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ON SOME MATHEMATICAL PRINCIPLES IN THE LINEAR THEORY OF DAMPED OSCILATIONS OF CONTINUA II[1)]

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§5. A fundamental theorem on a quadratic operator equation and some of its consequences.

5.1. We shall say that the space $\mathbb{L} \subset \mathbb{H}_0 = \mathbb{H}_1 \oplus \mathbb{H}_2$ has an <u>angular operator</u> K (with respect to $\mathbb{H}$), where K is some linear bounded operator, mapping $\mathbb{H}_1$ in $\mathbb{H}_2$, if $\mathbb{L} = \{x \oplus Kx;\ x \in \mathbb{H}_1\}$. It is easy to see that the theorem of Banach on the existence of a continuous inverse operator of a linear operator which maps one Banach space one-to-one and continuously on the other, admits to state that the space $\mathbb{L}(\subset\mathbb{H}_0)$ has an angular operator K if and only if the projector P_1 (projecting $\mathbb{H}_0$ orthogonally on $\mathbb{H}_1$) projects $\mathbb{L}$ one-to-one onto $\mathbb{H}_1$.

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1) <u>Note of the editor</u>: This paper is the second and last part of a paper published in Russian in "Proc. Int. Sympos. on Applications of the Theory of Functions in Continuum Mechanics, Tbilisi 1963, vol II: Fluid and Gas Mechanics, Math. Methods, Nauka, Moskow, 1965, pp. 283-322."
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LEMMA 5.1. *Let*

$$G = \begin{bmatrix} 0 & C_1 \\ -C_2 & -B \end{bmatrix} \qquad (C_1, C_2, B \in \mathbb{R})$$

be some operator, acting in $\mathbb{H}_0$, *and let* $\mathbb{L}(\subset \mathbb{H}_0)$ *be some subspace, having an angular operator* K. *In order that the subspace* $\mathbb{L}$ *will be invariant with respect to* G, *it is necessary* (*and if* $\overline{\mathbb{R}(C_1)} = \mathbb{H}_1$ *it is also sufficient*) *that the operator* $Z = KC_1$ *satisfies the relation*

$$(5.1) \qquad Z^2 + BZ + C_2C_1 = 0.$$

PROOF. If $\tilde{x} \in \mathbb{L}$, then $\tilde{x} = x \oplus Kx$ $(x \in \mathbb{H})$ and so $G\tilde{x} = C_1Kx \oplus (-C_2x - BKx)$. In order that $Gx \in \mathbb{L}$ it is necessary and sufficient that its second component is obtained from the first by applying the operator K, i.e., $-C_2x - BKx = KC_1Kx$. Therefore $G\mathbb{L} \subset \mathbb{L}$ if and only if

$$(F \equiv) \quad KC_1K + BK + C_2 = 0.$$

By multiplying this equation from the right by C_1, we get (5.1). Converseley, if (5.1) holds, then $FC_1 = 0$, and if $\overline{\mathbb{R}(C_1)} = \mathbb{H}$, then $F = 0$.

5.2. For further progress in the investigation of the quadratic equation $L(Z) = 0$ we need some facts from the geometry of spaces $\mathbb{H}_0$ with J-metric and the theory of operators (J-self-adjoint) in such spaces.

The subspace $\mathbb{L}(\subset \mathbb{H}_0)$ is called *J-non-negative* if $(J\tilde{x}, \tilde{x}) \geq 0$ for all $\tilde{x} \in \mathbb{L}$. It is called *maximal J-non-negative* if it is not a proper part of any other J-non-negative subspace.

The following proposition holds true (see[31, 2-3], and also [4]).

5.1°. *In order that some subspace* $\mathbb{L} \subset \mathbb{H}_0$ *will be maximal J-non-negative it is necessary and sufficient that it has an* angular operator K with $\|K\| \leq 1$.

The space $\mathbb{L} \subset \mathbb{H}_0$ is called J-isotropic if $(J\tilde{x}, \tilde{x}) = 0$ for all $\tilde{x} \in \mathbb{L}$.

If $\mathbb{L} \subset \mathbb{H}_0$ is a subspace with angular operator K, then the J-isotropicness of $\mathbb{L}$ means that $(Kx, Kx) = (x, x)$ $(x \in \mathbb{H}_1)$, i.e., it means that K is an isometric operator.

If $\tilde{H}$ is a bounded J-self-adjoint operator, acting in $\mathbb{H}_0$, then every root space $\mathbb{E}_\lambda$, corresponding to a non-real eigen-value, is J-isotropic. A more general proposition states that every two root spaces $\mathbb{E}_\lambda(\tilde{H})$ and $\mathbb{E}_\mu(\tilde{H})$ of the operator $\tilde{H}$ with $\mu \neq \bar{\lambda}$ are J-orthogonal, i.e., if $\mu \neq \bar{\lambda}$ (in particular, $\mu = \lambda \neq \bar{\lambda}$), then $(J\tilde{x}, \tilde{y}) = 0$ for $\tilde{x} \in \mathbb{E}_\lambda(\tilde{H})$ and $\tilde{y} \in \mathbb{E}_\mu(\tilde{H})$. This statement is a direct generalitzation of well-known propositions from linear algebra [25, 30, 10, 20]. From these propositions it follows that if Λ is some set of non-real eigen-values of the operator $\tilde{H}$, not containing any pair of conjugated complex numbers, then the linear span $\mathbb{E}$ of the root spaces $\mathbb{E}_\lambda(\tilde{H})$ with $\lambda \in \Lambda$ is an isotropic space. Clearly the closure $\bar{\mathbb{E}}$ will be a J-isotropic subspace too.

The following theorem holds true [20, 14] [1)]

5.2°. Let $\tilde{H}$ be a bounded J-self-adjoint operator having the compact component $\operatorname{Im}H = (\tilde{H} - \tilde{H}^*)/2i$, and let $\sigma_0(\tilde{H}) = \Lambda \cup \bar{\Lambda}$ be a decomposition of its non-real spectrum $\sigma_0(\tilde{H})$ into disjoint parts Λ and $\bar{\Lambda}$ which are symmetric with respect to the real axis. Then there exists a maximal J-non-negative subspace $\mathbb{L}_\Lambda (\subset \mathbb{H}_0)$ with the following properties: 1°) $\tilde{H}\mathbb{L}_\Lambda \subset \mathbb{L}_\Lambda$ and 2°) the non-real part of the spectrum of the restriction of the operator $\tilde{H}$ to $\mathbb{L}_\Lambda$ coincides with Λ.

Every such subspace $\mathbb{L}$ has another similar property:

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1) This theorem of H. Langer has to be considered as a generalization of a well-known theorem of L.S. Pontryagin [30] (see also [10]) on self-adjoint operators in spaces Π_κ. A simpler proof of Langer's theorem and a simultaneous generalization has been given by M.G. Krein [14]. This result allows to generalize the fundamental theorem 1.1 also. We remark that Langer's theorem in its complete version [20 - 21] also allows to prove the validity of theorem 1.1 in the case that B is an unbounded self-adjoint operator.

3°) $\mathbb{L}_\Lambda$ contains any root space $\mathbb{E}_\lambda(\tilde{H})$ corresponding to $\lambda \in \Lambda$:

With the aid of this theorem the following theorem 5.1, which is fundamental for what follows, can be proved without great difficulty.

THEOREM 5.1. Let $B = B^*$ ($\in \mathfrak{R}$), $C \in \mathfrak{S}_\infty$ and $C>0$. Then for any decomposition of the non-real spectrum $\sigma_0(L)$ of the pencil $L(\lambda) = \lambda^2 I + \lambda B + C$ into two disjoint parts Λ and $\bar{\Lambda} = \sigma_0(L)\setminus\Lambda$, situated symmetrically with respect to the real axis, the equation $L(Z) = 0$ has a root Z_Λ($\in \mathfrak{S}_\infty$) possessing the following properties: 1°) $Z_\Lambda^* Z_\Lambda \leq C$ and 2°) the non-real part of the spectrum $\sigma(Z_\Lambda)$ coincides with Λ.

