Synchronization of Interacting Automata

by

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The problem of synchronizing an automata chain, as posed by J. Myhill [1, 2, 3, 4, 5] provides us with a number of models which seem to be meaningful. We modify the statement of the problem as follows. Does there exist a finite automaton A such that a chain of n automata A would be synchronized at time $t = T$ after being "switched on" at time $t = 0$ by an initiating signal supplied to an arbitrarily chosen automaton?[1]

Each automaton is assumed to be connected with its two immediate neighbors, and the complexity of each automaton is assumed to be independent of the number n of automata in the chain. By synchronization we mean the simultaneous transition of all automata of the chain into the state called synchronized (terminal) provided each automaton enters this state only at time T.

We also consider Moore automata for which the output signal is the internal state. First we present a general idea of the solution of the problem as originally stated [3]. Basically an algorithm consists of organizing successive bisections of segments of the automata chain. Consider Figure 1. The first bisection of the chain is carried out as follows. The initiating signal puts the end automaton into the preterminal state and two signals p_1 and p_3 start to propagate down the chain from this automaton.

The first signal has unit velocity of propagation and the second one has velocity 1/3 (a signal propagates with velocity $1/m$ if it passes to a neighboring automaton after having stayed in the preceding one for m time units). The signal p_1 reaches the end of the chain, takes the other end automaton into the preterminal state and goes back with the same velocity. The meeting of the reflected signal with the signal p_3 occurs at the center of the chain, and the corresponding automaton (or two automata if the number of automata in the chain is even) passes to the preterminal state. If the reflected signal continues propagating down the chain with velocity 1 and if at the initial moment the first automaton emits a signal with propagation velocity 1/7 (signal p_7), these signals will meet at a distance of 1/4 from the beginning of the chain. Further, if every automaton on entering the preterminal state emits a sequence of signals which propagate with velocities

[1] In Myhill's problem the initating signal is supplied to an end automaton of the chain.

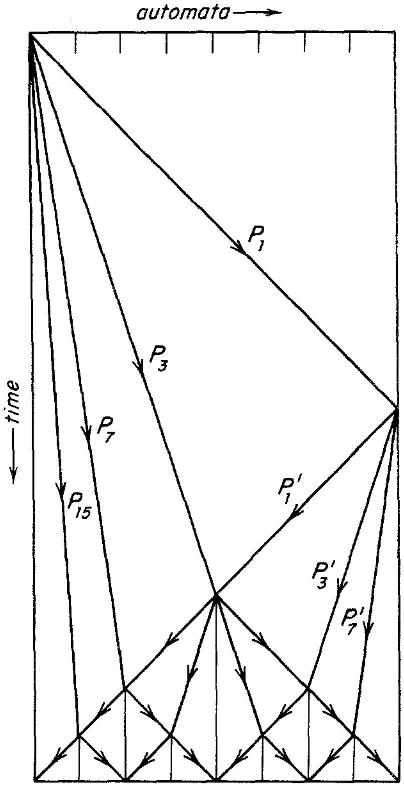

Figure 1

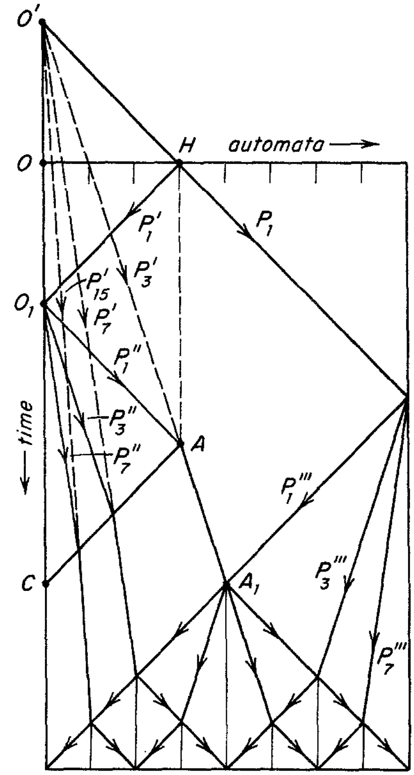

Figure 2

$1/(2^{m+1}-1)$ and if automata at the meeting points of the signals enter the preterminal state, then the process of successive bisections of the resulting segments of the chain will take place, as shown in Figure 1.

Now suppose that the initating signal is supplied to any one of the automata of the chain. The general picture of the propagation of signals is shown in Figure 2. After the initiating signal is sent, two signals p_1 and p'_1 begin to propagate in both directions from the initial automaton. Both signals have velocity 1. (The initial automaton does not pass into the preterminal state unless it is an end automaton.) When the signals p_1 and p'_1 reach the ends of the chain they take the end automata into the preterminal state and give rise to reflected signals which propagate with the same velocity. As stated above, an automaton which has passed into the preterminal state begins to generate a sequence of signals with propagation velocities $1/(2^{m+1}-1)$.

If the initiating signal had been supplied to the automaton 0 situated at the end of the chain closest to the initial automaton, the picture of the propagation of signals would have been the same as in Figure 1 with initial point $0'$. Then the signal p'_3, moving from the point $0'$ with velocity 1/3, would have met the reflected signal p'''_1 at the point A_1 (the center of the chain). It is not difficult to see that the propagation line of the signal p'_3 intersects the propagation line of the p''_1 at the point A which corresponds to the position of the initial automaton. Hence in order to effect the first bisection of the chain it is necessary to change the velocity of the reflected signal p_1'' from 1 to 1/3 at the point A. The propagation line of the reflected signal moving from the point $0'$ with velocity $1/(2^{m+1}-1)$ intersects the propagation line of the signal moving from the point 0_1 with velocity $1/(2^m-1)$; this intersection occurs at a point lying on the line AC which is the line of velocity switching. In order to achieve the correct succession of bisections, it is necessary that the velocity of every signal moving from the point 0_1 with velocity $1/(2^m-1)$ be changed to $1/(2^{m+1}-1)$. The slope of the switching line corresponds to a velocity of propagation equal to 1. Otherwise the picture of signal propagation in Figure 2 is identical to that in Figure 1. In order to set up the state transition table of an automaton, we introduce, as in Levenstein [3], the relation of contraposition of internal states and of transition functions of automata in the chain. Internal states $\overrightarrow{D}$ and $\overleftarrow{D}$, $\overrightarrow{C}$ and $\overleftarrow{C}$ are assumed to be opposite. All other states are opposite to themselves. Opposite values of the transition functions are $F(x_1, x_2, \cdots, x_n)$ and $F^*(x_n^*, x_{n-1}^*, \cdots, x_1^*)$, where x_n and x_n^* are opposite states of automaton. For the jth automaton of the chain we have the following relation:

$$(1) \qquad F_j(x_{j-1}, x_j, x_{j+1}) = F_j^*(x_{j+1}^*, x_j^*, x_{j-1}^*).$$

The transition functions are defined in Table 1 for only half of the sets, because the values of transition functions for the remaining sets can be obtained from (1).

