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REMARKS ON SCHRÖDINGER OPERATORS WITH VECTOR POTENTIALS*

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Three selfadjoint realizations $H_0 \geq H \geq H'$ of the formal Schrödinger operator in R^m with a singular vector potential $b(x)$ and a singular scalar potential $q(x)$ are constructed through quadratic forms, corresponding to the minimal, intermediate, and maximal forms. It is shown that if $b \in L^2_{loc}$ and $0 \leq q \in L^1_{loc}$, then $H_0 = H$ and the pointwise domination $|e^{-tH}u| \leq e^{t\Delta}|u|$ holds for $t \geq 0$ and $u \in L^2$. If in addition $b \in L^p_{loc}$ with some $p > m$, then $H' = H$ holds.

1. Introduction.

Consider the formal Schrödinger operator in R^m :

$$\text{(T)} \quad T = -(\nabla - ib(x))^2 + q(x) = -\sum_{k=1}^{m} (d_k - ib_k(x))^2 + q(x) ,$$

where $d_k = d/dx_k$, $b = (b_1, \ldots, b_m)$ is a real vector-valued function, and q is a real scalar-valued function on R^m.

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It was shown by Simon [5] that under certain conditions, a selfadjoint realization H of T in $\underline{H} = L^2(R^m)$ can be constructed with the property, among others, that

$$\text{(P)} \qquad |e^{-tH}u| \leq e^{t\Delta}|u| \quad \text{a.e. pointwise on } R^m ,$$

where $t \geq 0$, $u \in \underline{H}$, and Δ denotes the canonical realization in $\underline{H}$ of the Laplacian (with domain $\underline{D}(\Delta) = H^2(R^m)$, the Sobolev space). Simon assumes, for example, that $b \in L^4_{loc}$, $\operatorname{div} b = 0$ and $q = 0$, for $m = 3$, but conjectures that $b \in L^2_{loc}$ would suffice. (See [5] for various consequences of (P). See also Schechter [3] and Simon [4] for the essential selfadjointness of T .)

In what follows we shall prove (P) under the assumptions

$$\text{(A1)} \qquad b \in L^2_{loc}(R^m)^m ,$$

$$\text{(A2)} \qquad 0 \leq q \in L^1_{loc}(R^m) ,$$

thus verifying Simon's conjecture. Note that we assume nothing about $\operatorname{div} b$.

The definition of H is by no means trivial in this case, however, and several different choices are possible. We define H through <u>quadratic</u> <u>forms</u> (as Simon does), but this statement is still vague. To be more precise, it is convenient to define a linear operator D' from $\underline{H}$ to $\underline{H}^m$ by

$$(1.1) \qquad D'u = \nabla u - ibu .$$

The domain $\underline{D}(D')$ of D' consists of all $u \in \underline{H}$ for which the right member of (1.1), taken in the distribution sense, belongs to $\underline{H}^m$. It is essential here that $bu \in L^1_{loc}$ by (A1) so that (1.1) makes sense as a distribution.

It is easy to see that D' thus defined is a densely defined, closed operator from $\underline{H}$ to $\underline{H}^m$. Thus

$$B' = D'^{*}D' \tag{1.2}$$

is selfadjoint in $\underline{H}$. We then define

$$H' = B' \dotplus q \qquad \text{(form sum)}, \tag{1.3}$$

which means that H' is the selfadjoint operator associated with the densely defined, closed quadratic form

$$h'[u] = \|D'u\|^2 + \|q^{1/2}u\|^2 \tag{1.4}$$

with domain $\underline{D}(h') = \underline{D}(D') \cap \underline{D}(q^{1/2})$, where q is regarded as a nonnegative selfadjoint operator in $\underline{H}$.

