Decision Problems for ω-Automata

by

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1. Introduction

In [4], Hartmanis and Stearns investigated properties of sets of infinite sequences which can be defined by finite automata. In this paper we consider various definitions for machines of this type, including ones introduced by Büchi [1] and McNaughton [6]. For each type of finite automaton we classify the complexity of definable sets of sequences. More precisely, let Σ^ω be the set of ω-sequences on the finite set Σ. Consider the Borel hierarchy with respect to the product topology on Σ^ω. The complexity of a subset of Σ^ω is given by its position in the Borel hierarchy. It is shown that increasing the complexity of requirements for a sequence to be accepted by a finite automaton raises the level in the Borel hierarchy at which definable sets are found. Furthermore, procedures are given for deciding the complexity of sets defined by a large class of machines.

In [4], Σ is taken to be $\{0, 1\}$ and the usual topology on the real line is considered. We use the product topology because it is more natural when dealing with finite-state machines, in that it avoids the necessity of identifying infinite sequences (e.g., $100\cdots$ equals $011\cdots$ on the real line). Moreover, the product space Σ^ω is, in effect, an infinite tree with paths through the tree corresponding in a one-to-one fashion with points of Σ^ω. We believe that this analogy adds an intuitive flavor to the proofs.

The second section contains definitions and an outline of related results. Section 3 gives the hierarchy results. In Section 4 we give algorithms for deciding the complexity of sets defined by arbitrary machines. Relationships between the various machine types are also explored. In the last section we discuss reducibility relationships existing among various undecidable properties of Turing machines which accept infinite sequences.

2. Notation, Background and Basic Definitions

Let Σ be a finite set, the *input alphabet*. We use Σ^* [Σ^ω] to denote the set of all finite [infinite] sequences on Σ. If $x, y \in \Sigma^*$, xy is the concatenation

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of x and y. Let $\alpha = \sigma_1 \sigma_2 \sigma_3 \cdots (\sigma_i \in \Sigma)$ be a member of Σ^ω. Abbreviate $\sigma_1 \sigma_2 \cdots \sigma_i$ by $\bar{\sigma}_i$ and define the partial order $\prec$ on $\Sigma^* \cup \Sigma^\omega$ by $\bar{\sigma}_i \prec \bar{\sigma}_j \prec \alpha$ for $i < j < \omega$. Let $P(S)$ be the set of all subsets of the set S. Set inclusion is indicated by $\subseteq$, proper set inclusion by $\subset$, and $c(A)$ is the cardinality of the set A.

Definition 2.1. A finite automaton (f.a.) over Σ is a system $\mathcal{M} = \langle S, s_0, M\rangle$ where S is a finite set, the *set of states*, M is a function M: $S \times \Sigma \to S$ and $s_0 \in S$ is the *initial state*.

In the following, $\mathcal{M} = \langle S, s_0, M\rangle$ is a fixed but arbitrary f.a.

Definition 2.2. $\bar{M}$: $S \times \Sigma^* \to S$ is the extension of M given by $\bar{M}(s, x\sigma) = M(\bar{M}(s, x), \sigma)$ for $\sigma \in \Sigma$, $x \in \Sigma^*$. $R_{\mathcal{M}}$ is a function, $R_{\mathcal{M}}$: $\Sigma^* \to S$, given by $R_{\mathcal{M}}(x) = \bar{M}(s_0, x)$, called the *response function of* $\mathcal{M}$. (To simplify the notation we omit the subscript $\mathcal{M}$ in $R_{\mathcal{M}}$.)

Definition 2.3. $R|\alpha$ is the function R restricted to $\{x|\ x \prec \alpha \in \Sigma^\omega\}$. Let $\mathrm{In}(\alpha) = \{s | s \in S,\ c((R|\alpha)^{-1}(s)) = \omega\}$. Thus $\mathrm{In}(\alpha)$ is the set of states of $\mathcal{M}$ which are entered infinitely often while reading α.

To simplify the proofs we always assume that all states of $\mathcal{M}$ are accessible from the initial state. That is, for all $s \in S$, there is an $x \in \Sigma^*$ such that $R(x) = s$.

We may adjoin to $\mathcal{M}$ the following conditions for acceptance of sequences $\alpha = \sigma_1\sigma_2 \cdots$ of Σ^ω. The conditions are called *output conditions* or just outputs.