In Figure 3 we give an example of synchronization of a chain of 28 automata where the initiating signal is supplied to the tenth automaton.

Comparison of Figures 1 and 2 shows that in Figure 2 synchronization required an amount of time which is smaller by exactly the amount of time needed for a signal with velocity 1 to cover the distance from the point 0 to the point H. Thus in the case where the initiating signal is supplied to an arbitrary automaton the synchronization process requires $T = 2n-2-a_{\min}$ time units, where $a_{\min}$ is the distance from the initial automaton to the automaton at the nearest end.

From the point of view of practical applications of the synchronization problem, the following statement of the problem seems to be more natural.[2]

Suppose we have a chain of objects (devices) which have different starting times (latent periods), i.e., the ith object starts functioning τ_i time units after an initiating signal is applied. Time is assumed to be discrete, so that the τ_i are integers. The question arises whether automata which satisfy the following requirements exist:

1. Each object is associated with one automaton.
2. The automata form a chain in which every automaton is connected only with its two neighbors (except for end automata which are connected with only one neighbor).
3. The complexity of the automata depends only on the starting times of their associated objects and does not depend on the length of the chain or the starting times of the other objects.

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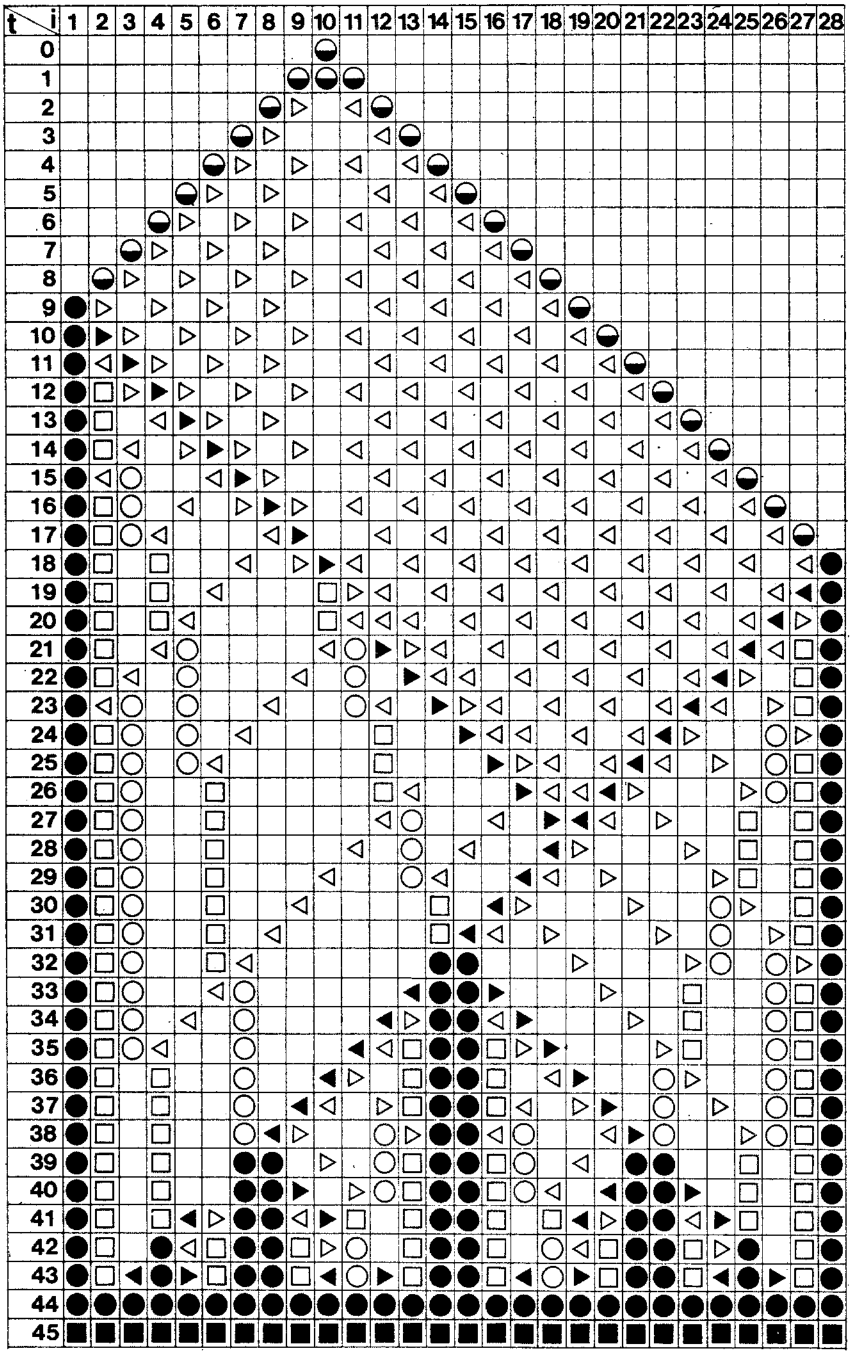
[2] In practical cases it is not very difficult to realize simultaneous transmission of signals to a number of objects, using, for example, common communication lines.



