-ary rather than an n-ary relation, and (ii) the two systems objects are taken to be sets of (generalized) time functions rather than arbitrary sets. The concepts of non-anticipation and initial states are introduced in Definitions 2.4 and 2.8, respectively. The question of the existence of sets of initial states for time systems is treated (Theorem 2.9) and shown to be inherently related to the property of non-anticipation of the system.

Remark 2.1. In the following, all sets denoted by single capital letters are assumed to be nonempty. A set G is said to be (strictly) *R-simply ordered* if the relation $R \subseteq (G \times G)$ satisfies: (i) $\neg(tRt)$; (ii) tRt' & $t'Rt'' \Rightarrow tRt''$; and (iii) $t \neq t' \Rightarrow tRt' \vee t'Rt$ $(t, t', t'' \in G)$. If G is R-simply ordered and $t \in G$, then the set $U(t) = \{t' \mid tRt'\}$ is the *t-section* of G. If tRt', then the set $(t, t'] = \{t'' \mid tRt''$ & $t''Rt'\} \cup \{t'\}$ is the *(t, t')-interval* of G. Evidently, if tRt' then $\{(t, t'], U(t')\}$ is a partition of the set $U(t)$.

Definition 2.2. A *time object* is any set $V \subseteq A^T = \{v \mid v: T \to A\}$ such that A and T are sets and $T = U(t_0)$ in some R-simply ordered set G $(t_0 \in G)$. If V is a time object and $v \in V$, then v is a *time function.* If $v: T \to A$ is a time function and $t \in T$, then the restriction $v/(t_0, t] = \{(t', v(t')) \mid t' \in (t_0, t]\}$ is the *t-initial segment* of v. If $t, t' \in T$ and tRt', then the restriction $v/(t, t']$ is the *(t, t')-segment* of v. A (2-ary) *time system* is any relation $S \subseteq (A^T \times B^T)$ such that A, B, and T are sets and $T = U(t_0)$ in some R-simply ordered set G. The time objects $\mathscr{D}S = \{x \mid (\exists y): xSy\}$ and $\mathscr{R}S = \{y \mid (\exists x): xSy\}$ are, respectively, the *input set* (or domain) and the *output set* (or range) of S. A, B, and T are the *input space*, the *output space*, and the *time set* of S, respectively. If $t \in T = U(t_0)$, then the relation $\left(\overset{t}{\underset{t_0}{\sim}}\right) \subseteq (\mathscr{D}S \times \mathscr{D}S)$ such that

$$x \overset{t}{\underset{t_0}{\sim}} x' \Leftrightarrow x/(t_0, t] = x'/(t_0, t]$$

is *t-equivalence* on $\mathscr{D}S$.

Remark 2.3. $\left(\overset{t}{\underset{t_0}{\sim}}\right)$ is an equivalence relation. If $x \in \mathscr{D}S$ and $t \in U(t_0)$, then $x \bmod \left(\overset{t}{\underset{t_0}{\sim}}\right)$ is the x-equivalence class in $\mathscr{D}S$ under $\left(\overset{t}{\underset{t_0}{\sim}}\right)$. Also,

$$\mathscr{D}S/\left(\overset{t}{\underset{t_0}{\sim}}\right) = \left\{x \bmod \left(\overset{t}{\underset{t_0}{\sim}}\right) \middle| \, x \in \mathscr{D}S\right\}$$

is a partition of $\mathscr{D}S$. We define

$$\mathscr{E}_S = \bigcup_{t \in U(t_0)} \mathscr{D}S/\left(\overset{t}{\underset{t_0}{\sim}}\right).$$

Definition 2.4. Henceforth, let $T = U(t_0)$ be a nonempty subset of the R-simply ordered set G ($t_0 \in G$). A time system $S \subseteq (A^T \times B^T)$ is *functional* if and only if $S: \mathscr{D}S \to \mathscr{R}S$. If S is functional, then S is *non-anticipatory* if and only if for all $x, x' \in \mathscr{D}S$ and all $t \in T$,

$$x \overset{t}{\underset{t_0}{\sim}} x' \Rightarrow S(x)(t) = S(x')(t).$$

If S is not functional, then S is *non-anticipatory* if and only if there exists a set $F \subseteq \mathscr{R}S^{\mathscr{D}S}$ of (into) functional non-anticipatory systems such that $S = \cup F = \{(x, y) | \, (\exists f): f \in F \;\&\; y = f(x)\}$.