The quadratic form h' may be called the maximal form associated with T . We define the minimal form h_0 as the closure of the restriction of h' to domain $C_0^\infty(R^m)$, and the associated operator H_0 . There is another form h of practical interest, such that

$$h_0 \subset h \subset h' ; \tag{1.5}$$

h is obtained from h' by restricting its domain to $\underline{D}(D) \cap \underline{D}(q^{1/2})$, where D is the closure of the restriction of D' to C_0^∞ . In other words, h is obtained by first letting $u \in C_0^\infty$ in (1.4), closing the two forms on the right separately and then taking the sum, whereas h_0 is obtained by closing the sum for $u \in C_0^\infty$. The operator H associated with h may thus be written

$$H = B \dotplus q , \qquad B = D^{*}D , \tag{1.6}$$

with D as defined above. Note that D is the minimal, and D' is the maximal, operator for $\nabla - ib$.

We note that (1.5) implies

$$H_0 \geq H \geq H' \tag{1.7}$$

in the sense of order relation for semibounded operators (see [1, Chapter 6]).

If b and q are well-behaved, the three operators H_0, H, and H' are identical. In general, there is no obvious reason why this should be the case. (cf. [1, Chapter 6, Section 1.6]).

The main results to be proved in this paper are

<u>THEOREM I.</u> $H_0 = H$.

<u>THEOREM II.</u> (P) <u>is true for</u> H .

We have no results regarding H' under the assumptions (A1), (A2), but we have

<u>THEOREM III.</u> <u>If</u> (A1) <u>is replaced by the stronger condition that</u> $b \in L^p_{loc}(R^m)^m$ <u>for some</u> $p > m$, <u>then</u> $H' = H = H_0$.

<u>REMARK 1.1.</u> In [5] Simon sets $q = 0$ and chooses H (rather than H') as the definition of the Hamiltonian. To prove (P), however, he assumes that $b \in L^4_{loc}(R^3)^3$; then $H' = H$ is true by Theorem III.

<u>REMARK 1.2.</u> Theorems I to III can be extended to the case when q is nonreal but satisfies $|\text{Im } q| \leq M \text{ Re } q \in L^1_{loc}(R^m)$ instead of (A2), where $M < \infty$ is a constant. In this case the operator q is m-sectorial (see [1, Chapters 5,6]), and the forms h_0, h, h' can be defined as above with obvious modifications. The proofs given below are valid in this case too, due to a generalization of the product formula by Simon (see [2]).

The proofs of the theorems are given in the following sections. After a preliminary study of the operator D (section 2), we first prove Theorem II when $q = 0$ (so that $H = B$) (section 3), and then in the general case using the Trotter product formula (section 4). Theorem I is proved using Theorem II (section 5). Theorem III is proved in the last section.

2. The operator D.

It is important to know when a given $u \in \underline{H}$ is in $\underline{D}(D)$.

LEMMA 2.1. *Let* $u \in \underline{H} \cap H^1_{loc}(R^m) \cap L^\infty_{loc}(R^m)$ *so that* $v = \nabla u - ibu \in L^2_{loc}$. *If* $v \in \underline{H}^m$, *then* $u \in \underline{D}(D)$ *and* $Du = v$.

Proof. The assumption implies that $u \in \underline{D}(D')$. We first truncate u by setting $u_n = \phi_n u$, $n = 1,2,\ldots,$ where $\phi_n(x) = \phi(x/n)$ and $\phi \in C_0^\infty(R^m)$ with $\phi = 1$ identically near the origin. Then $u_n \to u$ in $\underline{H}$ and $D'u_n = \phi_n v + (\nabla\phi_n)u \to v$ in $\underline{H}^m$. Hence it suffices to show that $u_n \in \underline{D}(D)$.

$w = u_n$ has the properties that

$w \in H^1(R^m) \cap L^\infty(R^m)$, with compact support.

Thus it suffices to show that any such w is in $\underline{D}(D)$. Let $w_\varepsilon = J_\varepsilon w$, where J_ε is the Friedrichs mollifier, so that $w_\varepsilon \in C_0^\infty(R^m)$ with a common compact support, $|w_\varepsilon| \leq M < \infty$ pointwise, and $w_\varepsilon \to w$ in $H^1(R^m)$ as $\varepsilon \to 0$. We have also $w_\varepsilon \to w$ a.e. pointwise along some sequence $\varepsilon = \varepsilon_n \to 0$. Thus $\nabla w_\varepsilon \to \nabla w$ and $bw_\varepsilon \to bw$ in $\underline{H}^m$ along the sequence. (The latter follows by dominated convergence theorem.) Hence $D'w_\varepsilon \to D'w$ in $\underline{H}^m$. Since $w_\varepsilon \in C_0^\infty \subset \underline{D}(D)$, we conclude that $w \in \underline{D}(D)$.