1. Let $D \subseteq S$. $\mathcal{M}$ accepts α with respect to D if $(\exists i)R(\bar{\sigma}_i) \in D$.
1′. Let $D \subseteq S$. $\mathcal{M}$ accepts α with respect to D if $(\forall i)R(\bar{\sigma}_i) \in D$.
2. Let $D \subseteq S$. $\mathcal{M}$ accepts α with respect to D if $\mathrm{In}(\alpha) \cap D \neq \varnothing$.
2′. Let $\mathcal{D} \subseteq P(S)$. $\mathcal{M}$ accepts α with respect to $\mathcal{D}$ if $(\exists D \in \mathcal{D})\ \mathrm{In}(\alpha) \subseteq D$.
3. Let $\mathcal{D} \subseteq P(S)$. $\mathcal{M}$ accepts α with respect to $\mathcal{D}$ if $(\exists D \in \mathcal{D})\ \mathrm{In}(\alpha) = D$.

Definition 2.4. An i-f.a. is a f.a. augmented by an output of type i. If $\mathcal{M}$ is an i-f.a., $T(\mathcal{M})$, *the set of sequences defined by* $\mathcal{M}$, is $\{\alpha|\ \alpha \in \Sigma^\omega,\ \alpha$ accepted by $\mathcal{M}\}$. (Of course the notion of acceptance is with respect to the designated set D or set of sets $\mathcal{D}$ and the output type. To simplify the text we use just "accept" whenever the meaning is clear.)

Definition 2.5. $A \subseteq \Sigma^\omega$ is *i-definable* if there is an i-f.a. which defines it.

Definition 2.6. The i-f.a. $\mathcal{M}_1$ is equivalent to the j-f.a. $\mathcal{M}_2$ if $T(\mathcal{M}_1) = T(\mathcal{M}_2)$.

1′-f.a. were studied by Hartmanis and Stearns [4]. 2-f.a. and 3-f.a. were introduced by Büchi [1] and McNaughton [6], respectively. In [1], non-deterministic 2-f.a. were used to obtain a decision procedure for the restricted second-order theory of the structure $\langle N, '\rangle$, where N is the set of natural numbers and $'$ is the successor function on N. In [6], non-deterministic 3-f.a., non-deterministic 2-f.a. and deterministic 3-f.a. are shown to define the same sets. This theorem can be used to simplify Büchi's decision procedure. In [2], a theorem about 3-f.a. is used, together with the results of [1] and [6], to obtain an algorithm for constructing finite automata from specifications given in the restricted second-order language of $\langle N, ' \rangle$ (see [7] for a discussion of these results). In [3], the hierarchy result below for 3-f.a. is presented and used to obtain a classification for decision problems for the restricted second-order theory of structures of the form $\langle N, ', Q \rangle$, where Q is a recursive predicate.

Definition 2.7. For $x \in \Sigma^*$, let $N_x = \{\alpha \mid \alpha \in \Sigma^*, x \prec \alpha\}$. $A \subseteq \Sigma^\omega$ is an open set of the product topology if there is a $B \subseteq \Sigma^*$ such that

$$A = \bigcup_{x \in B} N_x .$$

Hence $\{N_x \mid x \in \Sigma^*\}$ is a basis for the product topology on Σ^ω.

Definition 2.8. Let A be an open set. $B \subseteq \Sigma^*$ is a *basis for A* if $A = \bigcup_{x \in B} N_x$. B is a minimal basis for A if B is a basis and $(\forall x, y \in B)[x \preceq y \supset x = y]$.

Let F_0 and G_0 denote the class of subsets of Σ^ω which are both open and closed. F_1 $[G_1]$ is the class of closed [open] sets. F_2 $[G_2]$ is the class of sets which are denumerable unions [intersections] of closed [open] sets. F_3 $[G_3]$ contains denumerable intersections [unions] of sets in F_2 $[G_2]$. We define $F_3, F_4, \cdots$ $[G_3, G_4, \cdots]$ similarly. A^c denotes the complement of $A \subseteq \Sigma^\omega$.

It is well known that for all i, $F_i \subset F_{i+1}$, $G_i \subset G_{i+1}$ and $F_i \cup G_i \subset F_{i+1} \cap G_{i+1}$. Also, each F_i and G_i is closed under finite unions and intersections, and $A \in F_i$ if and only if $A^c \in G_i$.

Definition 2.9. $A \subseteq \Sigma^\omega$ is a *Borel set* if it belongs to $\bigcup_i F_i = \bigcup_i G_i$. The hierarchies $F_0, F_1, \cdots$ and $G_0, G_1, \cdots$ form the *Borel hierarchy.*

The complexity of a subset of Σ^ω is given by its position in the Borel hierarchy with respect to the product topology on Σ^ω.

Intuitively, Σ^ω is an infinite labelled tree where if $c(\Sigma) = n$, then each vertex has n successor vertices. Points of Σ^ω correspond to infinite paths of the tree. Vertices correspond to members of Σ^*. N_x $(x \in \Sigma^*)$ is the set of all paths through the vertex corresponding to x. An open set is the union of all paths through some set of vertices. The reader is urged to use this correspondence as an aid in motivating the proofs.