Remark 2.5. Definition 2.4 is simply a formalization of the usual idea of non-anticipation, i.e., the present value of any output of the system is independent of future values of the system input.

THEOREM 2.6. *A time system S is non-anticipatory if and only if $S = \cup F_S^*$, where*

$$F_S^* = \{f | \, f: \mathscr{D}S \to \mathscr{R}S \;\&\; f \subseteq S \;\&\; f \textit{ is non-anticipatory}\}.$$

Proof. The sufficiency is obvious. Let S be non-anticipatory. If S is functional, then $F_S^* = \{S\}$ and hence $S = \cup F_S^*$. If S is not functional, then $S = \cup F$, where $F \subseteq \mathscr{R}S^{\mathscr{D}S}$ contains only non-anticipatory elements. Clearly, $F \subseteq F_S^*$. Also, $\cup F_S^* \subseteq S$. Hence

$$S = \cup F \subseteq \cup F_S^* \subseteq S,$$

i.e., $S = \cup F_S^*$.

Remark 2.7. We next formalize the concept of initial states for a time system. Intuitively, a set of initial states for a time system is an auxiliary set used so that the system may be described as a function. With such a functional representation, the system becomes "predictable" if the initial state is known.

Definition 2.8. A set Z is a *set of initial states* for the time system $S \subseteq (A^T \times B^T)$ if and only if there exists a map $r: Z \times \mathscr{D}S \times T \to B$ such that (i) for all $x \in \mathscr{D}S$ and all $y \in \mathscr{R}S$,

$$xSy \Leftrightarrow (\exists z)(\forall t): y(t) = r(z, x, t) \qquad (z \in Z;\ t \in T),$$

and (ii) for all $x, x' \in \mathscr{D}S$ and all $t \in T$,

$$x \overset{t}{\underset{t_0}{\sim}} x' \Rightarrow (\forall z): r(z, x, t) = r(z, x', t) \qquad (z \in Z).$$

THEOREM 2.9. *If S is a time system, then there exists a set of initial states for S if and only if S is non-anticipatory.*

Proof. If $S \subseteq (A^T \times B^T)$ is non-anticipatory, then $S = \cup F$, where $F \subseteq \mathscr{R}S^{\mathscr{D}S}$ contains only non-anticipatory elements. Then, for the map $r: F \times \mathscr{D}S \times T \to B$ such that $r(f, x, t) = f(x)(t)$ we have

$$xSy \Leftrightarrow (\exists f): y = f(x) \Leftrightarrow (\exists f)(\forall t): y(t) = f(x)(t)$$
$$\Leftrightarrow (\exists f)(\forall t): y(t) = r(f, x, t)$$

and

$$x \overset{t}{\underset{t_0}{\sim}} x' \Rightarrow (\forall f): f(x)(t) = f(x')(t) \Rightarrow (\forall f): r(f, x, t) = r(f, x', t),$$

which proves F is a set of initial states for S. Conversely, given $r: Z \times \mathscr{D}S \times T \to B$ such that Z is a set of initial states for S, if we associate the map $r_z: \mathscr{D}S \to \mathscr{R}S$ with each $z \in Z$ such that

$$y = r_z(x) \Leftrightarrow (\forall t): y(t) = r(z, x, t),$$

we have

$$x \overset{t}{\underset{t_0}{\sim}} x' \Rightarrow (\forall z): r(z, x, t) = r(z, x', t) \Rightarrow (\forall z): r_z(x)(t) = r_z(x')(t).$$

Moreover, if $F = \{r_z \mid z \in Z\}$, then

$$xSy \Leftrightarrow (\exists z)(\forall t): y(t) = r(z, x, t) \Leftrightarrow (\exists z): y = r_z(x),$$

i.e., $S = \cup F$. Therefore, S is non-anticipatory.