For any $\lambda \in \Lambda$ the operator Z_Λ and the pencil L have the same Jordan chains.

If the system of Jordan chains of the pencil L, corresponding to all possible $\lambda \in \Lambda$, is complete in $\mathbb{H}$ then the root Z_Λ is defined in a unique way by the properties 1° and 2°. In that case

$$\lambda^2 I + B + C = (\lambda I - Z_\Lambda^*)(\lambda I - Z_\Lambda), \tag{5.2}$$

where $\quad B = - Z_\Lambda = Z_\Lambda^*, \qquad C = Z_\Lambda^* Z_\Lambda$.

PROOF. It has been remarked in §2 already that the operator $\tilde{H}$ defined in $\mathbb{H}$ by the equation (2.4) is J-self-adjoint, and according to (2.2) its imaginary component $\operatorname{Im}\tilde{H}$ is compact; in addition $\sigma(\tilde{H}) = \sigma(L)$ and consequently $\sigma_0(\tilde{H}) = \sigma_0(L)$.
On the basis of the propositions 5.1° and 5.2° there corresponds to $\tilde{H}$ an invariant subspace $\mathbb{L}_\Lambda$ with angular operator K_Λ such that $\|K_\Lambda\| \leq 1$, all root spaces $\mathbb{E}_\lambda(\tilde{H})$ with $\lambda \in \Lambda$ are in $\mathbb{L}_\Lambda$, and on $\mathbb{L}_\Lambda$ the restriction of $\tilde{H}$ has a non-real spectrum coinciding with Λ.

According to Lemma 5.1 we shall have for the operator $Z_\Lambda = K_\Lambda C^{\frac{1}{2}}$:

$$Z_\Lambda^2 + BZ_\Lambda + C = 0.$$

As $\|K_\Lambda\| \leq 1$, it follows that $\|Z_\Lambda x\| \leq \|C^{\frac{1}{2}}x\|$, i.e. $Z_\Lambda^* Z_\Lambda \leq C$.

Let us examine the spectrum of the obtained root Z_Λ($\in\mathfrak{S}_\infty$). Let λ_0 ($\neq 0$) be some eigen-value of Z_Λ and $\Psi_0, \Psi_1, \dots, \Psi_{p-1}$ a Jordan chain corresponding to λ_0:

(5.3) $Z_\Lambda \Psi_0 = \lambda_0 \Psi_0, \quad Z_\Lambda \Psi_j = \lambda_0 \Psi_j + \Psi_{j-1} \; (j = 1, \ldots, p-1).$

According to lemma 3.1 the vectors $\Psi_0, \Psi_1, \ldots, \Psi_{p-1}$ form a Jordan chain of the pencil L for the value λ_0, and then, according to lemma 2.1, the vectors (2.5) will form a Jordan chain of the operator $\tilde{H}$ for the same value λ_0. The vectors (2.5) can be written in the form

$$(1.4) \qquad \begin{bmatrix} C^{\frac{1}{2}} & \Psi_j \\ Z_\Lambda & \Psi_j \end{bmatrix} = \begin{bmatrix} C^{\frac{1}{2}} & \Psi_j \\ K_\Lambda & C^{\frac{1}{2}} \Psi_j \end{bmatrix} \quad (j = 0, 1, \ldots, p-1),$$

and therefore they belong to the subspace $\mathbb{L}_\Lambda$. Therefore, if $\lambda_0 \in \sigma(Z_\Lambda)$ is non-real, it follows that $\lambda_0 \in \Lambda$. Conversely, let $\lambda_0 \in \Lambda$ and let $\Psi_0, \Psi_1, \ldots, \Psi_{p-1}$ be some Jordan chain of the pencil L, corresponding to the value λ_0. Then the sequence (2.5) will be a Jordan chain of the operator $\tilde{H}$, corresponding to the value λ_0. As $\lambda_0 \in \Lambda$, the vectors of the chain (2.5) will belong to $\mathbb{L}_\Lambda$ and their components will be connected by means of the angular operator K_Λ, i.e., we have

$$K_\Lambda C^{\frac{1}{2}} \Psi_0 = \lambda_0 \Psi_0, \quad K_\Lambda C^{\frac{1}{2}} \Psi_j = \lambda_0 \Psi_j + \Psi_{j-1} \; (j = 1, 2, \ldots, p-1).$$

So the vectors $\Psi_0, \Psi_1, \ldots, \Psi_{p-1}$ form a Jordan chain of the operator Z_Λ, corresponding to the value λ_0. This completes the proof of the first statement of the theorem. The second statement will also be proved if we show that every root Z_Λ of the equation $L(Z) = 0$, which has the mentioned properties 1° and 2°, is obtained by the formula $Z_\Lambda = K_\Lambda C^{\frac{1}{2}}$, where K_Λ is an angular operator with $\| K_\Lambda \| \leq 1$ of some invariant subspace of the operator $\tilde{H}$, on which the non-real spectrum of the restriction of $\tilde{H}$ coincides with Λ.

If we analyze the previous arguments we find that they contain the following general conclusion.

5.3°. <u>Let $\mathbb{L}$ be some invariant subspace of the operator</u> $\tilde{H}$ <u>with angular operator</u> K. <u>Then the spectrum of the non-zero</u>

<u>eigen-values of the root</u> $Z = KC^{\frac{1}{2}}$ <u>of the equation</u> $\tilde{L}(Z)$ <u>coincides with the analogous spectrum of the restriction of</u> H <u>on</u> $\mathbb{L}$.

And what is more, from any Jordan chain of either $\tilde{H}$ or Z we can get by explicit formulae a Jordan chain of the other operator, where both chains correspond to one and the same eigen-value.

Therefore it remains to prove that if Z_0 is some root of the equation $L(Z) = 0$, possessing the property $Z_0^* Z_0 \leqq C$, then $Z_0 = K_0 C^{\frac{1}{2}}$, where K_0 is an angular operator of some subspace $\mathbb{L}_0$, which is invariant with respect to $\tilde{H}$, and with $\| K_0 \| \leqq 1$. This property means that $\| Z_0 x \| \leqq \| C^{\frac{1}{2}} x \|$ $(x \in \tilde{H})$ and, as $\mathbb{R}(C^{\frac{1}{2}}) = \mathbb{H}$, from this follows the existence of a unique operator K_0 $(\in \mathbb{R})$ with $\| K_0 \| \leqq 1$ such that $Z_0 x = K_0 C^{\frac{1}{2}} x$ $(x \in \mathbb{H})$, i.e. $Z_0 = K_0 C^{\frac{1}{2}}$. By lemma 5.1 the subspace $\mathbb{L} = \{x \oplus K_0 x;\ x \in \mathbb{H}\}$, having the angular operator K_0, will be invariant with respect to $\tilde{H}$.

The third statement of theorem 5.1 holds true because of the general theorem 3.2. So the proof of the theorem is complete.

<u>REMARK 5.1.</u> The third statement of the theorem means that in the expression $Z_\Lambda = KC^{\frac{1}{2}}$ the operator K is isometric and this in turn means that the subspace $\mathbb{L}(\subset \mathbb{H}_0)$ with the angular operator K_Λ is a maximal isotropic subspace in $\mathbb{H}_0$. On the basis of what is said about isotropic subspaces on page 3 it is possible to convince oneself directly of this fact. From the completeness of the system of root spaces $\mathbb{E}_\lambda(Z_\Lambda)$ $(\lambda \in \Lambda)$ it follows that $\mathbb{L}_\Lambda$ is the linear closed hull of the root space system $\mathbb{E}_\lambda(\tilde{H})$ $(\lambda \in \Lambda)$.