LEMMA 2.1. *A is a member of F_0 $[G_0]$ if and only if there are $x_1, \cdots, x_n$ such that*

$$A = \bigcup_{i=1}^{n} N_{x_i} .$$

Equivalently, $A \in F_0$ if and only if A has a finite minimal basis.

G_2 can be characterized as follows.

LEMMA 2.2. *$A \in G_2$ if and only if there is a $B \subseteq \Sigma^*$ such that $\alpha \in A$ if and only if $\exists$ $x_1 \prec x_2 \prec \cdots$, $x_i \in B$ and $x_i \prec \alpha$, $i = 1, 2, \cdots$.*

Proof. 1. Let $A \in G_2$. Then there are open sets $A_1 \supseteq A_2 \supseteq \cdots$ such that $A = \bigcap_i A_i$. Let B_1 be a minimal basis for A_1. Choose a minimal basis B_2 for A_2 which satisfies $B_2 \cap B_1 = \varnothing$. This is done by first picking a minimal basis $\bar{B}_2$ for A_2. If $x \in B_1 \cap \bar{B}_2$, replace x in $\bar{B}_2$ by $\{x\sigma_1, \cdots, x\sigma_n \mid \Sigma = \{\sigma_1, \cdots, \sigma_n\}\}$. B_2 is the modified $\bar{B}_2$. Similarly, define minimal bases $B_3, B_4, \cdots$ for $A_3, A_4, \cdots$ respectively, where $B_{i+1} \cap \bigcup_{j=1}^{i} B_j = \varnothing$.

Let $B = \bigcup_i B_i$. If $\alpha \in A$, then $\alpha \in A_i$, $i = 1, 2, \cdots$. Hence there are $x_i \in B_i$, $i = 1, 2, \cdots$, such that $x_i \prec \alpha$ (B_i is a basis for A_i) and $x_i \neq x_j$ for $i \neq j$. Choose a subset $\{x_{i_j}\}$ of $\{x_i\}$ so that $x_{i_1} \prec x_{i_2} \prec \cdots$.

If $\exists\ x_1 \prec x_2 \prec \cdots \prec \alpha$, $x_j \in B_{i_j}$, then $\alpha \in \bigcap_j A_{i_j}$. Then $\alpha \in A$ because the A_i are decreasing.

2. Let $\alpha \in A$ if and only if $\exists\ x_1 \prec x_2 \prec \cdots$, $x_i \in B$, $x_i \prec \alpha$, $i = 1, 2, \cdots$. Let $C_1 = B$ and define B_i, $i = 1, 2, \cdots$, as follows:

$$B_1 = \{x \mid x \in C_1,\ (\forall y)[y \in C_1 \wedge y \preceq x \supset y = x]\}.$$

Assume that B_i and C_i have been defined. Let $C_{i+1} = C_i - B_i$ and

$$B_{i+1} = \{x \mid x \in C_{i+1},\ (\forall y)[y \in C_{i+1} \wedge y \preceq x \supset y = x]\}.$$

Then $B = \bigcup_i B_i$ and each B_i is a minimal basis for an open set A_i. It is easy to show that $A = \bigcap_i A_i$, so that $A \in G_2$. This completes the proof.

Definition 2.10. If $A \in G_2$ and B is as in Lemma 2.2, then B is a G_2-*basis* for A.

3. Hierarchy Results

We show that 1-definable sets [1′-definable sets] are in G_1 [F_1], 2-definable [2′-definable] sets are in G_2 [F_2], and 3-definable sets are in $G_3 \cap F_3$.

THEOREM 3.1. *Every* 1-*definable set is in* G_1.

Proof. Let A be 1-definable. There is a 1-f.a. $\mathcal{M} = \langle S, s_0, M, D\rangle$ such that $A = T(\mathcal{M})$. Let $B = \{x \mid R(x) \in D\}$. Then $A = \bigcup_{x \in B} N_x$, so that B is a basis for A and $A \in G_1$.

COROLLARY 3.2. *Every* 1′-*definable set is in* F_1.

Proof. If A is 1′-definable, then A^c is 1-definable.

Corollary 3.2 was proved in [4] for the usual topology on the real line.

THEOREM 3.3. *Every* 2-*definable set is in* G_2.

Proof. Let A be 2-definable. There is a 2-f.a. $\mathcal{M} = \langle S, s_0, M, D\rangle$ such that $A = T(\mathcal{M})$. Let $B = \{x \mid R(x) \in D\}$. Then $A = \{\alpha \mid \mathrm{In}(\alpha) \cap D \neq \varnothing\}$ and this is just the set of α for which there are $x_1 \prec x_2 \prec \cdots$ such that $x_i \in B$, $x_i \prec \alpha$, $i = 1, 2, \cdots$. Hence B is a G_2-basis for A, so that $A \in G_2$.