3. Causal Time Systems. In this section, we introduce the concept of a causal time system and present our basic results. Causality is a term most often used to assert the existence of a "state set", a "state transition function", and an "output function" for the system. In terms of these auxiliary sets and functions, the behavior in an input-output sense of a system is revealed to be sequential or inductive. It is generally recognized that non-anticipation is a necessary condition for causality. Our finding (Theorem 3.6 below) is that it is also sufficient.

Definition 3.1. A time system $S \subseteq (A^T \times B^T)$ is *causal* if and only if there exists a 4-tuple of sets (Q, Q_0, τ, θ) such that

(i) $Q_0 \subseteq Q$

(ii) $\tau: Q_0 \times \mathscr{D}S \times (T \cup \{t_0\}) \to Q$

(iii) $\theta: Q \to B$

(iv) for all $x \in \mathscr{D}S$ and all $q \in Q_0$, $\tau(q, x, t_0) = q$

(v) for all $x, x' \in \mathscr{D}S$, all $q, q' \in Q_0$, and all $t, t' \in (T \cup \{t_0\})$ such that tRt',

$$\tau(q, x, t) = \tau(q', x', t) \;\&\; x/(t, t'] = x'/(t, t']$$
$$\Rightarrow \tau(q, x, t') = \tau(q', x', t').$$

(vi) for all $x \in \mathscr{D}S$ and all $y \in \mathscr{R}S$,

$$xSy \Leftrightarrow (\exists q)(\forall t): y(t) = \theta(\tau(q, x, t)) \qquad (q \in Q_0;\; t \in T).$$

The sets Q, τ, and θ are called a *state set*, a *state transition function*, and an *output function* for S, respectively. The condition (v) is called the *state property* of τ.

Definition 3.2. If S is causal with respect to (Q, Q_0, τ, θ) and if $q \in Q_0$ and $x \in \mathscr{D}S$, then the map $\tau_{(q,x)}: T \to Q$ which takes $t \to \tau(q, x, t)$ is a *state trajectory* of S. We define

$$K_\tau = \{\tau_{(q,x)} \mid q \in Q_0 \;\&\; x \in \mathscr{D}S\}.$$

Evidently, $K_\tau \subseteq Q^T$ and hence K_τ is a time object.

Definition 3.3. Let $S \subseteq (A^T \times B^T)$ be a time system. S is a *transition system* if and only if S is causal with respect to some 4-tuple (B, B_0, τ, I), where B is the output space of S and where $I = \{(b, b) \mid b \in B\}$ is the identity map on B. S is a *static system* if and only if there exists a map $c: A \to B$ such that for all $x \in \mathscr{D}S$ and all $y \in \mathscr{R}S$,

$$xSy \Leftrightarrow (\forall t): y(t) = c(x(t)).$$

If $S \subseteq (A^T \times B^T)$ and $S' \subseteq (C^T \times D^T)$ are time systems and $\mathscr{R}S \subseteq \mathscr{D}S'$, then the *series interconnection* of S and S' is the (composition) time system

$$(S' \circ S) = \{(x, u) \mid (\exists y): xSy \;\&\; yS'u\}.$$

The time object $\mathscr{R}S$ is called a *reticulation set* of $(S' \circ S)$; the motivation here is given by Mesarović [4].