As the root Z_Λ is compact the following consequence holds:

<u>COROLLARY 5.1.</u> <u>If the set of non-real eigen-values of the pencil</u> L <u>is infinite, then the point</u> 0 <u>is its unique limit point.</u>

5.3. A theorem of H. Weil [6, 7 : 2] yields a more precise result than is formulated in Corollary 5.1. According to that theorem for any operator $Z \in \mathfrak{S}_\infty$ and any continuous function $f(r)$ $(0 \leqq r < \infty,\ f(0) = 0)$, such that the function $f(e^t)$ $(-\infty < t < \infty)$ is downwards convex, the following inequality holds:

$$\sum_{j=1}^{n} f(|\lambda_j(Z)|) \leqq \sum_{j=1}^{n} f(s_j(Z)) \qquad (n = 1, 2, \ldots).$$

Here $\{\lambda_j(Z)\}$ is a complete sequence of non-zero eigen-values, in order of decreasing absolute values (taking into account their algebraic multiplicity) $(|\lambda_1(Z)| \geqq |\lambda_2(Z)| \geqq \ldots)$. If the function $f(e^t)$ is strictly convex and the sequence $\{\lambda_j(Z)\}$ is infinite, then the relation

$$\sum_{j=1}^{\infty} f(|\lambda_j(Z)|) = \sum_{j=1}^{\infty} f(s_j(Z))$$

assuming that the right side is finite is valid if and only if Z is a normal operator $(Z^*Z = ZZ^*)$. [1)]

Let us apply the theorem of Weil to the operator Z_Λ. As $Z_\Lambda^* Z_\Lambda \leqq C$ it follows that $\lambda_j(Z_\Lambda^* Z_\Lambda) \leqq \lambda_j(C)$ $(j = 1, 2, \ldots)$ and consequently

$$(5.5) \qquad s_j(Z_\Lambda) \leqq \lambda_j(C^{\frac{1}{2}}) = \sqrt{\lambda_j(C)} \qquad (j = 1, 2, \ldots)$$

From this we obtain the first statement of the following theorem.

THEOREM 5.2. *If for the continuous function* $f(r)$ $(0 \leqq r < \infty,\ f(0) = 0)$ *the corresponding function* $f(e^t)$ $(-\infty < t < \infty)$ *is downwards convex then the following relation holds for the root* Z_Λ, *given by Theorem 5.1*:

$$(5.6) \qquad \sum_{j=1}^{n} f(|\lambda_j(Z_\Lambda)|) \leqq \sum_{j=1}^{n} f(\sqrt{\lambda_j(C)}) \qquad (n = 1, 2, \ldots),$$

and consequently, if the sequence $\{\lambda_j(Z_\Lambda)\}$ *is infinite, then*

$$(5.7) \qquad \sum_{j=1}^{\infty} f(|\lambda_j(Z_\Lambda)|) \leqq \sum_{j=1}^{\infty} f(\sqrt{\lambda_j(C)}).$$

If the function $f(e^t)$ *is strictly convex and both sums in* (5.7) *are finite and equal, then* 1°) *the operators* B *and* C *are commutative*, 2°) $B^2 \leqq 4C$ *and* 3°) *the root* Z_Λ *is a complete*[2)] *normal*

1) This addition to the theorem of Weil(which strictly speaking has been formulated by Weil only in application to matrices) can be found in the book [6].
2) A normal operator $Z(\in \mathfrak{R})$ is called complete if it vanishes in 0 only or, what is that same, if $\overline{\mathfrak{R}(Z)} = \mathfrak{H}$.

operator for which

$$(5.8)\qquad |\lambda_j(Z_\Lambda)| = \sqrt{\lambda_j(C)} \qquad (j = 1, 2, \dots).$$

If the conditions 1° *and* 2° *are fulfilled, then for a given* Λ *always a root*

$$(5.9)\qquad Z_\Lambda = \tfrac{1}{2}(-B + i\sqrt{4C - B^2})$$

can be found for which the condition 3° *will be fulfilled.*

PROOF. If the sums in (5.7) are equal and finite and the function $f(e^t)$ is strictly convex, then it follows from (5.5) that

$$(5.10)\qquad s_j(Z_\Lambda) = \lambda_j(C^{\frac{1}{2}}), \quad \text{i.e.,}\ \lambda_j(Z_\Lambda^* Z_\Lambda) = \lambda_j(C)(j = 1,2,\dots),$$

and

$$(5.11)\qquad \sum_{j=1}^{\infty} f(|\lambda_j(Z_\Lambda)|) = \sum_{j=1}^{\infty} f(s_j(Z_\Lambda)).$$

From (5.11) it follows that Z_Λ is a normal operator.

As $Z_\Lambda^* Z_\Lambda \leqq C$, it follows from (5.10) that $Z_\Lambda^* Z_\Lambda = C$. Taking into account that $C > 0$ we conclude that Z_Λ is a complete normal operator for which the relation (5.8) is valid. From the completeness of the normal operator Z_Λ and the equality $Z_\Lambda^* Z_\Lambda = C$ the factorization (5.2) follows so that $B = -Z_\Lambda - Z_\Lambda^*$, therefore the commutativity of Z_Λ and Z_Λ^* implies the commutativity of B and C.

Simultaneously we get

$$4C - B^2 = 4Z_\Lambda^* Z_\Lambda - (Z_\Lambda + Z_\Lambda^*)^2 = \left[\frac{Z_\Lambda - Z_\Lambda^*}{i}\right]^2 \geqq 0.$$

So this proves the second statement of the theorem.

Let us pass on to the third statement. If the operators B and C are commutative, then in $\mathfrak{H}$ an orthonormal basis $\{e_j\}_1^\infty$ can be found, such that

$$(5.12) \quad Ce_j = \lambda_j(C)e_j, \; Be_j = \mu_j e_j \qquad (j = 1, 2, \ldots),$$

and so

$$(5.13) \quad (4C - B^2)e_j = (4\lambda_j(C) - \mu_j^2)e_j \qquad (j = 1, 2, \ldots).$$

Therefore, if $4C - B^2 \geqq 0$, then $4\lambda_j(C) - \mu_j^2 \geqq 0$ $(j = 1, 2, \ldots)$.

Consequently the equation

$$(5.14) \quad \lambda^2 + \mu_j\lambda + \lambda_j(C) = 0 \qquad (j = 1, 2, \ldots)$$

will have either a double real root ($=\frac{1}{2}\mu_j$) or a pair of non-real conjugated complex roots. In the first case the real root of the equation (5.14), and in the second case the complex root which belongs to is denotes by λ_j^0. We form the normal operator Z_Λ by putting $Z_\Lambda e_j = \lambda_j^0 e_j$ $(j = 1, 2, \ldots)$. As $(Z_\Lambda^2 + BZ_\Lambda + C)e_j = 0$, $(j = 1, 2, \ldots)$, it follows that $Z_\Lambda^2 + BZ_\Lambda + C = 0$; in view of the fact that $|\lambda_j^0| = \lambda_j(C)$ it follows that (5.8) holds true. By defining in a corresponding way the operator $\sqrt{4C - B^2}$ with alternating signs, it is easy to understand that the root Z_Λ can be written in the form (5.9).

This finishes the proof of the theorem.

COROLLARY 5.2. *Let $\{\lambda_j^+(L)\}_1^N$ $(N \leqq \infty)$ be a complete sequence of eigen values of the pencil L, lying inside the upper half-plane and (taking into account their algebraic multiplicity) ordered according to decreasing moduli $(|\lambda_1^+(L)| \geqq |\lambda_2^+(L)| \geqq \ldots.)$. Then*

$$(5.15) \quad \sum_{j=1}^{n} f(|\lambda_j^+(L)|) \leqq \sum_{j=1}^{\infty} f(\sqrt{\lambda_j(C)}) \qquad (n = 1, 2, \ldots, N),$$

where $f(r)$ $(0 \leqq r < \infty,\ f(0) = 0)$ is an arbitrary continuous function to which there is a corresponding downwards convex function $f(e^t)$ $(-\infty < t < \infty)$.