Notation

A	B	C	C	D	D	E_1	E_2	R	F
	◒	▶	◀	▷	◁	○	□	●	■

Figure 3

TABLE 1

Number		Rule		Condition
1	x	A	z	$x = A$; $z \in \{A, \vec{D}, E_1, E_2\}$ or $x \in \{\overleftarrow{C}, E_1\}$; $z = \vec{D}$ or $x = \vec{D}$; $z = \overleftarrow{D}$ or $x = E_1$; $z = E_2$ or $x = E_2$; $z = E_2$
		A		
2	x	$\vec{C}$	z	$x = A$; $z \in \{\vec{D}, \overleftarrow{D}\}$ or
		A		$x \in \{E_1, E_2\}$; $z = \vec{D}$
3	x	$\vec{D}$	z	$x \in \{B, \overleftarrow{C}, E_1\}$; $z = A$ or $x = E_2$; $z = \vec{C}$ or $x = A$; $z \in \{A, \vec{C}\}$
		A		
4	B	B	B	
		A		
5	x	E_1	$\overleftarrow{D}$	$x \in \{A, E_2\}$
		A		
6	A	y	z	$y = B$; $z = A$ or
		B		$y = A$; $z = B$
7	x	A	z	$x \in \{\vec{C}, R\}$; $z \in \{A, \vec{D}, E_1, E_2\}$
		$\vec{C}$		
8	$\vec{C}$	$\overleftarrow{C}$	z	$z \in \{A, E_1, E_2\}$
		$\vec{C}$		
9	x	$\vec{D}$	z	$x \in \{\vec{C}, R\}$; $z \in \{A, \overleftarrow{C}, \vec{D}, \overleftarrow{D}\}$
		$\vec{C}$		
10	x	$\overleftarrow{D}$	z	$x \in \{\vec{C}, \overleftarrow{D}\}$; $z = \overleftarrow{D}$ or
		$\vec{C}$		$x = R$; $z = A$
11	R	B	B	
		$\vec{C}$		

TABLE 1 (continued)

Number		Rule		Condition
12	x	A	z	$x = \overrightarrow{C};\ z \in \{\overrightarrow{C}, \overleftarrow{C}\}$ or
		R		$x = B;\ z = B$
13	x	$\overrightarrow{C}$	z	$x \in \{R, \overrightarrow{D}\};\ z = E_2$ or
		R		$x = \overleftarrow{D};\ z = E_1$
14	x	R	z	$x = \overleftarrow{C};\ z = \overrightarrow{C}$ or $x = B;\ z = B$ or $x = R;\ z \in \{A, \overrightarrow{C}, \overleftarrow{D}, E_2\}$ or $x \in \{A, \overleftarrow{D}\};\ z \in \{A, \overrightarrow{D}\}$ or $x = \overrightarrow{D};\ z \in \{A, \overrightarrow{D}, \overleftarrow{D}\}$ or $x = E_2;\ z = E_2$
		R		
15	x	y	z	$x = \overrightarrow{C};\ y = \overrightarrow{D};\ z \in \{E_1, E_2\}$ or
		R		$x = R;\ y \in \{B, \overrightarrow{D}, \overleftarrow{D}, E_2\};\ z = R$
16	$\overrightarrow{C}$	y	z	$y = E_1;\ z \in \{A, \overrightarrow{D}, E_2\}$ or
		R		$y = E_2;\ z \in \{A, R\}$
17	A	B	z	$z \in \{B, \overrightarrow{D}\}$
		$\overrightarrow{D}$		
18	x	A	z	$x \in \{\overleftarrow{C}, \overrightarrow{D}\};\ z \in \{A, E_1, E_2\}$ or
		$\overrightarrow{D}$		$x = \overrightarrow{D};\ z = \overrightarrow{D}$
19	x	$\overleftarrow{C}$	z	$x \in \{A, \overrightarrow{C}, \overleftarrow{D}, E_1\};\ z \in \{R, \overleftarrow{D}\}$
		$\overrightarrow{D}$		
20	$\overleftarrow{D}$	$\overrightarrow{C}$	z	$z \in \{A, \overleftarrow{C}, \overrightarrow{D}, E_2\}$
		$\overrightarrow{D}$		
21	x	$\overleftarrow{D}$	$\overleftarrow{D}$	$x \in \{A, B\}$
		$\overrightarrow{D}$		

TABLE 1 (continued)

Number		Rule		Condition
22	x	$\overleftarrow{D}$	z	$x \in \{\overrightarrow{C}, \overleftarrow{D}\}$; $z \in \{A, B\}$ or
		$\overrightarrow{D}$		$x = \overrightarrow{D}$; $z \in \{E_1, E_2\}$
23	$\overrightarrow{D}$	E_2	z	$z \in \{A, R\}$
		$\overrightarrow{D}$		
24	$\overleftarrow{D}$	$\overrightarrow{C}$	$\overleftarrow{D}$	
		E_1		
25	x	$\overrightarrow{D}$	E_2	$x \in \{A, \overrightarrow{D}\}$
		E_1		
26	x	E_1	z	$x = A$; $z \in \{A, \overrightarrow{D}, E_2\}$ or
		E_1		$x = \overleftarrow{C}$; $z = \overrightarrow{D}$
27	x	$\overrightarrow{D}$	E_1	$x \in \{A, \overrightarrow{D}\}$
		E_2		
28	x	E_2	z	$x = A$; $z \in \{A, \overrightarrow{C}, \overrightarrow{D}, R\}$ or
		E_2		$x = R$; $z \in \{E_1, \overrightarrow{D}\}$
29	x	$\overrightarrow{C}$	$\overleftarrow{D}$	$x \in \{R, \overrightarrow{D}\}$
		E_2		
30	x	$\overrightarrow{D}$	R	$x \in \{\overleftarrow{C}, E_1\}$
		E_2		
31	R	R	R	
		F		
32	F	F	F	
		A		

4. When the initiating signal is supplied to some automaton in the chain at time $t = 0$ the objects start operating simultaneously at time T.

It is not our intention to present a solution which would be optimal with respect to the time required for synchronization or to the number of internal states. We shall show the possibility of existence of a solution. In order for the chain of automata to perform the given task, it is necessary that ith automaton generate an activating signal for the ith object τ_i time units before the moment of synchronization.