THEOREM 3.4. *A time system S is causal if and only if S is the series interconnection of some transition system S' and some static system S''.*

Proof. If $S' \subseteq (A^T \times B^T)$ is causal with respect to the 4-tuple (B, B_0, τ, I) and $S'' \subseteq (C^T \times D^T)$ is static with respect to the map $c: C \to D$ and $\mathscr{R}S' \subseteq$

$\mathscr{D}S''$, then $S = (S'' \circ S')$ is causal with respect to the 4-tuple (B, B_0, τ, c). Conversely, let $S \subseteq (A^T \times B^T)$ be causal with respect to the 4-tuple (Q, Q_0, τ, θ). Define the time system $S' \subseteq (A^T \times Q^T)$ such that for all $x \in \mathscr{D}S$ and all $u \in Q^T$,

$$xS'u \Leftrightarrow (\exists q)(\forall t)\colon u(t) = \tau(q, x, t) \qquad (q \in Q_0;\ t \in T).$$

S' is clearly causal with respect to (Q, Q_0, τ, I), where I is the identity map on Q. Therefore, S' is a transition system. Define the time system $S'' \subseteq (Q^T \times B^T)$ such that for all $u \in Q^T$ and all $y \in B^T$,

$$uS''y \Leftrightarrow (\forall t)\colon y(t) = \theta(u(t)) \qquad (t \in T).$$

S'' is clearly static. Moreover, $\mathscr{R}S' \subseteq Q^T = \mathscr{D}S''$. Finally, $S = (S'' \circ S')$.

COROLLARY 3.5. *If the time system S is causal with respect to (Q, Q_0, τ, θ), then the set of state trajectories K_τ is a reticulation set of S.*

THEOREM 3.6. *A time system is causal if and only if it is non-anticipatory.*

Proof. Let $S \subseteq (A^T \times B^T)$ be causal with respect to the 4-tuple (Q, Q_0, τ, θ). Then Q_0 is a set of initial states for S. In fact, consider the composition $(\theta \circ \tau)$. For all $x \in \mathscr{D}S$ and all $y \in \mathscr{R}S$,

$$xSy \Leftrightarrow (\exists q)(\forall t)\colon y(t) = \theta(\tau(q, x, t)) \Leftrightarrow (\exists q)(\forall t)\colon y(t) = (\theta \circ \tau)(q, x, t).$$

Moreover, for all $x, x' \in \mathscr{D}S$ and all $t \in T$,

$$x \underset{t_0}{\overset{t}{\sim}} x' \Rightarrow (\forall q)\colon q = q \ \&\ x \underset{t_0}{\overset{t}{\sim}} x'$$
$$\Rightarrow (\forall q)\colon \tau(q, x, t_0) = \tau(q, x', t_0) \ \&\ x/(t_0, t] = x'/(t_0, t]$$
$$\Rightarrow (\forall q)\colon \tau(q, x, t) = \tau(q, x', t) \Rightarrow (\forall q)\colon \theta(\tau(q, x, t)) = \theta(\tau(q, x', t))$$
$$\Rightarrow (\forall q)\colon (\theta \circ \tau)(q, x, t) = (\theta \circ \tau)(q, x', t),$$

i.e., Q_0 is a set of initial states for S. Therefore, S is non-anticipatory (by Theorem 2.9). Conversely, let $S \subseteq (A^T \times B^T)$ be non-anticipatory. Then, by Theorem 2.9, there exists a map $r\colon Z \times \mathscr{D}S \times T \to B$ such that Z is a set of initial states for S. Let $\mathscr{E}_S = \bigcup_{t \in T} \mathscr{D}S/\left(\underset{t_0}{\overset{t}{\sim}}\right)$, let $\varnothing$ be the empty set, and consider the 4-tuple (Q, Q_0, τ, θ) such that