THEOREM 3.4. *Every* 2′-*definable set is in* F_2.

Proof. Let $A = T(\mathcal{M})$ where $\mathcal{M} = \langle S, s_0, M, \mathcal{D}\rangle$ is a 2′-f.a. Assume that $\mathcal{D} = \{D\}$. For each $x \in \Sigma^*$, let $A_x = \{\alpha \mid x \prec \alpha,\ (\forall y)[x \preceq y \prec \alpha \supset R(y) \in D]\}$. It is easy to see that A_x is closed. Then $A = \bigcup_{x \in \Sigma^*} A_x$, so that A is in F_2. If $\mathcal{D} = \{D_1, \cdots, D_n\}$, then A is a finite union of members of F_2, so that A is still in F_2.

The following two theorems were proved in [3].

THEOREM 3.5. *Every* 3-*definable set is in* $G_3 \cap F_3$.

Proof. Let $A = T(\mathcal{M})$, where $\mathcal{M} = \langle S, s_0, M, \mathcal{D}\rangle$ is a 3-f.a. Assume that $\mathcal{D} = \{D\}$. Then, by Theorem 3.4, $A_D = \{\alpha \mid \mathrm{In}(\alpha) \subseteq D\}$ is in F_2. For each $E \subset D$, $A_E = \{\alpha \mid \mathrm{In}(\alpha) \subseteq E\}$ is also in F_2. Hence $A = A_D \cap \left(\bigcup_{E \subset D} A_E\right)^c$ is in the Boolean algebra over F_2 and therefore $A \in F_3 \cap G_3$. If $\mathscr{D} = \{D_1, \cdots, D_n\}$, then A is a union of members of $F_3 \cap G_3$, so that A is in $F_3 \cap G_3$.

COROLLARY 3.6. *Every 3-definable set is in the Boolean algebra over G_2.*

In the following, assume that $\Sigma = \{0, 1\}$. This will simplify the notation.

Definition 3.1. Let $A^\dagger$ consist of those members of Σ^ω in which a finite number of 1's occur; i.e., $A^\dagger = \{\alpha | c\{x | x \in \Sigma^*, x1 \prec \alpha\} < \omega\}$.

LEMMA 3.1. *$A^\dagger$ is in F_2 but not in G_2.*

Proof. Assume that $A^\dagger \in G_2$ with G_2-basis B. Obtain a contradiction by constructing an $\alpha \in \Sigma^\omega$ which contains an infinite number of elements of B as initial segments but which is not in $A^\dagger$.

Choose n_1 such that $0^{n_1} \in B$. We know that n_1 exists because $0^\omega \in A^\dagger$. Choose n_2 such that $0^{n_1} 10^{n_2} \in B$. Now n_2 exists because $0^{n_1} 10^\omega \in A^\dagger$. Choose $n_3, n_4, \cdots$ similarly. Let $\alpha = 0^{n_1} 10^{n_2} 10^{n_3} 1 \cdots$. We have $c\{x | x \in B, x \prec \alpha\} = \omega$, but $\alpha \notin A^\dagger$. Hence B is not a G_2-basis for $A^\dagger$ and $A^\dagger \notin G_2$:

$$A^{\dagger c} \in G_2 \text{ with } G_2\text{-basis } \{x1 | x \in \Sigma^*\}.$$

LEMMA 3.2. *Let $A^{\neq} = \{\alpha | 0 \prec \alpha, \alpha \in A^\dagger\} \cup \{\alpha | 1 \prec \alpha, \alpha \in A^{\dagger c}\}$. Then $A^{\neq}$ is in neither G_2 nor F_2.*

Proof. This is similar to the proof of Lemma 3.1.

It is easy to show the following theorem.

THEOREM 3.7. *$A^\dagger$ is 2′-definable, $A^{\dagger c}$ is 2-definable and $A^{\neq}$ is 3-definable.*

Theorem 3.7 demonstrates that Theorems 3.3–3.5 give the best possible characterizations of 2-, 2′- and 3-definability. In the next section we show that 2-f.a. and 2′-f.a. differ from 3-f.a. only on $G_3 \cap F_3$. An interesting open problem is that of obtaining "natural" output conditions which enable finite machines to define sets above $G_3 \cap F_3$ in the Borel hierarchy.

4. Algorithms for Determining Complexity

In this section we give an effective procedure for determining the complexity (with respect to the Borel hierarchy) of a set defined by an arbitrary 3-f.a. The complexity of sets defined by other types of f.a. can be calculated by first constructing an equivalent 3-f.a. and then applying the given decision method.