Suppose we have a chain of automata of the type considered above which enters the terminal state simultaneously with the chain of objects. Any three adjacent automata in the chain can be in the preterminal state only at the moment which precedes the moment of transition into the terminal state by three time units. The states of these automata at this moment depend on their states and on the states of their right-hand and left-hand neighbors at the previous moment, i.e., on the states of five adjacent automata. Generally the states of any $2k+1$ adjacent automata at time t are determined by the states of $2k+3$ automata at the moment $t-1$. Hence the state of any automaton at time t is determined by its own state and by the states of τ_i of its right-hand and left-hand neighbors at the moment $t-\tau_i$. Thus we can determine the moment which precedes the moment of synchronization by τ_i time units by observing the states of $2\tau_i+1$ adjacent automata.

Consider a chain of $2\sum_{i=1}^{n} \tau_i+n$ automata which solves the synchronization problem. Decompose it into subchains of length $2\tau_i+1$. Every such subchain will be considered as a single automaton with the operating conditions of each automaton preserved. Further, an original automaton will be called a subautomaton and a subchain of $2\tau_i+1$ subautomata will be called an automaton. By an input to an automaton we mean the states of nearest end subautomaton of neighboring automata. By combining $2\tau_i+1$ subautomata into one automaton, we can observe the states of all $2\tau_i+1$ subautomata simultaneously. Thus we see that if we observe the state of the automaton formed by $2\tau_i+1$ subautomata, we can determine the moment which precedes the moment of synchronization of the whole chain of $2\sum_{i=1}^{n} \tau_i+n$ subautomata by τ_i time units. Detection of the states of automata which occur τ_i time units before the moment of synchronization of the whole chain can be realized by a logical network whose inputs are the states of the subautomata. Thus there exists a solution of the problem of synchronization of the system of objects with different starting times by means of the chain of automata. The method of solution can be summarized as follows. We consider a chain of $2\tau_i+1$ subautomata as the automaton associated with the ith object. These subautomata solve the problem of synchronization of the chain. This general chain of length $2\sum_{i=1}^{n} \tau_i+n$ is formed by subautomata of all the automata. A starting signal for an object is generated by the logical network whose inputs are the states of the subautomata of the given automaton. Note that automata continue to operate after supplying the starting signal to the object and all subautomata enter the synchronized state at the same time that objects start to operate. The proposed construction of automata satisfies the conditions of the problem.

Suppose that the initiating signal, supplied to the ith automaton, is received by its middle subautomaton. Then the time interval between transmission of the initiating signal and the beginning of operation of all the objects is equal to

$$4\sum_{i=1}^{n}\tau_i+2n-2-\tau_i-\min\left\{\left[2\sum_{j=1}^{i-1}\tau_j+(i-1)\right],\left[2\sum_{j=i+1}^{n}\tau_j+(n-i-1)\right]\right\}.$$

At $\tau_i = 0$ for all $1 \le i \le n$, $T = 2n-2-a_{\min}$.

Now we examine the requirements for the states of the automata at the moment which precedes the moment of synchronization by τ_i time units.

1. At the moment which precedes the moment of synchronization by τ_i time units, at least one subautomaton of the subautomata chain of length $2\tau_i+1$ will be in the preterminal state (state R: see Table 1).

At the kth division of the subautomata chain, which is accomplished $[(n-1)/2^k]$ time units before the moment of synchronization, the distance between two non-adjacent automata in state R is equal to $[(n-1)/2^k]$. We can choose k such that

(2) $$\left[\frac{n-1}{2^k}\right] \le \tau_i \le \left[\frac{n-1}{2^{k-1}}\right].$$

Let $n-1 = \alpha 2^k+\beta$, where $\beta<2^k$. Then $[(n-1)/2^k] = \alpha$, and

$$\left[\frac{n-1}{2^{k-1}}\right] = 2\alpha+\left[\frac{\beta}{2^{k-1}}\right] \le 2\alpha+1,$$

i.e.,

$$\left[\frac{n-1}{2^{k-1}}\right] \le 2\left[\frac{n-1}{2^k}\right]+1$$

or

(3) $$\left[\frac{n-1}{2^k}\right] \ge \frac{1}{2}\left[\frac{n-1}{2^{k-1}}\right]-\frac{1}{2}.$$

It is seen from (2) and (3) that

$$\tau_i \ge \frac{1}{2}\left[\frac{n-1}{2^{k-1}}\right]-\frac{1}{2} \quad\text{and}\quad 2\tau_i+1 \ge \left[\frac{n-1}{2^{k-1}}\right].$$

From this it follows that there is at least one automaton in a segment of length $2\tau_i+1$ which will enter state R at the moment which precedes the moment of synchronization by $[(n-1)/2^{k-1}]>\tau_i$ time units. Hence, it is always possible to select a chain of subautomata of length τ_i+1 from the chain of length $2\tau_i+1$ such that the first one has at least one end subautomaton in state R. If two adjacent subautomata are in state R, then only one of them is included in the chain of length τ_i+1. Further, only such chains will be considered.

2. If both end automata are in the preterminal state R and neither of these is in state $\overrightarrow{C}$, $\overleftarrow{C}$, or R, the moment of synchronization will occur in τ_i time units.

If the left [right] end subautomaton is in the preterminal state and the other end subautomaton is in state $\overleftarrow{C}$ [$\overrightarrow{C}$] and none of the subautomata are in states $\overrightarrow{C}$, $\overleftarrow{C}$, R, then the synchronization moment will occur in τ_i time units.

These two statements can be seen directly from Figure 3.

3. If a subautomata chain of length τ_i+1 does not satisfy the conditions of

Section 2 and the left [right] end subautomaton is in state R, then there is a subautomaton in this chain which is in state $\vec{C}$ [$\overleftarrow{C}$] and a subautomaton in state E_1 or E_2 among the subautomata which are in state $\vec{C}$ [$\overleftarrow{C}$] and the right [left] end subautomaton.