(i) $Q = Z \times (\mathscr{E}_S \cup \{\varnothing\}) \times (T \cup \{t_0\})$

(ii) $Q_0 = Z \times \{\varnothing\} \times \{t_0\}$

(iii) $\tau\colon Q_0 \times \mathscr{D}S \times (T \cup \{t_0\}) \to Q$ is such that

$$\tau\ ((z, \varnothing, t_0), x, t) = \begin{cases} (z, \varnothing, t_0) \text{ if } t = t_0 \\ (z, x \bmod \left(\underset{t_0}{\overset{t}{\sim}}\right), t) \text{ otherwise} \end{cases}$$

(iv) $\theta: Q \to B$ is such that

$$\theta(z, E, t) = \begin{cases} b \text{ if } E = \varnothing \text{ or } t = t_0, \text{ or both} \\ r(z, \rho_t(E), t) \text{ otherwise,} \end{cases}$$

where b is any representative of B and where ρ_t is any choice function on $\mathscr{D}S/\left(\frac{t}{\widetilde{t_0}}\right)$ into $\mathscr{D}S$, i.e., any map such that $\rho_t(E) \in E$ for all $E \in \mathscr{D}S/\left(\frac{t}{\widetilde{t_0}}\right)$. The existence of ρ_t is guaranteed by the axiom of choice. Properties (i)–(iv) of Definition 3.1 are satisfied by definition. Consider (v): If $t = t_0$ then

$$\tau((z,\varnothing, t_0), x, t_0) = \tau((z',\varnothing, t_0), x', t_0) \;\&\; x/(t_0, t'] = x'/(t_0, t']$$

$$\Rightarrow (z = z') \;\&\; x \frac{t'}{\widetilde{t_0}} x' \Rightarrow \left(z, x \bmod \left(\frac{t'}{\widetilde{t_0}}\right), t'\right) = \left(z', x' \bmod \left(\frac{t'}{\widetilde{t_0}}\right), t'\right)$$

$$\Rightarrow \tau((z,\varnothing, t_0), x, t') = \tau((z',\varnothing, t_0), x', t').$$

Otherwise, if $t \neq t_0$, then

$$\tau((z,\varnothing, t_0), x, t) = \tau((z',\varnothing, t_0), x', t) \;\&\; x/(t, t'] = x'/(t, t']$$

$$\Rightarrow \left(z, x \bmod \left(\frac{t}{\widetilde{t_0}}\right), t\right) = \left(z', x' \bmod \left(\frac{t}{\widetilde{t_0}}\right), t\right) \;\&\; x/(t, t'] = x'/(t, t']$$

$$\Rightarrow (z = z') \;\&\; x/(t_0, t] = x'/(t_0, t] \;\&\; x/(t, t'] = x'/(t, t']$$

$$\Rightarrow (z = z') \;\&\; x/(t_0, t'] = x'/(t_0, t']$$

$$\Rightarrow \left(z, x \bmod \left(\frac{t'}{\widetilde{t_0}}\right), t'\right) = \left(z', x' \bmod \left(\frac{t'}{\widetilde{t_0}}\right), t'\right)$$

$$\Rightarrow \tau((z,\varnothing, t_0), x, t') = \tau((z',\varnothing, t_0), x', t').$$

Hence, (v) is satisfied. Finally, for all $x \in \mathscr{D}S$ and all $y \in \mathscr{R}S$, we have

$$xSy \Leftrightarrow (\exists z)(\forall t)\colon y(t) = r(z, x, t)$$

$$\Leftrightarrow (\exists z)(\forall t)\colon y(t) = r\left(z, \rho_t\left(x \bmod \left(\frac{t}{\widetilde{t_0}}\right)\right), t\right)$$

$$\Leftrightarrow (\exists z)(\forall t)\colon y(t) = \theta\left(z, x \bmod \left(\frac{t}{\widetilde{t_0}}\right), t\right)$$

$$\Leftrightarrow (\exists z)(\forall t)\colon y(t) = \theta(\tau((z,\varnothing, t_0), x, t)),$$

i.e., (vi) is satisfied. This proves S is causal and hence the theorem is proved.