In the following, let $\mathscr{M} = \langle S, s_0, M, \mathscr{D} \rangle$ be a fixed but arbitrary 3-f.a.

Definition 4.1. For $x, y \in \Sigma^*$, $x \prec y$, let $\mathscr{R}(x, y) = \{R(z) | x \preceq z \preceq y\}$. For $s \in S$, let $Ac(s) = \{q | q \in S, (\exists x)\bar{M}(s, x) = q\}$. Call $Ac(s)$ the *set of states accessible from* s and $\mathscr{R}(x, y)$ the *state path determined by the interval* x, y.

Definition 4.2. For $q \in S$, let $\mathscr{H}q = \{\mathscr{R}(x, y) | R(x) = R(y) = q, x, y \in \Sigma^*\}$, *the set of realizable cycles.*

Notation. For $z \in \Sigma^*$, $z^i = \overbrace{z \wedge \cdots \wedge z}^{i}$.

THEOREM 4.1. *$T(\mathscr{M})$ is open if and only if every non-empty $\mathscr{H}s$ satisfies*
(a) *$\mathscr{H}s \cap \mathscr{D} = \varnothing$ or*
(b) *for all $q \in Ac(s)$, $\mathscr{H}q \subseteq \mathscr{D}$.*

Proof. 1. Assume that $T(\mathscr{M}) \in G_1$ and that $\mathscr{H}s \cap \mathscr{D}$ is non-empty. Then there is an $x \in \Sigma^*$ such that $R(x) = s$ and $N_x \subseteq T(\mathscr{M})$. Let $q \in Ac(s)$ and $D \in \mathscr{H}q$. Prove that $D \in \mathscr{D}$. By the definition of $\mathscr{H}q$, Ac, there is a β of the form xyz^ω, where $R(xy) = q$ and for all i, $\mathscr{R}(xyz^i, xyz^{i+1}) = D$. Since $N_x \subseteq T(\mathscr{M})$, $D \in \mathscr{D}$.

2. Assume that (a) or (b) is true for any non-empty $\mathscr{H}s$. Let

$$B = \{x \mid \mathscr{H}(R(x)) \cap \mathscr{D} \neq \varnothing\}.$$

Prove that B is a basis for $T(\mathscr{M})$ so that $T(\mathscr{M}) \in G_1$. Assume that $T(\mathscr{M}) \neq \varnothing$ (the empty set is open). Let $\alpha \in T(\mathscr{M})$ so that $D = \mathrm{In}(\alpha) \in \mathscr{D}$. Choose a $y \prec \alpha$ such that $R(y) \in D$. Then $D \in \mathscr{H}(R(y)) \cap \mathscr{D}$, and thus $\mathscr{H}(R(y)) \cap \mathscr{D}$ is non-empty and $y \in B$.

If $\alpha \in N_y$, $y \in B$, then $\mathscr{H}(R(y)) \cap \mathscr{D} \neq \varnothing$, so that by (b), for all $q \in Ac(R(y))$, $\mathscr{H}q \subseteq \mathscr{D}$. Therefore $\mathrm{In}(\alpha) \in \mathscr{D}$ so that $\alpha \in T(\mathscr{M})$ and B is a basis. This completes the proof.

THEOREM 4.2. *$T(\mathscr{M})$ is in G_2 if and only if for all $s \in S$, $D \in \mathscr{D} \cap \mathscr{H}s$ and $E \in \mathscr{H}s$ implies $D \cup E \in \mathscr{D}$.*

Proof. 1. Assume that $T(\mathscr{M}) \in G_2$ with G_2-basis B (Definition 2.10). Let $s \in S$, $D \in \mathscr{D} \cap \mathscr{H}s$ and $E \in \mathscr{H}s$. Prove that $D \cup E \in \mathscr{D}$. This is done by defining an $\alpha \in T(\mathscr{M})$ for which $\mathrm{In}(\alpha) = D \cup E$. Choose x, y_1, z_1, w_1, to satisfy $R(x) = R(xy_1) = R(xy_1w_1) = s$, $\mathscr{R}(x, xy_1) = D$, $\mathscr{R}(xy_1, xy_1w_1) = E$, and $z_1 \in B$, $z_1 \prec xy_1$. Now y_1, z_1 and w_1 exist because $D \in \mathscr{H}s$, $D \in \mathscr{D}$ and B is a G_2-basis for $T(\mathscr{M})$ and $E \in \mathscr{H}s$. Choose y_2, z_2, w_2 such that $R(xy_1w_1y_2) = R(xy_1w_1y_2w_2) = s$, $\mathscr{R}(xy_1w_1, xy_1w_1y_2) = D$, $\mathscr{R}(xy_1w_1y_2, xy_1w_1y_2w_2) = E$ and z_2 satisfies $z_1 \prec z_2 \prec xy_1w_1y_2$, $z_2 \in B$. Similarly, choose y_i, z_i, w_i, $i = 3, 4, \cdots$. Let $\alpha = xy_1w_1y_2w_2\cdots$. We have $\alpha \in T(\mathscr{M})$ because $z_i \in B$, $z_i \prec \alpha$, $i = 1, 2, \cdots$ and B is a G_2-basis for $T(\mathscr{M})$. But $\mathrm{In}(\alpha) = D \cup E$, so that $D \cup E \in \mathscr{D}$.