In particular

$$(5.16) \quad \sum_{j=1}^{N} f(|\lambda_j^+(L)|) \leqq \sum_{j=1}^{\infty} f(\sqrt{\lambda_j(C)}).$$

If the function $f(e^t)$ is strictly convex and the right side in (5.16) is finite, then in (5.16) the equality sign holds true *if and only if the operators* B *and* C *are commutative and* $4C - B^2 > 0$. *In that case* $N = \infty$ *and* $|\lambda_j^+(L)| = \sqrt{\lambda_j(C)}$ $(j=1,2,\ldots)$.

PROOF. Indeed, if we put $\Lambda = \{\lambda_j^+(L)\}$, then the sequence $\{\lambda_j^+(L)\}_1^N$ will be a part of the sequence $\{\lambda_j(Z_\Lambda)\}$ and the inequal--ities (5.15) will be consequences of the inequalities (5.6).

If the function $f(e^t)$ is strictly convex and if in (5.16) the equality sign holds, then it is clear that $N = \infty$, $\{\lambda_j^+(L)\}_1^\infty = \{\lambda_j(Z_\Lambda)\}_1^\infty$ and in (5.7) the equality sign holds. By what has been proved the latter implies the commutativity of B, C, the normality of the operator Z_Λ and the relations (5.8). Let us choose in $\mathbb{H}$ an orthonormal basis $\{e_j\}_1^\infty$ such that

$$Z_\Lambda e_j = \lambda_j(Z_\Lambda)e_j, \qquad Z_\Lambda^* e_j = \overline{\lambda_j(Z_\Lambda)}e_j \quad (j = 1, 2, \ldots).$$

Then

$$Ce_j = Z_\Lambda^* Z_\Lambda e_j = |\lambda_j(Z_\Lambda)|^2 e_j, \quad Be_j = -(Z_\Lambda + Z_\Lambda^*)e_j = -2\,\mathrm{Re}\lambda_j(Z_\Lambda)e_j,$$

$$B^2 e_j = 4[\mathrm{Re}\,\lambda_j(Z_\Lambda)]^2 e_j \qquad (j = 1, 2, \ldots),$$

and consequently

$$(5.17) \quad (4C - B^2)e_j = \{4\,|\lambda_j(Z_\Lambda)|^2 - [\mathrm{Re}\lambda_j(Z_\Lambda)]^2\}e_j \quad (j = 1, 2, \ldots)$$

from which it follows that $4C - B^2 > 0$.

Conversely, if the operators B and C are commutative, then, by choosing an orthonormal basis $\{e_j\}$ such that the equalities (5.12) are valid, we have

$$(5.18) \quad (\lambda^2 I + \lambda B + C)x = \sum_{j=1}^{\infty} [\lambda^2 + \mu_j\lambda + \lambda_j(C)]\,(x, e_j)e_j.$$

If, in addition, $4C - B^2 > 0$, then according to (5.17) $4\lambda_j(C) - \mu_j^2 > 0$ $(j = 1, 2, \ldots)$, and every equation $\lambda^2 + \mu_j\lambda + \lambda_j(C) = 0$ will have a pair of non-real conjugated complex roots:

$$\lambda_j^{\pm} = \tfrac{1}{2}[-\mu_j \pm i\sqrt{4\lambda_j(C) - \mu_j^2}] \qquad (j = 1, 2, \ldots).$$

If for some $\Psi_0 \neq 0$ and $\lambda = \lambda_0$ we have $L(\lambda_0)\Psi_0 = 0$, then it follows from (5.18) that the number λ_0 coincides with one of the numbers $\lambda_j^{\pm}$, and if we assume for concreteness that $\text{Im}\lambda_0 > 0$, we have

$$\Psi_0 = \sum_{\lambda_j^{+}=\lambda_0} c_j e_j .$$

It is clear that conversely for every $\Psi_0 (\neq 0)$ of this form we shall always have $L(\lambda_0)\Psi_0 = 0$.

By means of (5.18) it is easy to verify that the pencil L has no adjoint vectors. So $\{\lambda_j^{+}(L)\} = \{\lambda_j^{+}\}$, and as $|\lambda_j^{+}| = \lambda_j(C)$ $(j = 1, 2, \ldots)$, this finishes the proof of the corollary.

We remark that the function $f(r) = r^{2q}$ $(q > 0)$ has a corresponding strictly convex function $f(e^t) = e^{2qt}$. Therefore for any $q > 0$ the following relation holds:

$$(5.19) \qquad \Sigma|\lambda_j^{+}(L)|^{2q} \leqq \text{Sp } C^q.$$

If some $q > 0$ we have $\text{Sp } C^q < \infty$, then in (5.19) the equality sign holds if and only if B and C are commutative and $4C - C^2 > 0$.

§6. A weakly damped pencil.

6.1. A pencil L is called <u>weakly damped</u> if the following condition is fulfilled:

$$(6.1) \qquad (Bx, x)^2 < 4(Cx, x)(x, x) \qquad \text{for } x \neq 0.$$

Clearly the condition (6.1) is equivalent to the condition of positiveness of the expression $(L(\lambda)x, x)$ for any $x \neq 0$ and real λ. In other words the condition (6.1) is equivalent to the positiveness of the operator $L(\lambda) = \lambda^2 I + \lambda B + C$ for any real λ.

We leave to the reader to prove that if $C \in \mathfrak{S}_\infty$, then it follows from the condition (6.1) that also $B \in \mathfrak{S}_\infty$ and in addition:

$$s_j^2(B) < 4\lambda_j(C) \qquad (j = 1, 2, \ldots, s_j(B) = \lambda_j^{\frac{1}{2}}(B^2)).$$

The latter inequality is easily obtained on the basis of the minimax properties of the eigen-values of self-adjoint compact operators. In this connection the following proposition holds:

6.1°. If $B = B^* \in \mathfrak{S}_\infty$, $C \in \mathfrak{S}_\infty$, $C > 0$, then the condition (6.1) of being weakly damped is equivalent to the condition of absence of real eigen-values in the pencil L.

Indeed, if (6.1) is fulfilled then for any real λ the operator $L(\lambda)$ is positive and consequently the pencil L has no real eigen-values. But if for some $x_0 \neq 0$ the inequality $(Bx_0, x_0)^2 \geqq 4(Cx_0, x_0)(x_0, x_0)$ is fulfilled, then the set of those λ for which there are $x \neq 0$ such that $(L(\lambda)x, x) = 0$ is not empty; it contains the points

$$\lambda_{1,2} = [-(Bx_0, x_0) \pm \sqrt{(Bx_0, x_0)^2 - 4(Cx_0,x_0)(x_0, x_0)}]/(x_0,x_0).$$

So according to a general theorem of P.H. Müller [29] the pencil has a real eigen-value.

We remark that if $C \in \mathfrak{S}_\infty$ and the operators B and C are commutative, then the condition (6.1) is equivalent to the condition $4C - B^2 > 0$ (see also 5.3). Generally the latter condition is more restictive than the condition (6.1) (see remark 2.1).

6.2. A compact operator Z will be called complete if the system of all its root spaces $\mathfrak{L}_\lambda(Z)$, corresponding to the non-zero eigen-values is complete and the operator Z^* has the same property.