COROLLARY 3.7. *If S is a time system, then the following statements are equivalent:*

(i) *S is non-anticipatory;*

(ii) *S is causal;*

(iii) *S is the series interconnection of some transition system S' and some static system S''.*

COROLLARY 3.8. *If the time system S is causal with respect to the 4-tuple (Q, Q_0, τ, θ), then Q_0 is a set of initial states for S.*

4. Time System Appearances and State Transitions. In this final section, we introduce the concept of "appearances" (at different times and under different input histories) of time systems. We use this concept to verify formally the intuitive idea of state transitions; namely, we prove that the image of the set of initial states under the state transition function (at any given time and initial input segment) is a set of initial states for the subsequent "appearance" of the system.

Definition 4.1. Let $S \subseteq (A^T \times B^T)$ be a time system. If $x \in \mathscr{D}S$ and $t \in T$, then the *appearance of S at t under* x is the relation $S_t^x \subseteq (A^{U(t)} \times B^{U(t)})$ such that

$$S_t^x = \left\{(x'/U(t), y/U(t)) \mid x'Sy \ \&\ x' \underset{t_0}{\overset{t}{\sim}} x\right\}.$$

Also, we define $S_{t_0}^x = S$ for all $x \in \mathscr{D}S$.

Remark 4.2. S_t^x is a time system. That is, $U(t)$ is a time set. In general,

$$\text{(i)} \ \mathscr{D}S_t^x = \left\{(x'/U(t)) \mid x' \underset{t_0}{\overset{t}{\sim}} x\right\} \qquad (t \neq t_0)$$

$$\text{(ii)} \ x \underset{t_0}{\overset{t}{\sim}} x' \Rightarrow S_t^x = S_t^{x'}.$$

THEOREM 4.3. *Let $S \subseteq (A^T \times B^T)$ be a causal time system. If $\tau: Q_0 \times \mathscr{D}S \times (T \cup \{t_0\}) \to Q$ is a state transition function for S, then for all $x \in \mathscr{D}S$ and all $t \in (T \cup \{t_0\})$ the set $Q_t^x = \tau(Q_0 \times \{x\} \times \{t\})$ is a set of initial states for S_t^x.*

Proof. For any $x \in \mathscr{D}S$, $\tau(Q_0 \times \{x\} \times \{t_0\}) = Q_0$, which is evidently a set of initial states for $S_{t_0}^x$ $(= S)$. Choose $x \in \mathscr{D}S$ and $t \in T$. Consider the relation

$$r_t^x = \{(\tau(q, x, t), x'/U(t), t', \theta(\tau(q, x', t'))) \mid q \in Q_0 \ \&\ x' \underset{t_0}{\overset{t}{\sim}} x \ \&\ t' \in U(t)\}.$$

r_t^x is a function on $Q_t^x \times \mathscr{D}S_t^x \times U(t)$ into B. To see this, let $q, q' \in Q_0$, $x', x'' \in x \bmod \left(\underset{t_0}{\overset{t}{\sim}}\right)$, and $t', t'' \in U(t)$ be such that (i) $\tau(q, x, t) = \tau(q', x, t)$, (ii) $x'/U(t) = x''/U(t)$, (iii) $t' = t''$. Since $x', x'' \in x \bmod \left(\underset{t_0}{\overset{t}{\sim}}\right)$, we have by property (v) of Definition 3.1,

$$\tau(q, x', t) = \tau(q, x, t) = \tau(q', x, t) = \tau(q', x'', t).$$

Moreover,

$$\begin{aligned}
\tau(q, x', t) = \tau(q', x'', t) \ \&\ x'/U(t) = x''/U(t) \\
&\Rightarrow (\forall t'): \tau(q, x', t) = \tau(q', x'', t) \ \&\ x'/(t, t'] = x''/(t, t'] \\
&\Rightarrow (\forall t'): \tau(q, x', t') = \tau(q', x'', t') \\
&\Rightarrow (\forall t'): \theta(\tau(q, x', t')) = \theta(\tau(q', x'', t')),
\end{aligned}$$

which proves r_t^x: $Q_t^x \times \mathscr{D}S_t^x \times U(t) \to B$. Now for all $u \in A^{U(t)}$ and all $v \in B^{U(t)}$,

$$uS_t^x v \Leftrightarrow (\exists q)(\forall t'): v(t') = r_t^x(\tau(q, x, t), u, t') \qquad (q \in Q_0;\ t' \in U(t)).$$