2. Assume that for all $s \in S$, $D \in \mathscr{D} \cap \mathscr{H}s$, $E \in \mathscr{H}s$ imply $D \cup E \in \mathscr{D}$. For $s \in S$ let

$$B_1^s = \{x \mid R(x) = s,\ (\forall\ y \prec x)\ R(y) \neq s\}$$

$$B_{i+1}^s = \{x \mid R(x) = s,\ (\exists\ y \in B_i^s)\ [y \prec x \wedge \mathscr{R}(y, x) \in \mathscr{D} \wedge (\forall w)\ [y \prec w \prec x \supset R(w) \neq s \vee \mathscr{R}(y, w) \notin \mathscr{D}]]\};$$

i.e., B_{i+1}^s is the set of shortest extensions of each $y \in B_i^s$ which cause $\mathscr{M}$ to reach s after traversing a member of $\mathscr{D}$. Let $B = \bigcup_{s \in S} \bigcup_i B_i^s$. Show that B is a G_2-basis for $T(\mathscr{M})$.

(a) Assume there are $x_1 \prec x_2 \prec \cdots$ such that $x_i \prec \alpha$, $x_i \in B$, for $i = 1, 2, \cdots$. Prove that $\alpha \in T(\mathscr{M})$. Choose a subset $y_1 \prec y_2 \prec \cdots$ of $\{x_i\}$ such that $\{y_i\} \subseteq \bigcup_i B_i^s$ for some s. Let $z_1 \prec z_2 \prec \cdots$ be a subset of the $\{y_i\}$ such that $\mathscr{R}(z_i, z_{i+1}) = \mathrm{In}(\alpha)$. But by the definition of the B_i^s sets, $\mathscr{R}(z_i, z_{i+1}) = \mathrm{In}(\alpha)$ is a union of sets in $\mathscr{D} \cap \mathscr{H}s$, so that $\mathrm{In}(\alpha) \in \mathscr{D}$ by hypothesis and thus $\alpha \in T(\mathscr{M})$.

(b) Assume that $\alpha \in T(\mathscr{M})$ and prove that there are $x_1 \prec x_2 \prec \cdots$ such that $x_i \prec \alpha$, $x_i \in B$, for $i = 1, 2, \cdots$. Let $D = \mathrm{In}(\alpha)$ and fix $s \in D$. Then there is an $x_1 \in B_1^s$, $x_1 \prec \alpha$. Assume that $x_i \prec \alpha$, $x_i \in \bigcup_j B_j^s$, has been determined, and choose x_{i+1} as follows: Let z and y be such that $x_i \prec y \prec z$, $R(z) = R(y) = s$

and $\mathscr{R}(y, z) = D \in \mathscr{D}$. Since $\mathscr{R}(x_i, y)$ and $\mathscr{R}(y, z)$ are in $\mathscr{H}s$, the hypothesis implies $\mathscr{R}(x, z) \in \mathscr{D}$. Hence by the definition of the B_j^s sets, some $\bar{z}$, $x_i \prec \bar{z} \preceq z$, is in $\bigcup_j B_j^s$. Let x_{i+1} be $\bar{z}$. The proof is finished.

It is easy to see that the following lemmas obtain.

LEMMA 4.1. *There is an effective procedure for obtaining the $\mathscr{H}s$ and $Ac(s)$ sets from $\mathscr{M}$.*

LEMMA 4.2. *If $\mathscr{M} = \langle S, s_0, M, \mathscr{D}\rangle$, then $T(\mathscr{M})^c = T(\langle S, s_0, M, P(S) - \mathscr{D}\rangle)$.*

LEMMA 4.3. *Given a 1-, 1′-, 2- or 2′-f.a., an equivalent 3-f.a. can be effectively obtained.*

Theorems 4.1–4.2 and Lemmas 4.1–4.2 imply the following theorem.

THEOREM 4.3. *There is an effective procedure for deciding the complexity of $T(\mathscr{M})$, with respect to the Borel hierarchy, for any 3-f.a. $\mathscr{M}$; i.e., we can decide whether $T(\mathscr{M})$ is in G_1, F_1, G_2, F_2 or $G_3 \cap F_3$.*

By Theorem 4.3 and Lemma 4.3, we have the following result.