If the operator Z is dissipative, $Z \in \mathfrak{S}_\infty$ and $Sp(-ReZ) < \infty$ then according to a theorem of M.S. Livsic (applied once before in (2.5)):

$$(6.2) \qquad -\sum \operatorname{Re} \lambda_j(Z) \leqq Sp(-ReZ),$$

where the equality sign is valid if and only if the system of root spaces $\mathfrak{L}_\lambda(Z)$ is complete in $\mathbb{H}$. But if this condition is satisfied, it is also satisfied for Z^*; so in that case the system of spaces $\mathfrak{L}_\lambda(Z^*)$ $(\lambda \in \sigma(Z^*))$ is also complete. For a dissipative operator Z we have always $\mathfrak{T}(Z) = \mathfrak{T}(Z^*)$, as for such operator the equation $Zx = 0$ is equivalent to the equations $(\mathrm{Re}Z)x = (\mathrm{Im}Z)x = 0$, and hence to $Z^*x = 0$. So a dissipative operator $Z(\in \mathfrak{S}_\infty)$ is complete if $\mathfrak{T}(Z) = \{0\}$, $\mathrm{Sp}(-\mathrm{Re}Z) < \infty$ and in (6.2) holds the equality sign.

In the following propositions it is assumed that in the pencil L the coefficient $C \in \mathfrak{S}_\infty$, $C > 0$.

THEOREM 6.1. *Let L be a weakly damped pencil with $B \geq 0$ and $\mathrm{Sp}\, B < \infty$. Then the system of root spaces of the root Z_Λ of the equation $L(Z) = 0$ is complete in $\mathbb{H}$ if and only if*

$$(6.3) \qquad -\sum_{\lambda\in\sigma(L)} \mathrm{Re}\,\lambda = \mathrm{Sp}\, B.$$

If this condition is fulfilled for any choice of Λ $(\bar{\Lambda} \cup \{0\} = \sigma(L) \setminus \Lambda)$ the roots Z_Λ and $Z_{\bar{\Lambda}}$ will be complete dissipative operators and will form a pair of solutions of the equation $L(Z) = 0$.

PROOF. Let us denote by R the projector which projects $\mathbb{H}$ orthogonally on the linear closed hull $\hat{\mathbb{H}}$ of all root spaces of the operator Z_Λ. We put $\hat{Z}_\Lambda = RZ_\Lambda R$; from $RL(Z_\Lambda)R = 0$ we get easily that $\hat{Z}_\Lambda^2 + \hat{B}\hat{Z}_\Lambda + \hat{C} = 0$, where $\hat{B} = RBR$; $\hat{C} = RCR$. Theorem 3.2 is applicable to the pencil $\hat{L}(\lambda) = \lambda^2\hat{I} + \lambda\hat{B} + \hat{C}$ and the root $\hat{Z}_\Lambda$, considered in $\hat{\mathbb{H}}$.

Hence

$$\hat{Z}_\Lambda + \hat{Z}_\Lambda^* = -\hat{B} \leq 0.$$

So the operator $\hat{Z}_\Lambda$ is dissipative and the application of the relation (6.2) on Z gives

(6.4) $\left(-\sum_{\lambda\in\sigma(L)} \mathrm{Re}\lambda =\right) -2 \sum_{\lambda\in\Lambda} \mathrm{Re}\lambda \leq \mathrm{Sp}\,\hat{B}\ (\leq \mathrm{Sp}B)$

If the system of root spaces of the operator Z_Λ is complete, then $R = I$, $RBR = B$ and according to a theorem of M.S. Livsic the equality sign holds true everywhere in (6.4). Conversely, if the equality sign holds in (6.3), then it follows from (6.4) that $\mathrm{Sp}\hat{B} = \mathrm{Sp}B$, hence $QBQ = 0$ ($Q = I - R$) and $QB = BQ = 0$. Multiplying each term of the equation $Z_\Lambda^2 + BZ_\Lambda + C = 0$ on the left and the right by Q we get: $QZ_\Lambda^2Q = -QCQ$. But then $QZ_\Lambda^{*2}Q = -QCQ$ as well. As $QZ_\Lambda^*Q = Z_\Lambda^*Q$, it follows that $(Z_\Lambda^*Q)^2 = -QCQ$. As the operator Z_Λ^* has no non-zero eigen-value in $Q\mathbb{H} = \hat{\mathbb{H}}^\perp$, any eigen-value of the operator Z_Λ^*Q and therefore also any eigen-value of the non-negative operator $QCQ = -(Z_\Lambda^*Q)^2$ equals zero. Therefore $QCQ = 0$ and as by our hypothesis the operator C is positive it follows that $Q = 0$, $R = I$.

As by assumption $C > 0$, it follows that the kernel $\mathfrak{Z}(Z)=\{0\}$ for any solution Z of the equation $L(Z) = 0$.

So, if the condition (6.3) is fulfilled the operators Z_Λ and $Z_{\overline{\Lambda}}$ are complete and dissipative. They form a complete pair of solutions by Lemma 4.3 and Remark 4.1.

This completes the proof of the theorem.

A simple comparison of the Theorems 2.1 and 6.1 leads to the following conclusion.

THEOREM 6.2. *Let* L *be a weakly damped pencil for which* $B \geq 0$, $\mathrm{Sp}\ B < \infty$ *and* $\liminf n^2\lambda_n(C) = 0$. *Then for any choice of* Λ *the roots* Z_Λ *and* $Z_{\overline{\Lambda}}$ *are complete and dissipative operators which form a complete pair of solutions of the equation* $L(Z) = 0$.

In addition we formulate the following propositions which is a complement to Theorem 2.2 (for $\kappa = 1$):

THEOREM 6.3. *Let* L *be a pencil for which* $B \geq 0$, $4C - B^2 > 0$ *and* $\lim n^2\lambda_n(C) = 0$. *Then any solution* Z_0 *of the equation* $L(Z) = 0$ *satisfying the condition* $Z_0^*Z_0 \leq C$, *is a complete and dissipative operator. For any choice of* Λ *the roots* Z_Λ *and* $Z_{\overline{\Lambda}}$ *form a complete pair of solutions of the equation* $L(Z) = 0$.

For the sake of brevity we omit the proof of this proposition. Let us comment upon the conditions of the theorem. As we know, the condition $B^2 < 4C$ implies the pencil L is weakly damped. From this condition (like, as a matter of fact, from the condition of being weakly damped) it follows that $\lambda_n(B) \leqq 2\lambda_n^{1/2}(C)$ $(n = 1, 2, \ldots)$, and therefore if $\lim n^2\lambda_n(C) = 0$, then $\lim n\lambda_n(B) = 0$. The latter condition is also fulfilled in case the condition $\mathrm{Sp}\, B < \infty$ $(B \geqq 0)$ is satisfied, to which condition it is very near, though it is nevertheless somewhat weaker.

§7. A strongly damped pencil.

7.1. The pencil L is called <u>strongly damped</u> if

$$(7.1)\quad (Bx, x) > 2\sqrt{(Cx, x)(x, x)} \qquad \text{for } x \neq 0 .$$

In this case the equation

$$((L(\lambda)x, x) =)\quad (x, x)\lambda^2 + (Bx, x)\lambda + (Cx, x) = 0$$

(for any $x \neq 0$) has two different negative roots $\lambda_{1,2} = p_\pm(x)$, where

$$(7.2)\quad p_\pm(x) = \frac{1}{2(x, x)}[-(Bx, x) \pm \sqrt{(Bx, x)^2 - 4(Cx, x)(x, x)}].$$

It is clear that

$$p_-(x) < p_+(x) < 0, \qquad p_+(x)p_-(x) = (Cx, x)/(x, x).$$

If for some $\Psi_0 \neq 0$ and a complex λ_0 we have $L(\lambda_0)\Psi_0 = 0$, then $(L(\lambda_0)\Psi_0, \Psi_0) = 0$, and therefore λ_0 coincides with one number $p_\pm(\Psi_0)$ or with the other. From this the following proposition follows:

7.1°. <u>Every eigen-value of a strongly damped pencil</u> L <u>is negative</u>.