This can be proved by considering Figure 4. The left end subautomaton, in state R, generates a signal $\vec{C}$ which travels to the right with velocity 1. Since $\tau_i > [(n-1)/2^k]$, the signal $\vec{C}$ meets the signal E_1 or the signal E_2 moving in the

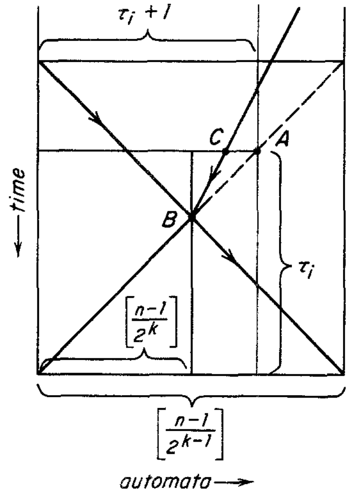


Figure 4

opposite direction. The subautomaton situated at the meeting point enters state R. At the moment preceding the moment of synchronization by τ_i time units, the signal E_1 or E_2 is in a chain of length τ_i+1, since it was moving toward the meeting point with velocity less than 1. Therefore, there must be a fictitious signal to the left which moves from the point A to the meeting point with velocity 1. The number of subautomata by which the signal E_1 or E_2 is shifted before meeting the signal $\vec{C}$ is equal to the number of signals $\vec{D}$ moving toward E_1 or E_2 and situated between E_1 or E_2 and the approaching $\vec{C}$.

Thus the distance from the meeting point B to a left end subautomaton is

$$\delta = d(E_1 \vee E_2, R) - n_D,$$

where $d(E_1 \vee E_2, R)$ is the distance between the subautomaton in the state E_1 or E_2 and the left end subautomaton, and n_D is the number of subautomata in state D.

The time interval before synchronization consists of the time interval δ and the interval of time required for the signal $\vec{C}$ to reach the meeting point, i.e.,

$$\delta + [\delta - d(R, \vec{C})] = 2\delta - d(R, \vec{C}) = 2d(E_1 \vee E_2, R) - 2n_D - d(R, \vec{C}),$$

where $d(R, \vec{C})$ is the distance between the subautomata in state R and a subautomaton in state $\vec{C}$.

The condition for transmitting the starting signal to an object can now be written as

$$\tau_i = 2d(E_1 \vee E_2, R) - 2n_D - d(R, \vec{C}).$$

We now consider a version of the synchronization problem for the case in which transmission of a signal from one automaton to another requires τ time units, i.e., communication lines between automata all have delay τ. We are interested in a solution in which the complexity of the automata does not depend on τ. The general idea of the solution is the following. Each automaton is a combination of a pair of subautomata (Figure 5).

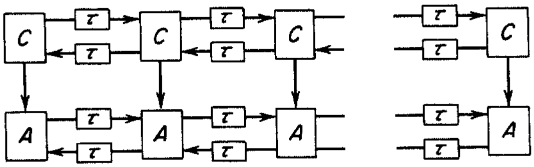


Figure 5

The subautomata C have to generate gating signals with time interval $\tau+1$ and the subautomata A solve the synchronization problem in the way presented above in the rhythm of the gating signals, i.e., the subautomata C are supplying signals for state changes to the subautomata A. In order to solve the synchronization problem, it is sufficient to solve the problem of organizing co-phased periodic operation of the subautomata C. The solution is trivial for the period $2(\tau+1)$. In order to put the subautomata C into periodic operation with period $\tau+1$, we use an algorithm of sequential approach of signals. Consider two automata C. After an initiating signal is sent to the automaton C_1 at time $t = 0$, it sends three signals a, b, c to the automaton C_2 at three consecutive moments of time. The automaton C_2 sends these signals back, delaying the signal a by 1 time unit, the signal b by 2 time units and the signal c by 3 time units. The automaton C_1 takes analogous actions with these signals. As a result, in $(\tau+1)(\tau+2)$ time units one of the automata will emit a pair of signals $\{a, c\}$ which is identified with the signal b. An automaton emits the first synchronized signal when it receives the signal b at its input after having emitted it. This occurs simultaneously for both automata at time $(\tau+2)^2$. From this moment on, an automaton which receives a signal turns it back. Thus, signals with period $\tau+1$ start circulating in the system.

The given algorithm can be explained by the picture of synchronization of two automata with delay $\tau = 5$ in the communication line (see Figure 6). Each automaton has 12 internal states and its complexity does not depend on the value of τ. The rules for changing internal states and formation of output signals for automata are given in Table 2.

We now return to the original problem. The process of synchronization begins after transmission of the initiating signal to an arbitrary automaton and results in synchronization of this automaton with its right-hand and left-hand neighbors according to the above algorithm. At the moment of synchronization of this chain of three automata, the initial automaton passes into a state which corresponds to the starting state in the problem of synchronization of a chain without