3. The operator B.

THEOREM 3.1. (P) *is true for* H *replaced by* B .

Proof. This is known if b is smooth, say $b \in C^1(R^m)$; see [5] for the proof. We shall prove Theorem 3.1 by a limiting process, approximating b with a sequence b_n of smooth functions such that

(3.1) $$b_n \to b \quad \text{in} \quad L^2_{loc}(R^m)^m , \quad n \to \infty .$$

Let D_n and $B_n = D_n^* D_n$ be defined as above with b replaced by b_n. Since (P) is true for H replaced by B_n as remarked above, Theorem 3.1 will follow if we can show that $e^{-tB_n} \to e^{-tB}$ strongly as $n \to \infty$. As is well known, for this it suffices to prove

LEMMA 3.2. *$(1+B_n)^{-1} \to (1+B)^{-1}$ strongly, $n \to \infty$.*

Since $\|(1+B_n)^{-1}\| \leq 1$, it suffices to show that

(3.2) $$u_n = (1+B_n)^{-1}f \to (1+B)^{-1}f$$

for all f in a dense set in $\underline{H}$. Thus we may assume that $f \in \underline{H} \cap L^\infty$. The proof will be given in several steps.

LEMMA 3.3. $\|u_n\| \leq \|f\|$, $\|D_n u_n\| \leq \|f\|$, *and* $|u_n| \leq g \leq M < \infty$ *pointwise, where* $g \in \underline{H} \cap L^\infty$ *is independent of* n.

Proof. (3.2) implies

(3.3) $$f = (1+B_n)u_n .$$

Hence
$$(f,u_n) = \|u_n\|^2 + \|D_n u_n\|^2$$

because $B_n = D_n^* D_n$. The first two inequalities in the lemma follow immediately. To prove the last one, we note that the results of [5] apply to B_n. Hence (see [5, equation (6)])

$$|u_n| = |(1+B_n)^{-1}f| \leq (1-\Delta)^{-1}|f| \equiv g \in \underline{H} \cap L^\infty \quad \text{pointwise;}$$

note that $g \in L^\infty$ because $f \in L^\infty$.

LEMMA 3.4. $\{u_n\}$ is bounded in $H^1_{loc}(R^m)$.

Proof. We have $\nabla u_n = D_n u_n + ib_n u_n$ by (1.1). In view of Lemma 3.3 and the fact that the L^2-norm on any bounded set in R^m of the b^n are bounded by (3.1), the lemma follows immediately.

LEMMA 3.5. $\{u_n\}$ contains a subsequence (hereafter denoted again by $\{u_n\}$) such that there is $u \in \underline{H} \cap H^1_{loc}(R^m) \cap L^\infty(R^m)$ with the properties: $u_n \to u$ in $\underline{H}$ as well as a.e. pointwise, and $u_n \rightharpoonup u$ in $H^1_{loc}(R^m)$. ($\rightharpoonup$ denotes weak convergence.)

Proof. In view of Lemma 3.4, one can use the diagonal process to extract a subsequence $\{u_n\}$ weakly convergent in H^1_{loc} to a $u \in H^1_{loc}$. This implies $u_n \to u$ in L^2_{loc} strongly, by Rellich's theorem. Since $|u_n| \leq g \in \underline{H} \cap L^\infty$ by Lemma 3.3, we have $u \in \underline{H} \cap L^\infty$ and $u_n \to u$ in $\underline{H}$.

LEMMA 3.6. $b_n u_n \to bu$ in $L^2_{loc}(R^m)^m$.

Proof. We have $b_n u_n - bu = (b_n - b)u_n + b(u_n - u)$. Since $|u_n| \leq M$, $(b_n - b)u_n \to 0$ in L^2_{loc} by (3.1). Since $|b(u_n - u)| \leq 2M|b|$, $b(u_n - u) \to 0$ in L^2_{loc} by dominated convergence.