We shall introduce a series of general definitions for the pencil L with $B = B^*$, $C > 0$. Let $L(\lambda_0)\Psi_0 = 0$ $(\Psi_0 \neq 0)$, then there are three possible cases: the value of $|\lambda_0|^2$ can be equal, less or more than the quotient $(C\Psi_0, \Psi_0) / (\Psi_0, \Psi_0)$. In correspondence to these cases the eigen-vector Ψ_0 is called <u>neutral</u>, of the <u>first</u> kind or of the <u>second</u> kind.

If λ_0 is non-real, then the eigen-vector Ψ_0 clearly will be neutral.

If all eigen-vectors corresponding to one and the same eigen-value λ_0 are of one and the same kind (first or second), then the eigen-value is called definite and, according to the case occuring it is either called an <u>eigen-value of the first kind</u> or of the <u>second kind</u>.

If the condition (7.1) is satisfied, any eigen-vector Ψ_0 will belong to one kind or the other, namely: it will be of the first kind if $\lambda_0 = p_+(\Psi_0)$ and of the second kind if $\lambda_0 = p_-(\Psi_0)$.

Let us put

(7.3) $\alpha_>(L) = - \sup p_-(x), \qquad \alpha_<(L) = - \inf p_+(x).$

It turns out that

(7.4) $p_-(x) < p_+(y) \qquad (x, y \in \mathbb{H} \; ; \; x, y \neq 0),$

hence

(7.5) $(0 <) \qquad \alpha_<(L) \leqq \alpha_>(L).$

and therefore the following proposition holds true

7.2° <u>Every eigen-value of a strongly damped pencil is definite</u>; <u>it is either</u> $\geqq -\alpha_<(L)$ <u>or</u> $\leqq -\alpha_>(L)$; <u>if it is</u> $\geqq -\alpha_<(L)$ <u>and</u> $> -\alpha_>(L)$, <u>then it is of the first kind</u>, <u>but if it is</u> $\leqq -\alpha_>(L)$ <u>and</u> $< -\alpha_<(L)$, <u>then it is of the second kind</u>.

In the algebraic case ($\mathbb{H}$ finite-dimensional) all statements mentioned above have been established before by R. Duffin [8]; in that case always $\alpha_<(L) < \alpha_>(L)$. From this result the inequality (7.5) follows immediately, even in our case ($\mathbb{H}$ infinite dimensional)

Indeed, let us denote by $\mathbb{H}^{(0)}$ the linear span of the pairs of elements x, y $\in \mathbb{H}$, x, y $\neq 0$ and by P the orthogonal projector projecting $\mathbb{H}$ on $\mathbb{H}^{(0)}$. It is clear that the restriction L_0 of the pencil PLP in $\mathbb{H}^{(0)}$ is a strongly damped pencil again. For the corresponding functionals $p_+^{(0)}$ [8, Theorem 4] we have $p_-^{(0)}(x) < p_+^{(0)}(y)$; on the other hand $p_\pm^{(0)}(z) = p_\pm(z) \quad (z \in \mathbb{H}^{(0)})$.

7.3° <u>If the condition</u> (7.1) <u>is satisfied the operator</u> B <u>is uniformly positive, namely</u>

$$(7.6) \qquad (Bx, x) \geqq \alpha_<(L)\,(x, x) \qquad (x \in \mathbb{H}).$$

Indeed, according to (7.2) and (7.3):

$$(Bx, x) \,/\, (x, x) \geqq -p_-(x) \geqq \alpha_<(L) \qquad (x \in \mathbb{H},\ x \neq 0).$$

7.2. In the previous conclusion, except in the condition (7.1), we used only the fact that C is a positive operator from $\mathbb{R}$ [1].

If we suppose that the positive operator $C \in \mathfrak{S}_\infty$, then it is possible to state on the basis of Theorem 5.1 that the equation $L(Z) = 0$ has a pair of solutions Z_1 and $Z_2 = -B - Z_1^*$ for which $Z_1^* Z_1 \leqq C$.

For any $x \in \mathbb{H}$, $x \neq 0$ we put

$$q(x) = (BZ_1x, Z_1x)/2\|C^{\frac{1}{2}}Z_1x\|\cdot\|Z_1x\| \qquad (> 1).$$

As for any $x \in \mathbb{H}$

$$((Z_1^2 + BZ_1 + C)x, Z_1x) = 0, \quad (BZ_1x, Z_1x) = -(Z_1^2x, Z_1x) - (Cx, Z_1x),$$

and

$$|(Cx, Z_1x)| \leqq \|C^{\frac{1}{2}}x\|\cdot\|C^{\frac{1}{2}}Z_1x\|,$$

$$|(Z_1^2x, Z_1x)| = |(Z_1^*Z_1Z_1x, x)| \leqq \|(Z_1^*Z_1)^{\frac{1}{2}}Z_1x\|\cdot\|(Z_1^*Z_1)^{\frac{1}{2}}x\| \leqq$$
$$\leqq \|C^{\frac{1}{2}}Z_1x\|\cdot\|C^{\frac{1}{2}}x\|,$$

[1] The authors, however, have succeeded in generalizing in that case a series of successive conclusions as well, which will be shown elsewhere.

it follows that [1)]

$$2q(x)\| C^{\frac12}Z_1x \|\cdot\|Z_1x\| = (BZ_1x, Z_1x) \leqq 2\| C^{\frac12}Z_1x\|\cdot\| C^{\frac12}x\| ,$$

(7.7) $$\| Z_1x \| \leqq \frac{1}{q(x)}\| C^{\frac12}x \| \qquad (x \in \mathbb{H}).$$

Hence

(7.8) $$((C - Z_1^*Z_1)x, x) \geqq (1 - \frac{1}{q^2(x)})\ (Cx, x) > 0 \ (x \in \mathbb{H}, x \neq 0).$$

From the relation $Z_2^*Z_1 = C$ it follows that

$$\|C^{\frac12}x\|^2 = (Z_1x, Z_2x) \leqq \| Z_1x\|\cdot\| Z_2x \| .$$

Comparing this with (7.7) we get

$$\| Z_2x \| \geqq q(x)\ \|C^{\frac12}x\| \qquad (x \in \mathbb{H}).$$

Therefore

(7.8) $$((Z_2^*Z_2 - C)x, x) \geqq (q^2(x) - 1)\ (Cx, x) \quad (X \in \mathbb{H}, x \neq 0).$$

On the other hand it follows from $Z_2 = - B - Z_1^*$ that the positive operator $H_2 = Z_2^*Z_2 - C$ can be represented in the form $H_2 = B^2 + T$, where $T \in \mathfrak{S}_\infty$.

Therefore, if μ is the greatest lower bound of the spectrum of the operator B^2 ($\mu \geqq \alpha_<^2(L)$), then for any $\varepsilon > 0$ the spectrum left to the point μ-ε of the positive operator H_2 consists of a finite number of isolated eigen-values of finite multiplicity, which are positive because of (7.9).

So the operator $H_2 = Z_2^*Z_2 - C$ is <u>uniformly positive:</u> $m(H_2) = \inf[(H_2x, x) / (x, x)] > 0$.

At the same time we conclude that the operator Z_2 is continuously invertible, as

$$\|Z_2x\|^2 = (H_2x,x) + (Cx, x) \geqq (H_2x, x) \geqq m(H_2)\ \|x\|^2 (x \in \mathbb{H}).$$

Now the proof of the following fundamental theorem does not take us much trouble.

1) Continuing this argument we can prove that $\| Z_1x \| \leqq (q(x) - \sqrt{q^2(x) - 1})\ \|C^{\frac12}x\|$.