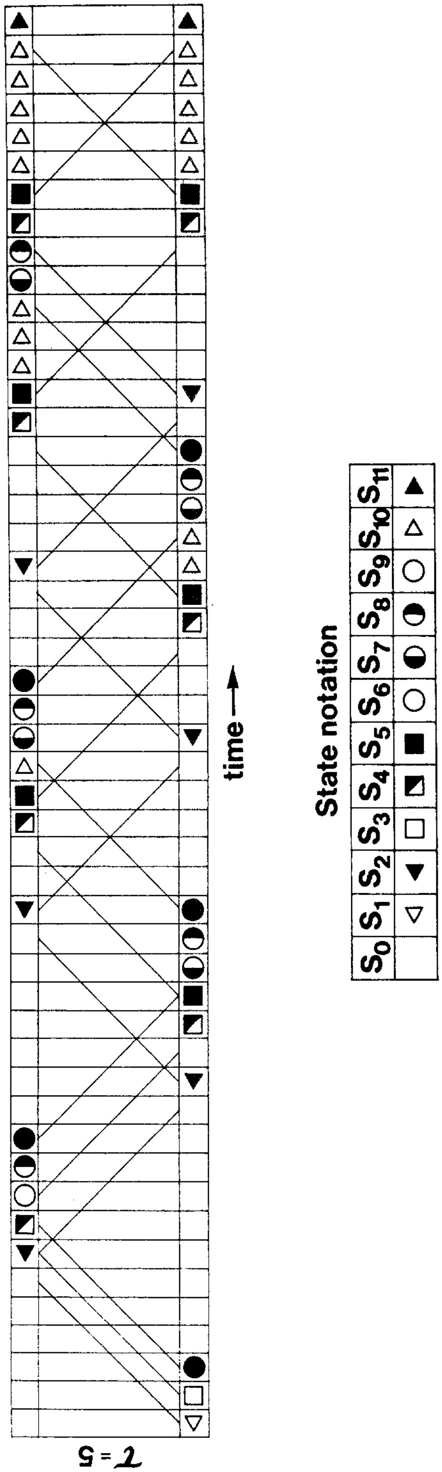


Figure 6

TABLE 2

	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}
S_0	S_0	S_2	S_2	—	S_0	S_4	S_4	—	—	—	S_0	S_0
S_1	S_3	—	—	—	—	—	—	—	—	—	—	—
S_2	S_0	—	—	S_4	S_0	—	—	—	—	—	—	—
S_3	S_9	—	S_4	—	—	—	—	—	—	—	—	—
S_4	S_5	—	—	—	S_5	—	—	S_5	S_5	S_6	S_5	—
S_5	S_{10}	—	—	—	S_{10}	S_{11}	—	S_{10}	S_{10}	S_7	S_{10}	—
S_6	S_8	—	—	—	—	—	—	—	—	—	—	—
S_7	S_8	—	—	—	—	—	—	—	—	—	—	—
S_8	S_9	—	S_4	—	—	—	—	—	—	—	—	—
S_9	S_0	—	—	—	—	—	—	—	—	—	—	—
S_{10}	S_{10}	—	—	—	S_{10}	S_{11}	—	S_{10}	S_{10}	S_7	S_{10}	—
S_{11}	—	—	—	—	—	—	—	—	—	—	S_0	S_0

delays in the communication lines. Further changes of state occur only at gated time moments. Thus every automaton in the chain can change state only once during $\tau+1$ time units, and the synchronization problem is solved with this gating in the same way as in the synchronization problem for a chain without delays.

In the solution of the above problems a constant rigid structure was essential for the connections between automata. It is of interest to examine the possibility of synchronization of a collection of automata in the case of random pair interaction.

Consider a collection of N identical automata. Random pair interaction comes about as follows. At each moment, independent equiprobable partition of N automata into $N/2$ pairs is performed (for simplicity N is assumed to be even). In every pair thus formed, an output signal from an automaton is an input signal for its partner in the pair. Further, we consider Moore automata for which an input signal is the index of an internal state. Let $x_i(t)$ denote the internal state of an automaton A at time t. Then if the automata A^i and A^j form a pair at time t,

$$x_i(t+1) = F[x_i(t);\ x_j(t)]$$

and

$$x_j(t+1) = F[x_j(t);\ x_i(t)].$$

If the transition function is symmetric, then $x_i(t+1) = x_j(t+1)$. Select the state $x = 0$, called an initial state, and $x = n$, called a terminal (synchronized) state.

Then $F[0, 0] = 0$, i.e., in an encounter (pairing) between automata both in the initial state, these automata do not change state.

Let $r_j(t)$ denote the number of automata which are in state j at time t. The segment of automata $\rho_j(t) = r_j(t)/N$ which are in state j at time t will be called the filling number. Let $\bar{\rho}_j(t)$ denote the mathematical expectation of $\rho_j(t)$. For sufficiently large N the expectation of the segment of automata forming pairs with both automata in state j is $\bar{\rho}_j^2(t)$.[3] Similarly, the mathematical expectation of the segment of the automata forming a pair with one automaton in state j and the other in state i is $2\rho_j(t)\rho_i(t)$.

Now consider the problem of synchronization of the collection of automata described above. Automata collections will be called ϵ-synchronizable if after transmission of the initiating signal to any one randomly selected automaton at time $t = 0$,

$$\Delta\bar{\rho}_n(t) \geq 0 \qquad \text{for all } t>0$$

$$\lim_{t\to\infty} \bar{\rho}_n(t) = 1$$

and there exists T such that

$$\bar{\rho}_n(t) \leq \epsilon \text{ with } t \leq T$$

$$\bar{\rho}_n(t) \geq 1-\epsilon \text{ with } t \geq T+1.$$

Hence ϵ-synchronizability means the existence of a moment at which not less than $1-2\epsilon$ automata enter the terminal state simultaneously. Also of interest are constructions of automata which guarantee ϵ-synchronization and asymptotic behavior of the number of states of such automata as $N\to\infty$ and $\epsilon\to 0$.

We consider two possible constructions of automata.

$$\begin{aligned}
&1.\quad x_i(t+1) = x_j(t+1) = \max[x_i(t); x_j(t)]+1, && \text{if } \max[x_i(t); x_j(t)] \neq 0,\\
& && \max[x_i(t); x_j(t)] \neq n\\
(4)\quad & \quad\;\; x_i(t+1) = x_j(t+1) = n && \text{if } \max[x_i(t); x_j(t)] = n\\
& \quad\;\; x_i(t+1) = x_j(t+1) = 0 && \text{if } \max[x_i(t); x_j(t)] = 0.
\end{aligned}$$

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[3] The probability of formation of the pair (j, j) is

$$\frac{r_j}{N}\cdot\frac{r_j-1}{N-1} = \frac{r_j}{N}\cdot\frac{r_j}{N}\cdot\frac{N}{N-1} - \frac{r_j}{N(N-1)} = \rho_j^2\,\frac{N}{N-1} - \frac{\rho_j}{N-1}$$

$$= \rho_j^2 + \frac{\rho_j^2}{N-1} - \frac{\rho_j}{N-1} = \rho_j^2 - \frac{\rho_j(1-\rho_j)}{N-1}.$$

Similarly the probability of formation of the pair (j, i) is

$$2\rho_j\rho_i + \frac{2\rho_j\rho_i}{N-1}.$$

The initiating signal transfers an automaton from state 0 into state 1. Consider the first time units of operation of the system:

(1) $\rho_0(0) = 1,\ \rho_j(0) = 0 \quad \text{for all } j>0;$

(2) $\rho_0(1) = 1-\frac{1}{N},\quad \rho_1(1) = \frac{1}{N},\quad \rho_j(1) = 0 \quad \text{for all } j>1;$

(3) $\rho_0(2) = 1-\frac{2}{N},\quad \rho_1(2) = 0,\quad \rho_2(2) = \frac{2}{N},\quad \rho_j(2) = 0 \quad \text{for all } j>2;$

(4) $\bar{\rho}_0(3) = \rho_0^2(2) = \left(1-\frac{2}{N}\right),\quad \rho_1(3) = 0,\quad \rho_2(3) = 0,$

$$\rho_3(3) = 1-\left(1-\frac{2}{N}\right)^2,\quad \rho_j(3) = 0 \quad \text{for all } j>3.$$