LEMMA 3.7. $u = (1+B)^{-1}f$.

Proof. Let $\phi \in C_0^\infty(R^m)$. (3.3) implies

(3.4) $$(f,\phi) = (u_n,\phi) + (D_n u_n, D_n\phi) .$$

But

(3.5) $$D_n\phi = \nabla\phi - ib_n\phi \to \nabla\phi - ib\phi = D\phi \quad \text{in} \quad \underline{H}^m$$

by (3.1). Similarly

$$D_n u_n = \nabla u_n - ib_n u_n \rightharpoonup \nabla u - ibu \quad \text{in} \quad L^2_{loc}$$

by Lemmas 3.5 and 3.6. Hence

$$\|\nabla u - ibu\|_\Omega \leq \liminf_{n\to\infty} \|D_n u_n\| \leq \|f\| \quad ,$$

where $\| \ \|_\Omega$ denotes the L^2-norm on a bounded set $\Omega \subset R^m$. Since this is true for any such Ω, we have $\nabla u - ibu \in \underline{H}^m$. Since $u \in H^1_{loc} \cap L^\infty$, it follows from Lemma 2.1 that $u \in \underline{D}(D)$ with $Du = \nabla u - ibu$. Thus

$$(3.6) \qquad D_n u_n \rightharpoonup Du \in \underline{H}^m \quad \text{in} \quad L^2_{loc} \quad .$$

Since ϕ and the $D_n\phi$ have a common compact support, it follows from (3.4-6) that $(f,\phi) = (u,\phi) + (Du,D\phi)$. Since this is true for all $\phi \in C_0^\infty$, which is a core for D, we conclude that $Bu = D^*Du$ exists and equals $f-u$. Hence $(1+B)u = f$ and $u = (1+B)^{-1}f$. This completes the proof of Lemma 3.2 and Theorem 3.1.

4. The operator H.

We can now complete the proof of Theorem II. Since $H = B \dotplus q$, a general convergence theorem for the Trotter product formula given in [2] can be applied to give

$$(4.1) \qquad e^{-tH}u = \lim_{n\to\infty} [e^{-(t/n)B}e^{-(t/n)q}]^n u \quad , \qquad u \in \underline{H} \quad .$$

Lemma 4.1. For $s \geq 0$ and $n = 1,2,\ldots,$ we have

$$(4.2) \qquad |(e^{-sB}e^{-sq})^n u| \leq e^{ns\Delta}|u| \qquad \text{pointwise} \ .$$

Proof. By induction. (4.2) is obvious for $n = 0$. Suppose it is true for an n. Then, writing $v = (e^{-sB}e^{-sq})^n u$,

$$|(e^{-sB}e^{-sq})^{n+1}u| = |e^{-sB}e^{-sq}v| \leq e^{s\Delta}|e^{-sq}v|$$
$$\leq e^{s\Delta}|v| \leq e^{s\Delta}(e^{ns\Delta}|u|) = e^{(n+1)s\Delta}|u| \quad ,$$

where we have used Theorem 3.1, the positivity preserving property of $e^{s\Delta}$ twice, and (4.2) (induction hypothesis).

Lemma 4.1 shows that the vector following lim sign in (4.1) is dominated by $e^{t\Delta}|u|$ pointwise. Hence (P) follows as required.

5. The operator H_0 .

We now prove Theorem I. It suffices to show that $h_0 = h$, and this is true if (and only if) h has a core contained in $\underline{D}(h_0)$. We shall show that $\underline{E} = (1+H)^{-1}C_0^\infty(R^m)$ is such a core. Obviously $\underline{E}$ is a core for H, hence for h too. It remains to show that $\underline{E} \in \underline{D}(h_0)$.