THEOREM 7.1. *For a strongly damped pencil the following statements are true:*

1°) *the quadratic equation* $L(Z) = 0$ *has one and only one root* Z_1 *with the property*[1)] $Z_1^* Z_1 \leqq C$;

2°) *the root* Z_1 *and the accompanying root* $Z_2 = -B - Z_1^*$ *are symmetrized by one and the same uniformly positive operator*

$$S = B + Z_1 + Z_1^* = -(B + Z_2 + Z_2^*) = Z_1 - Z_2 ;$$

3°) *the root* Z_1 *is similar to a negative compact operator; its spectrum lies on the segment* $[-\alpha_<(L), 0]$; *the eigen-vectors (-values) of this root are exhausting all the pencil's eigen-vectors (-values) of the first kind;*

4°) *the root* Z_2 *is similar to a negative bounded operator; its spectrum lies on the segment* $[-\|B\| - \sqrt{\|C\|}, -\alpha_>(L)]$; *the eigen-vectors (-values) of this root are exhausting all the pencil's eigen-vectors (-values) of the second kind;*

5°) *the spectrum* $\sigma(L) = \sigma(Z_1) \cup \sigma(Z_2)$;

6°) *the roots* Z_1 *and* Z_2 *form a complete pair of operators.*

PROOF. Temporarily ignoring statement 1° we continue the investigation of the root Z_1 (having the property $Z_1^* Z_1 \leqq C$) and the accompanying root $Z_2 = -B - Z_1^*$, the existence of which is guaranteed by Theorem 5.1.

From $Z^2 + BZ + C = 0$ it follows that $(Z + Z^* + B)Z = Z^* Z - C$. Putting $Z = Z_k$ ($k = 1, 2$) we get

$$SZ_k = -H_k \qquad (k = 1, 2;\ H_k = (-1)^k (Z_k^* Z_k - C)). \tag{7.10}$$

So each root Z_k is symmetrized by the operator S:

$$SZ_k = Z_k^* S \qquad (k = 1, 2),$$

and also by its operator H_k:

$$H_k Z_k = Z_k^* H_k = -Z_k^* S Z_k = -Z_k S Z_k^* \quad (k = 1, 2). \tag{7.11}$$

1) It is possible to prove that the accompanying root $Z_2 = -B - Z_1^*$ is completely defined by its property $Z_2^* Z_2 \geqq C$.

The latter means that the operator Z_k is symmetric with respect to the scalar product $(x, Y)_k = (H_k x, y)$ $(k = 1, 2)$.

As the operator H_2 is uniformly positive, the scalar product $(x, y)_2$ is topologically equivalent to the scalar product (x, y) given in $\mathbb{H}$. With respect to the scalar product $(x, y)_2$ the operator Z_2 is a negative self-adjoint operator with the following spectral decomposition

$$(7.12) \qquad Z_2 = \int_a^b \lambda dE_2(\lambda),$$

where, respectively, a and b are the smallest and the largest number respectively of the spectrum $\sigma(Z_2)$, for which

$$(7.13) \qquad (-\tfrac{3}{2}\|B\| \leq) \qquad -\|B\|-\sqrt{\|C\|} \leq a < b \leq -\ \alpha_<(L).$$

Let us explain where the inequalities (7.13) come from. As $Z_2 = -B - Z_1^*$ and $Z_1 \in \$_\infty$, the condensation spectrum of the operator Z_2 coincides with the condensation spectrum of the operator $-B$, and according to (7.6) the entire spectrum $\sigma(-B)$ is contained in the interval $(-\infty, -\alpha_<(L))$. On the other hand every $\lambda_0 \in \sigma(Z_2)$ not belonging to the condensation spectrum of Z_2 is an eigen-value of the operator Z_2 of finite dimension, and if Ψ_0 is a corresponding eigen-vector $(Z_2\Psi_0 = \lambda_0\Psi_0)$, then $|\lambda_0|^2\|\Psi_0\|^2 = \|Z_2\Psi_0\|^2 > (C\Psi_0,\Psi_0)$.

So λ_0 is an eigen-value of the second kind of the pencil L, and therefore $\lambda_0 \leq -\alpha_>(L)$.

From $Z_2 = -B - Z_1^*$ it follows that $\|Z_2\| \leq \|B\| + \|Z_1\|$ and from $Z_1^*Z_1 \leq C$ and (7.1) we get $\|Z_1\| \leq \|C^{\frac{1}{2}}\| = \sqrt{\|C\|} \leq \frac{1}{2}\|B\|$. It still remains to remark that the number $-a$ coincides with the norm $\|Z_2\|_2 = -\inf[(Z_2x, x)_2 / (x, x)_2]$, and as the operator Z_2 is symmetric with respect to the scalar product $(.,.)_2$ we have according to a general theorem [16]: $-\alpha \leq \|Z_2\|$.

From (7.11) it follows that $G = H_2^{\frac{1}{2}}Z_2H_2^{-\frac{1}{2}} = H_2^{-\frac{1}{2}}Z_2^*H_2^{\frac{1}{2}} = G^* < 0$, and therefore it is possible to state that the operator Z_2 is similar to the self-adjoint negative operator G.

Let us show that the operator S is uniformly positive. As $S = -H_2Z_2^{-1}$ it follows that

$$(Sx, x) = -(H_2Z_2^{-1}x, x) = -(Z_2^{-1}x, x)_2 \geqq \frac{1}{|a|}(x, x)_2 \geqq \frac{m(H_2)}{|a|}(x, x).$$

From (7.8) and (7.10) it follows that $(SZ_1x, x) = (-H_1x, x) < 0$ $(x \in \mathbb{H}, x \neq 0)$.

So the root Z_1 is similar to a negative compact self-adjoint operator.

Consequently the root Z_1 possesses a system of eigen-vectors $\{\Psi_j^{(1)}\}_1^\infty$ which form a Riesz basis in $\mathbb{H}$:

$$Z_1 \Psi_j^{(1)} = \lambda_j^{(1)}\Psi_j^{(1)} \quad (j = 1, 2, \ldots).$$

If $Z_1\Psi = \lambda\Psi$ $(\Psi \neq 0)$ then $|\lambda|^2 \|\Psi\|^2 = \|Z_1\Psi\|^2 < (C\Psi, \Psi)$. So all vectors $\Psi_j^{(1)}$ (numbers $\lambda_j^{(1)}$) are eigen-vectors (eigen-values) of the first kind of the pencil L.

As $L(\lambda) = (\lambda I - Z_2^*)(\lambda I - Z_1)$, it follows that $L^{-1}(\lambda) = (\lambda I - Z_1)^{-1}(\lambda I - Z_2^*)^{-1}$,

Taking into account that the operator Z_2^* is similar to the operator Z_2 and therefore $\sigma(Z_2^*) = \sigma(Z_2)$ and also that the intersection of the spectra $\sigma(Z_1) \cap \sigma(Z_2)$ may consist of one point only (the minimal eigen-value of Z_1), which always belongs to the spectrum $\sigma(L)$, we conclude that $\sigma(L) = \sigma(Z_1) \cup \sigma(Z_2)$.

From this it follows already that the eigen-vectors (eigen-values) of the operator Z_1 are exhausting all eigen-vectors (eigen-values) of the first kind of the pencil L.

As by what has been proved the eigen-vectors of the first kind form a complete system in $\mathbb{H}$, it follows that by the equations $Z\Psi = \lambda\Psi$ (Ψ is an eigen-vector of the first kind of the pencil L, λ is the corresponding eigen-value) the root Z_1 with the property $Z_1^*Z_1 \leqq C$ is completely defined.

So all statements $1^\circ - 5^\circ$ have been proved. It remains to remark that statement 6° is contained in statement 2°, according to which the operator $S = Z_1 - Z_2$ is uniformly positive.

This completes the proof of the theorem.