It is evident that $\bar{\rho}_0(t)$ is equal to the mathematical expectation of the number of automata which are in state 0 at time $t-1$ and at this moment are pairing up with an automaton in state 0, i.e., $\bar{\rho}_0(t) = \bar{\rho}_0^2(t-1)$.[4]

The solution of this difference equation under the above initial conditions is $\bar{\rho}_0(t) = (1-2/N)^{2^{t-2}}$. Taking into account the principles of state changes, for sufficiently large N we may write

$$\bar{\rho}_0(t) = \exp\left(-\frac{2^{t-1}}{N}\right); \qquad \rho_t(t) = 1-\exp\left(-\frac{2^{t-1}}{N}\right);$$

$$\rho_j(t) = 0 \qquad (0<j \neq t).$$

Then, obviously,

$$\rho_n(t) = \begin{cases} 0 & \text{for } t < n \\ 1-\exp\left(-\dfrac{2^{t-1}}{N}\right) & \text{for } t \geq n. \end{cases} \tag{5}$$

In order to achieve ϵ-synchronization it is necessary that

$$1-\exp\left(-\frac{2^{t-1}}{N}\right) \geq 1-\epsilon, \tag{6}$$

i.e.,

$$n \geq \log_2 N + \log_2 \ln\frac{1}{\epsilon} + 1.$$

Hence, for the realization of ϵ-synchronization, it is necessary that the memory capacity of automata increase as N increases.

2. $x_i(t+1) = x_j(x+1) = \min[x_i(t); x_j(t)]+1$ if $\min[x_i(t); x_j(t)] \neq n$, $\max[x_i(t); x_j(t)] \neq 0$;

$x_i(t+1) = x_j(t+1) = n$ if $\min[x_i(t); x_j(t)] = n$,

$x_i(t+1) = x_j(t+1) = 0$ if $\max[x_i(t); x_j(t)] = 0$.

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[4] It is evident that this difference equation describes the process only approximately. We will use this approximate description here and disregard the influence of the variance of the distribution of $\rho_0(t)$.

The initiating signal transfers an automaton from state 0 into state 1. We introduce the new variable $\gamma_j(t) = \sum_{i=j}^{n} \rho_i(t)$. It is not difficult to see that only those automata will be in states with index greater than $j-1$ at time t which were in states not less than $j-1$ at moment $t-1$, i.e.,

(7) $$\bar{\gamma}_j(t) = \gamma_{j-1}^2(t-1) \qquad (j \geq 2).$$

Consider the following initial conditions for the system of difference equations (7):

$$\gamma_0 = 1, \qquad \gamma_1(0) = 0, \qquad \gamma_1(1) = \frac{1}{N}, \qquad \gamma_1(2) = \frac{2}{N};$$

as in the previous case $\gamma_1(t) = 1-\exp(-2^{t-1}/N)$. Note that $\gamma_2(1) = 0, \gamma_3(2) = 0$ and generally $\gamma_j(j-1) = 0$. The solution of the system (7) under the above initial conditions is

$$\gamma_j(t) = \begin{cases} 0 & \text{for } t \leq j+1 \\ [\gamma_j(t-j+1)]^{2^{j-1}} & \text{for } t > j+1. \end{cases}$$

Note that $\gamma_n(t) = \rho_n(t)$ and

(8) $$\gamma_n(t) = \begin{cases} 0 & \text{for } t \leq n-1 \\ \left[1-\exp\left(-\dfrac{2^{t-n}}{N}\right)\right]^{2^{n-1}} & \text{for } t > n+1. \end{cases}$$

It should be emphasized that in this construction the automata are allowed to leave the terminal state. Despite this it is easy to see that $\gamma_n(t)$ is a monotonically increasing function of t, and $\lim_{t\to\infty} \gamma_n(t) = 1$. We introduce the added notation $\exp(-2^{t-n}/N) = x$ and $2^{n-1} = 1/\alpha$, and consider the system of inequalities

(9) $$\begin{aligned} \bar{\gamma}_n(t) &= 1-x)^{1/\alpha} \leq \epsilon \\ \bar{\gamma}_n(t+1) &= (1-x^2)^{1/\alpha} \geq 1-\epsilon. \end{aligned}$$

We determine the domain of existence of solutions of system (9):

(10) $$\begin{aligned} 1-x &\leq \epsilon^\alpha, & x &\geq 1-\epsilon^\alpha; \\ 1-x^2 &\geq (1-\epsilon)^\alpha, & x^2 &\leq 1-(1-\epsilon)^\alpha. \end{aligned}$$

System (10) is equivalent to (9). For $\alpha < 1$,

$$(1-\epsilon)^\alpha = 1-\alpha\epsilon - \frac{(1-\alpha)}{2}\alpha\epsilon^2 - \frac{(1-\alpha)(2-\alpha)}{2\cdot 3}\alpha\epsilon^3 - \cdots = 1-\alpha\epsilon - R(\epsilon),$$

where $R(\epsilon) > 0$, and consequently $1-(1-\epsilon)^\alpha > \alpha\epsilon$. Expanding $(1-\epsilon^\alpha)$ into a Taylor series in α, we obtain

$$(1-\epsilon^\alpha) = \alpha \ln\frac{1}{\epsilon} - \frac{\alpha^2 \ln^2 1/\epsilon}{2} + \frac{\alpha^3 \ln^3 1/\epsilon}{2\cdot 3} - \cdots$$

and therefore $(1-\epsilon^\alpha) < \alpha \ln 1/\epsilon$ for $\alpha < 1/(\ln 1/\epsilon)$. Then the doman of existence of solutions of the system

(11) $$\begin{aligned} x &\geq \ln 1/\epsilon \\ x &\leq \sqrt{\alpha\epsilon} \\ \alpha &< \frac{1}{\ln 1/\epsilon} \end{aligned}$$