Let $u \in \underline{E}$, so that $u = (1+H)^{-1}f$ for some $f \in C_0^\infty$. Then

$$(5.1) \qquad |u| \leq (1-\Delta)^{-1}|f| \in \underline{H} \cap L^\infty(R^m)$$

by Theorem II, which implies such a pointwise domination by [5]. Then we can repeat the arguments used in the proof of Lemma 2.1 to construct, by truncation and mollification, a sequence $\{w_n\}$ such that

$$(5.2) \qquad w_n \in C_0^\infty , \quad w_n \to w \quad \text{and} \quad Dw_n \to Dw \quad \text{in} \quad \underline{H} .$$

On the other hand, we have $u \in \underline{D}(H) \subset \underline{D}(h) \subset \underline{D}(q^{1/2})$ so that $q^{1/2}u \in \underline{H}$. Recalling that the w_n have been constructed by truncation and mollification from u, which is a <u>bounded function</u> by (5.1), we see easily by dominated convergence theorem that

$$(5.3) \qquad q^{1/2}(w_n-u) \to 0 \quad \text{in} \quad \underline{H} ,$$

by going over to a subsequence if necessary. (Here the fact that $q^{1/2}u \in \underline{H}$ is essential for truncation, and $u \in L^\infty$ is essential for mollification.)

It follows from (5.2-3) that

$$h[w_n-u] = \|D(w_n-u)\|^2 + \|q^{1/2}(w_n-u)\|^2 \to 0 \quad .$$

Since $w_n \in C_0^\infty$, this proves $u \in \underline{D}(h_0)$ as required.

6. The operator H'.

We now prove Theorem III. It suffices to show that $D' \subset D$ (so that $D' = D$). Let $u \in \underline{D}(D')$.

We truncate u to $u_n = \phi_n u$ as in the proof of Lemma 2.1. Then $u_n \to u$ and $D'u_n \to D'u$ in $\underline{H}$. Thus it suffices to show $u_n \in \underline{D}(D)$.

$w = u_n$ has the property that it has compact support and $w \in \underline{H}$, $D'w \in \underline{H}$. We shall show that all such w are in $\underline{D}(D)$. We have

$$\nabla w = D'w + ibw \in L^{p_1}, \quad p_1{}^{-1} = 2^{-1} + p^{-1} \quad ,$$

because $D'w \in L^2$, $b \in L^p_{loc}$, $w \in L^2$ and w has compact support. It follows from the Sobolev imbedding theorem that

$$w \in L^{p_2}, \; p_2{}^{-1} = p_1{}^{-1} - m^{-1} = 2^{-1} - \theta$$

where $\theta = m^{-1} - p^{-1} > 0$, provided $p_2 < \infty$. This argument can be repeated until we have

$$\nabla w \in L^2, \; w \in L^{p_0}, \; p_0{}^{-1} = 2^{-1} - m^{-1}$$

if $m \geq 3$, $w \in L^r$ for any $r < \infty$ if $m = 2$, and $w \in L^\infty$ if $m = 1$.

Now we mollify w to w_ε as in the proof of Lemma 2.1. Then $\nabla w_\varepsilon \to \nabla w$ in $\underline{H}$ and $w_\varepsilon \to w$ in L^{p_0}, hence $bw_\varepsilon \to bw$ in L^s where $s^{-1} = p_0{}^{-1} + p^{-1} = 2^{-1} - \theta$ so that $bw_\varepsilon \to bw$ in $\underline{H}$ too (with slight modifications for $m \leq 2$). Thus $D'w_\varepsilon \to D'w$ in $\underline{H}$. Since $w_\varepsilon \in C_0^\infty$ and $w_\varepsilon \to w$ in $\underline{H}$, we have proved that $w \in \underline{D}(D)$.

REFERENCES

[1] Kato, T.: Perturbation theory for linear operators, Second Edition, Berlin-Heidelberg-New York, Springer 1976.

[2] Kato, T.: Trotter's product formula for an arbitrary pair of selfadjoint contraction semigroups, Advances in Math., to appear.

[3] Schechter, M.: Essential self-adjointness of the Schrödinger operator with magnetic vector potential, J. Functional Anal. 20 (1975), 93-104.

[4] Simon, B.: Schrödinger operators with singular magnetic vector potentials, Math. Z. 131 (1973), 361-370.

[5] Simon, B.: Kato's inequality and the comparison of semigroups, to appear.

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