<u>REMARK 7.1.</u> We remark that it follows from statement 6° of theorem 7.1 that, if the condition (7.1) if fulfilled, then any twice continuously differentiable (in the strong sense)

solution $v(t)$ of the equation

$$\ddot{v} + B\dot{v} + Cv = 0$$

can be written in the following form:

$$v = \exp(tZ_1)x_1 + \exp(tZ_2)x_2.$$

The elements x_1 and x_2 are uniquely defined by the initial conditions $v_0 = x_1 + x_2$ and $v_0' = Z_1x_1 + Z_2x_2$, where $v_0, v_0' \in \mathbb{H}$ may be given arbitarily.

With the associated equation

$$C\ddot{v} + B\dot{v} + v = 0 \tag{7.14}$$

the situation is different.

In an generalized sense for any $x_1, x_2 \in \mathbb{H}$ the vector function

$$v = \exp(tZ_1^{-1})x_1 + \exp(tZ_2^{-1})x_2 \tag{7.15}$$

will be a solution of the equation (7.14). This solution will have one(or two) continuous derivatives if $x_1 \in \mathbb{R}(Z_1)$ (or $x_1 \in \mathbb{R}(Z_1{}^2)$). If $x_1 \in \mathbb{R}(Z_1{}^2)$ the vector function (7.15) will be a solution of the equation in the usual sense. If the condition $v_0 - Z_2v_0' \in \mathbb{R}(Z_1)$ is fulfilled, then the equation (7.14) will have a continuously differentiable solution of the form (7.14) satisfying the initial conditions $v(0) = v_0$, $\dot{v}(0) = v_0'$.

<u>REMARK 7.2.</u> From the uniform positiveness of the operator $H_2 = Z_1^*Z_1 - C$ and the equation $Z_2^*Z_1 = C$ it is not difficult to deduce that there exists a positive $\rho < 1$ such that $Z_1^*Z_1 \leqq \rho^2 C$. But then $\|Z_1\| \leqq \rho\sqrt{\|C\|}$, and consequently $a \geqq -\|B\| - \rho\sqrt{\|C\|}$ so that in (7.13) the lower bound is never reached.

<u>REMARK 7.3.</u> For the sake of brevity we omit propositions concerning the behavior of eigen-values of one kind or the other for a strongly damped pencil L under monotone alteration of the operator B (see the corresponding algebraic propositions in [8]). On the basis of these propositions, in particular for

sequences of eigen-values of the first kind of the pencil L ($\lambda_1^{(1)}(L) \leqq \lambda_2^{(2)}(L) \leqq \ldots$ (<0)) it is possible to obtain the following upper bound:

$$-\lambda_n^{(1)}(L) \geqq \frac{\|B\| - \sqrt{\|B\|^2 - 4\lambda_n(C)}}{2} = \frac{2\lambda_n(C)}{\|B\| + \sqrt{\|B\|^2 - 4\lambda_n(C)}} =$$

$$= \frac{\lambda_n(C)}{\|B\|}[1 + o(1)] > \frac{\lambda_n(C)}{\|B\|} \quad (n = 1, 2, \ldots)$$

and the following lower bound:

$$-\lambda_n^{(1)}(L) \leqq \frac{2\lambda_n(C)}{m(B) + \sqrt{m^2(B) - 4\lambda_n(C)}} = \frac{\lambda_n(C)}{m(B)}[1 + o(1)]$$

which are valid beginning with those n for which

$$4\lambda_n(C) < m^2(B) = \inf[(Bx, x)/(x, x)]^2.$$

If the operator B has the form

(7.16) $\qquad B = \beta I + T \qquad (T \in \$_\infty)$,

then the following asymptotic equality holds

(7.17) $\qquad \lambda_n^{(1)}(L) = -\dfrac{\lambda_n(C)}{\beta}[1 + o(1)] \qquad (n \to \infty)$.

REMARK 7.4. The authors will prove elsewhere that with some complications, the results of this section can be generalized to the case where instead of the condition of being strongly damped (7.1) the weaker condition that the operator B is uniformly positive is fulfilled or the even weaker condition contained in the requirement that the condensation spectrum of the operator B is positive.

Among other things if the latter condition is fulfilled (and $C > 0$, $C \in \$_\infty$), then the pencil L always has no more than a finite number of non-real eigen-values.

In particular this condition is fulfilled if B has the form (7.16) with any $\beta > 0$ and $T \in \mathfrak{S}_\infty$. In this case, even when the condition (7.1) of being strongly damped is violated, the asymptotic formula (7.17) remains valid as soon as the following additional condition is fulfilled:

$$\text{(7.18)} \qquad \lim_{n\to\infty} \frac{\lambda_{n+1}(C)}{\lambda_n(C)} = 1 .$$

REMARK 7.5. Recently it was proved by S.G. Krein [18] that the problem of small vibrations of a viscous fluid contained in an immovable vessel and having a free surface can be reduced to the equation

$$\text{(7.19)} \qquad y = \mu G y + \frac{1}{\mu} H y ,$$

where $G, H \in \mathfrak{S}_\infty$, $G > 0$, $H \geq 0$, μ is a complex parameter (stemming from the expression $v(\xi,\eta,\zeta, t) = e^{-\mu t}\varphi(\xi,\eta,\zeta)$ for the velocity vector of a fluid particle with Euler coordinates ξ,η,ζ).

The substitution $\mu = -\lambda^{-1} - a$ $(a > 0)$ transforms the equation (7.19) into the following equation:

$$\lambda^2(a^2G + aI + H)y + \lambda(2aG + I)y + Gy = 0.$$

Whatever the operator $H = H^*$ $(\in \mathfrak{R})$ may be (none of the conditions $H \in \mathfrak{S}_\infty$, $H \geq 0$ is necessary), for sufficiently large a the operator $F = a^2G + H + aI$ will be uniformly positive. By taking such a and by substituting in the equation (7.19) $x = F^{1/2}y$, we transform this equation into the equation $L_a(\lambda)x = 0$, where

$$L_a(\lambda) = \lambda^2 I + \lambda B_a + C,$$

$$B_a = F^{-1/2}(2aG + I)F^{-1/2}, \qquad C = F^{-1/2}GF^{-1/2}.$$

It is easy to understand that the obtained pencil L_a is strongly damped if and only if [1)]

(7.20) $$4(Gx, x)(Hx, x) < (x, x)^2 \quad (x \in \mathbb{H}, x \neq 0).$$

and that in this case all of the previous theory can be applied to (7.19).

If in addition, $H \in \mathfrak{S}_\infty$, then the condition (7.16) is fulfilled for B_a with $\beta = 1/a$, and if $\lambda_{n+1}(G)/\lambda_n(G) \to 1$ for $n \to \infty$, then the condition (7.18) will be fulfilled for B_a and $\lambda_n(B_a)/\lambda_n(G) \to 1/a$.

All this allows one to obtain a series of essential extensions to the articles [1, 22]. For instance, in case the condition (7.20) and the conditions $G, H \in \mathfrak{S}_\infty$, $G > 0$ are fulfilled, it is possible to state that the equation (7.20) has a Riesz basis consisting of the eigen-vectors of the first kind[2)] of the equation (7.19), and that for the corresponding complete sequence $\mu_1^{(1)} \leqq \mu_2^{(1)} \leqq \dots$ of eigen-values of the first kind of the equation (7.19) the following asymptotic formula holds true

$$\mu_n^{(1)} = \frac{1}{\lambda_n(G)}[1 + o(1)] \qquad (n \to \infty).$$

1) We remark that for $H \leq 0$ the condition (7.20) is automatically fulfilled; and if $H = H_+ - H_-$ $(H_\pm \geqq 0)$, then the condition (7.20) will e.g. be fulfilled for $4\ \|G\| \cdot \|H_+\| < 1$.

2) It is not difficult to understand how to translate the notion of eigen-vector (-value) of the first or the second kind for the equation (7.19).

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