THEOREM 4.4. *There is an effective procedure for deciding the complexity of $T(\mathscr{M})$ for any i-f.a.*

The next two theorems show that 2- and 2′-f.a. differ from 3-f.a. only on $G_3 \cap F_3$.

THEOREM 4.5. *If $A \in G_2$ is 3-definable, then it is 2-definable.*

Proof. Let $T(\mathscr{M}) \in G_2$, where $\mathscr{M} = \langle S, s_0, M, \mathscr{D}\rangle$ is a 3-f.a. Now a 2-f.a. $\mathscr{M}^*$ satisfying $T(\mathscr{M}) = T(\mathscr{M}^*)$ is defined as follows. For each $s \in S$, let $\mathscr{M}_s$ be an f.a. which for any input sequence α satisfies:

(a) $\mathscr{M}_s$ enters a designated state ϵ the first time $\mathscr{M}$ would enter s in reading α;

(b) $\mathscr{M}_s$ reënters ϵ each time and only at such times that (1) $\mathscr{M}$ is in state s and (2) the set of states entered by $\mathscr{M}$ in reading α, since the previous time $\mathscr{M}_s$ was in ϵ, is in $\mathscr{D}$.

$\mathscr{M}^*$ is $\langle \mathscr{M}_{s_1} \times \cdots \times \mathscr{M}_{s_n}, D^*\rangle$, where $S = \{s_1, \cdots, s_n\}$, $\times$ is the usual product operation on machines and

$$D^* = \{(d_1, \cdots, d_n) \mid d_i \text{ a state of } \mathscr{M}_{s_i}, (\exists_j)(d_j = \epsilon)\}$$

is the output condition.

Note that $\mathscr{M}$ is built into each $\mathscr{M}_s$. It is clear that a finite automaton can be designed to satisfy (a) and (b). In the following, $\text{In}(\alpha)$ and $\mathscr{H}s$ always refer to $\mathscr{M}$.

1. $T(\mathscr{M}) \subseteq T(\mathscr{M}^*)$. Let $\alpha \in T(\mathscr{M})$ and $\text{In}(\alpha) \in \mathscr{D}$. Choose $s \in \text{In}(\alpha)$. $\mathscr{M}_s$ enters ϵ the first time $\mathscr{M}$ enters s, while reading α. This occurs because $s \in \text{In}(\alpha)$.

Assume $\mathscr{M}_s$ enters ϵ for the nth time at time t and let $E_1, E_2, \cdots$ be the sets of states which $\mathscr{M}$ enters between successively entering s after time t ($E_i \neq \varnothing$, $i = 1, 2, \cdots$, since $s \in \text{In}(\alpha)$). There is a finite sequence $E_j, E_{j+1}, \cdots, E_{j+k}$ such that $\text{In}(\alpha) = \bigcup_{l=0}^{k} E_{j+l}$. But then, since $\mathscr{H}s \cap \mathscr{D} \neq \varnothing$, Theorem 4.2 implies that $\bigcup_{l=1}^{j+k} E_l \in \mathscr{D}$. Hence $\mathscr{M}_s$ enters ϵ an $(n+1)$st time. This proves that if $\alpha \in T(\mathscr{M})$, then some $\mathscr{M}_s$ enters ϵ infinitely often, so that $\alpha \in T(\mathscr{M}^*)$.

2. $T(\mathscr{M}^*) \subseteq T(\mathscr{M})$. Let $\alpha \in T(\mathscr{M}^*)$, so that there is an s such that $\mathscr{M}_s$ enters ϵ infinitely often while reading α. Let E_i $(i = 1, 2, \cdots)$ be the set of states entered by $\mathscr{M}$ between the ith and $(i+1)$st times $\mathscr{M}_s$ enters ϵ. Then $E_i \in \mathscr{H}s \cap \mathscr{D}$ $(i = 1, 2, \cdots)$ by the definition of $\mathscr{M}_s$. In(α) must be equal to a finite union of E_i's, but since $\mathscr{H}s \cap \mathscr{D} \neq \varnothing$, Theorem 4.2 implies that any finite union of E_i's is in $\mathscr{D}$. Hence $\alpha \in T(\mathscr{M})$.

THEOREM 4.6. *A is 2-definable if and only if A^c is 2′-definable.*

Proof. Let $\mathscr{M} = \langle S, s_0, M, \dot{D}\rangle$ 2-define A. The 2′-f.a. $\langle S, s_0, M, P(S-D)\rangle$ 2′-defines A^c. If $\mathscr{M}$ 2′-defines A, first obtain a 3-f.a. $\mathscr{M}_1$ which defines A. Then modify $\mathscr{M}_1$ to a 3-f.a. $\mathscr{M}_2$ which 3-defines A^c. $A^c \in G_2$, so by Theorem 4.5 there is a 2-f.a. which defines it.