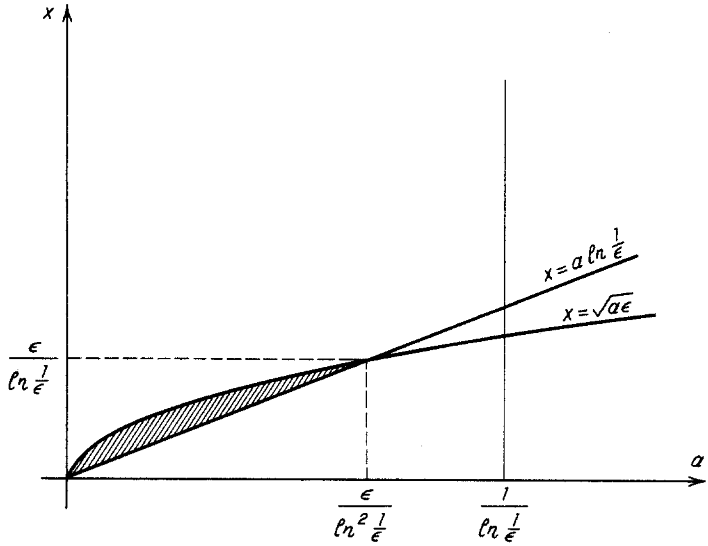

Figure 7

is already the domain of existence of solutions of the system (10), so that solutions of the system (11) are solutions of the system (10). The functions $x = \alpha \ln 1/\epsilon$ and $x = \sqrt{\alpha\epsilon}$, and the domain of existence of solutions of the system (11) formed by these curves, are shown in Figure 7. Obviously a solution exists only when

$$\alpha < \frac{\epsilon}{\ln^2 1/\epsilon}.$$

Hence

(12) $$n \geq \log_2 \frac{1}{\epsilon} + 2 \log_2 \ln \frac{1}{\epsilon} + 1.$$

Now let $\hat{n} = \log_2 1/\delta + 2 \log_2 \ln 1/\delta + 1 \geq \log_2 1/\epsilon + 2 \log_2 \ln 1/\epsilon + 1$, i.e., $\delta \leq \epsilon$. Consider the moment

$$\hat{t} = \log_2 N + \hat{n} + \log_2 \left(\ln \frac{1}{\delta} + \ln \ln \frac{1}{\delta} \right).$$

Then

$$\exp\left(-\frac{2^{\hat{t}-\hat{n}}}{N} \right) = \frac{\delta}{\ln 1/\delta} \quad \text{and} \quad 2^{\hat{n}-1} = \frac{1}{\delta}\left(\ln \frac{1}{\delta} \right)^2.$$

Consequently,

$$\bar{\gamma}_n(t) = \left(1 - \frac{\delta}{\ln 1/\epsilon} \right) \frac{1}{\delta} \left(\ln \frac{1}{\delta} \right)^2.$$

Taking into account the fact that $\epsilon/(\ln 1/\delta)$ is small, we may assume that

$$\bar{\gamma}_n(t) \cong \exp\left(-\ln \frac{1}{\delta} \right) = \delta \leq \epsilon.$$

On the other hand,

$$\bar{\gamma}_n(\hat{t}+1) = \left(1 - \frac{\delta^2}{(\ln 1/\delta)^2} \right) \frac{1}{\delta} \left(\ln \frac{1}{\delta} \right)^2 \approx \exp(-\delta) \approx 1 - \delta \geq 1 - \epsilon.$$

Thus we can assert that there exists $n_0(\epsilon)$ such that for all $n > n_0(\epsilon)$ we have simultaneous passage into the synchronized state of more than $1-2\epsilon$ automata at the moment $\hat{t}(N, n)$ independently of N and for arbitrarily small $\epsilon > 0$. Since $\gamma_n(t)$ is a monotonically increasing function of time and $\gamma_n(t) \leq 1$, this moment is unique.

Our solution of this problem was based on the following two assumptions: (1) For large N it is possible to set up the equations for mean values only, ignoring the variance, and (2) terms of order $1/N$ can be neglected. The validity of these assumptions is not obvious. Therefore, in order to examine the correctness of the results obtained, we simulated the behavior of a collection of 1024 automata on the computer. For each n ($n = 3, 4, \cdots, 16$), 200 experiments were made. The results of the experiments are shown in Figure 8. The upper unbroken curve represents the mean value of the maximal number of automata operating simultaneously ($\Delta\rho_{\max}$). The lower unbroken curve represents the root-mean-square deviation [$\sigma(\Delta\rho_{\max})$] and the dashed curve represents $1-2\epsilon(n)$. It is not difficult to see that the experiment does not contradict the results obtained.

Thus, for the above models we were able to satisfy the requirements of synchronization: all automata to within ϵ enter the synchronized state in exactly one time unit. In the second version of the problem we were able to preserve the important feature of the determined synchronization problem—independence of the required complexity of each automaton from the total number of automata.

The problems considered here naturally do not exhaust all possible methods of organization of interaction, but the authors hope that the results presented

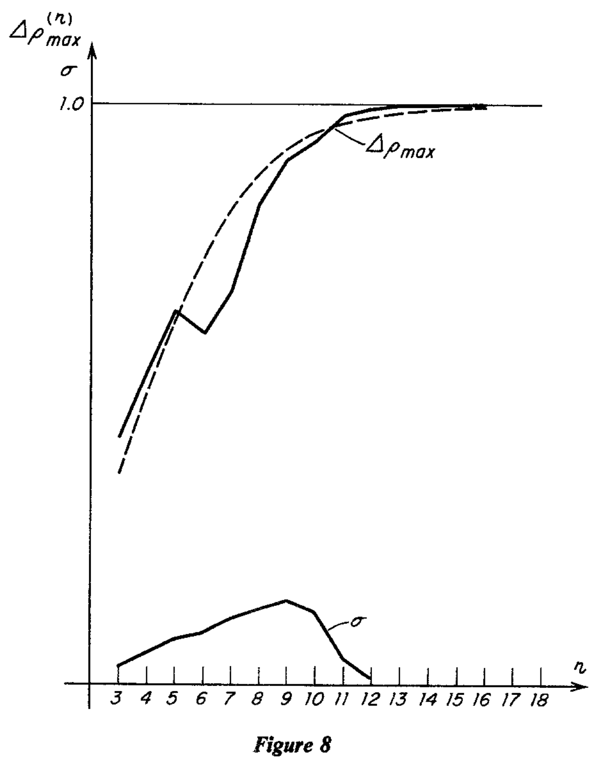

Figure 8

above make it possible to a certain extent to appraise possibilities and general features of organizing interaction in automata collections.

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