In fact, if for some $q \in Q_0$,

$$v(t') = r_t^x(\tau(q, x, t), u, t')$$

for all $t' \in U(t)$, then

$$v(t') = \theta(\tau(q, x', t')),$$

where $x' \in x \bmod \left(\underset{t_0}{\overset{t}{\sim}}\right)$ is the unique element such that $u = x'/U(t)$. Now clearly there exists some $y \in \mathscr{R}S$ such that for all $t \in T$,

$$y(t) = \theta(\tau(q, x', t)).$$

Moreover, $x'Sy$, where $u = x'/U(t)$ and $v = y/U(t)$. This proves $uS_t^x v$. Conversely, if $uS_t^x v$, then there exists some $x' \in x \bmod \left(\underset{t_0}{\overset{t}{\sim}}\right)$ and some $y \in \mathscr{R}S$ such that (i) $x'Sy$, (ii) $u = x'/U(t)$ and (iii) $v = y/U(t)$. Moreover, since S is causal with respect to (Q, Q_0, τ, θ), the condition (i) implies there exists some $q \in Q_0$ such that for all $t \in T$, $y(t) = \theta(\tau(q, x', t))$. Then, for all $t' \in U(t)$,

$$v(t') = y(t') = \theta(\tau(q, x', t')) = r_t^x(\tau(q, x, t), u, t').$$

Thus, it is proved that

$$uS_t^x v \Leftrightarrow (\exists q)(\forall t'): v(t') = r_t^x(\tau(q, x, t), u, t').$$

Finally, let $u', u'' \in \mathscr{D}S_t^x$ be such that for some $t' \in U(t)$, $u' \underset{t}{\overset{t'}{\sim}} u''$. Let $x', x'' \in x \bmod \left(\underset{t_0}{\overset{t}{\sim}}\right)$ be the unique elements such that $u' = x'/U(t)$ and $u'' = x''/U(t)$. Clearly, $x' \underset{t_0}{\overset{t'}{\sim}} x''$. Therefore, for all $q \in Q_0$,

$$\theta(\tau(q, x', t')) = \theta(\tau(q, x'', t')),$$

which implies that

$$r_t^x(\tau(q, x, t), x'/U(t), t') = r_t^x(\tau(q, x, t), x''/U(t), t').$$

Therefore, for all $u', u'' \in \mathscr{D}S_t^x$ and all $t' \in U(t)$,

$$u' \underset{t}{\overset{t'}{\sim}} u'' \Rightarrow (\forall q): r_t^x(\tau(q, x, t), u', t') = r_t^x(\tau(q, x, t), u'', t'),$$

and this proves that Q_t^x is a set of initial states for S_t^x.

COROLLARY 4.4. *A time system is causal (non-anticipatory) if and only if all of its appearances are causal (non-anticipatory), i.e.,* $S \subseteq (A^T \times B^T)$ *is causal if and only if for all* $x \in \mathscr{D}S$ *and all* $t \in (T \cup \{t_0\})$, S_t^x *is causal.*

COROLLARY 4.5. *If Q is a state set of the time system $S \subseteq (A^T \times B^T)$, then for every $x \in \mathscr{D}S$ and every $t \in (T \cup \{t_0\})$, there exists a subset $Q_t^x \subseteq Q$ which is a set of initial states for S_t^x.*

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