COROLLARY 4.7. *If $A \in F_2$ is 3-definable, then A is 2′-definable.*

In [6], McNaughton proves (in effect) that non-deterministic 2-f.a. define the same sets as 3-f.a. Hence Theorem 4.5 shows that non-deterministic 2-f.a. differ from deterministic 2-f.a. only on $G_3 \cap F_3$.

5. Undecidable Problems

In this section we consider decision problems for Turing machines which define sets of ω-sequences. The model employed is the one-tape, on-line Turing machine augmented by the various output conditions of Section 2.

Definition 5.1. A one-tape, on-line Turing machine (T.M.) is a Turing machine having (1) a single two-way infinite work tape with a read-write head and (2) a one-way infinite input tape, with a read-only head, which is to contain members of Σ^ω.

An i-T.M. is a Turing machine (as above) augmented by an output of type i (as in the definition of an i-f.a.). Definitions 2.4 and 2.5 with T.M. replacing f.a. define "Turing machine $\mathscr{M}$ i-defines $A \subseteq \Sigma^\omega$" and "$A$ is i-definable by a Turing machine". In [3], it is shown that every 3-T.M. defines a set in $F_3 \cap G_3$. In fact the method of proof for Theorems 3.1–3.5 is immediately applicable to the class of T.M.'s and indeed to any class of machines augmented by the corresponding output type.

Let $\mathscr{P}(C)$ stand for the problem of determining whether an arbitrary 3-T.M. defines a set in $C \subseteq P(\Sigma^\omega)$. By reducing the emptiness problem for ordinary Turing machines to $\mathscr{P}(G_i)$ $(i = 0, 1, 2)$, we obtain

THEOREM 5.1. *$\mathscr{P}(G_i)$ and $\mathscr{P}(F_i)$ are undecidable for $i = 0, 1, 2$.*

Hartmanis and Hopcroft [5] have investigated the relationship of undecidable problems for various types of machines with respect to Turing reducibility. The following theorems compare problems for 3-T.M.'s with problems on ordinary Turing machines.

Let M be the standard class of one-tape Turing machines which accept ($J \in M$ accepts the set of finite sequences on which it halts) sets of finite sequences on Σ. M^* is the class obtained from M by allowing machines to have an oracle which, given an index x for a machine in M, decides whether machine x halts on x.

Definition 5.2. $\underline{\mathscr{P}}_F$, $\underline{\mathscr{P}}_{\Sigma^*}$ and $\underline{\mathscr{P}}_R$ are respectively the problems of deciding whether an arbitrary $J \in M$ accepts a finite set, all of Σ^*, or a recursive set. If $\underline{\mathscr{P}}$ is a problem on M, then $\underline{\mathscr{P}}^*$ is the corresponding problem on M^*.

Notation. $\mathscr{P}_1 \leq \mathscr{P}_2$ means that the problem $\mathscr{P}_1$ is (Turing) reducible to the problem $\mathscr{P}_2$. $\mathscr{P}_1 \equiv \mathscr{P}_2$ if $\mathscr{P}_1 \leq \mathscr{P}_2$ and $\mathscr{P}_2 \leq \mathscr{P}_1$. $\mathscr{P}_1 < \mathscr{P}_2$ if $\mathscr{P}_1 \leq \mathscr{P}_2$ but $\mathscr{P}_2 \not\leq \mathscr{P}_1$.

It is easy to show

THEOREM 5.2. $\mathscr{P}(F_i) \equiv \mathscr{P}(G_i)$, $i = 0, 1, 2$.

THEOREM 5.3. $\underline{\mathscr{P}}_F \leq \mathscr{P}_\varnothing$ *(emptiness problem for* 3-*T.M.).*

The following theorem is proved by employing the notion of a "valid computation of a Turing machine" (see [5] for definition).

THEOREM 5.4. $\underline{\mathscr{P}}_F^* \leq \mathscr{P}(G_i)$, $i = 0, 1$.

It is well known [5, 8] that

$$\underline{\mathscr{P}}_\varnothing \text{ (emptiness problem for } M) < \underline{\mathscr{P}}_F \equiv \underline{\mathscr{P}}_{\Sigma^*} < \underline{\mathscr{P}}_F^* \equiv \underline{\mathscr{P}}_R.$$

Our results show that problems such as $\mathscr{P}(G_i)$ are at least as difficult as $\underline{\mathscr{P}}_R$. A general open problem is that of characterizing the relative degrees of unsolvability of problems on 3-T.M.

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