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OPERATORS THAT ARE POINTS OF SPECTRAL CONTINUITY

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In this paper a characterization is obtained of those bounded operators on a Hilbert space at which the spectrum is continuous, where the spectrum is considered as a function whose domain is the set of all operators with the norm topology and whose range is the set of compact subsets of the plane with the Hausdorff metric. Similar characterizations of the points of continuity of the Weyl spectrum, the spectral radius, and the essential spectral radius are also obtained.

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Let $B(H)$ denote the algebra of bounded linear operators on a separable complex Hilbert space H. If $A \in B(H)$, then $\sigma(A) = \{\lambda \in \mathbb{C} : \lambda - A \text{ is not invertible}\}$ denotes the spectrum of A, while $r(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ denotes the spectral radius of A. If S is the collection of compact subsets of $\mathbb{C}$ equipped with the Hausdorff metric, then $\sigma : B(H) \to S$ is a mapping from one metric space onto another.

A well-known example due to Kakutani ([13], Solution 87) gives a sequence of nilpotent operators, i.e., operators whose spectrum is the singleton $\{0\}$, which converges to an operator whose spectrum is the closed unit disk, $\{z : |z| \leq 1\}$. This example shows that σ and r both have points of discontinuity and it leads one to ask: "What are the points of continuity of σ (or r)?" In this paper, we consider and answer both of the questions above as well as several other related ones.

Recall that if $B_0(H)$ is the ideal of compact operators on H and $\pi : B(H) \to B(H)/B_0(H)$ is the natural projection map, then the essential spectrum of A is defined by $\sigma_e(A) = \sigma(\pi(A))$ and the essential spectral radius is defined by $r_e(A) = \max\{|\lambda| : \lambda \in \sigma_e(A)\}$. The points of continuity of $r_e : B(H) \to [0,\infty)$ are characterized in this paper as well as the points of continuity of the Weyl spectrum $\sigma_w : B(H) \to S$ defined by $\sigma_w(A) = \cap\{\sigma(A + K) : K \in B_0(H)\}$.

Let F denote the Fredholm operators and SF the collection of semi-Fredholm operators in $B(H)$ [11].

Let $P_{\pm}(A) = \{\lambda \in \mathbb{C} : \lambda - A \in SF \text{ and } \operatorname{ind}(\lambda - A) \neq 0\}$, where $\operatorname{ind}(T)$ is the index of an operator T in SF defined by $\operatorname{ind}(T) = \dim[\ker T] - \dim[\ker T^*]$. Let $\sigma_p^0(A)$ be the collection of isolated eigenvalues of A for which the corresponding spectral projection (via the Riesz functional calculus) has finite rank. For each A in $\mathcal{B}(H)$, $\sigma_{\ell e}(A)$ and $\sigma_{re}(A)$ denote the left and right spectrum of $\pi(A)$ in $\mathcal{B}(H)/\mathcal{B}_0(H)$. So, $\sigma_e(A) = \sigma_{\ell e}(A) \cup \sigma_{re}(A)$. Finally put $\sigma^0(A) = \sigma_p^0(A) \cup [\sigma_{\ell e}(A) \cap \sigma_{re}(A)]$. In Theorem 3.1 it is shown that $\sigma : \mathcal{B}(H) \to S$ is continuous at A if and only if every nonempty relatively open subset of $\sigma^0(A)\setminus[P_{\pm}(A)]^-$ contains a component of $\sigma(A)$.

The characterization of the points of continuity of σ_w (Theorem 3.6) is couched in similar terms. To characterize the points of continuity of r and r_e (Theorems 2.6 and 2.15), certain auxilary scalar-valued functions that are related to $P_{\pm}(A)$ and $\sigma^0(A)$ are introduced.

To be sure, this paper has its predecessors. Newburgh [18] seems to be the first to have systematically investigated the continuity of the spectrum. He showed that the spectrum of an element of a Banach algebra is upper semicontinuous and that the spectrum is continuous at any element with totally disconnected spectrum. In addition, he showed that the spectrum is continuous on an abelian Banach algebra. Moreover, he proved that if a sequence of operators $\{A_n\}$ in $\mathcal{B}(H)$ converges to A and each A_n satisfies a growth condition on its resolvent, then $\{\sigma(A_n)\}$ converges to $\sigma(A)$ in S . It is a corollary of this last result that if $A_n \to A$ and each A_n is normal, then $\sigma(A_n) \to \sigma(A)$ in S . Newburgh also proves several results concerning the continuity of the spectrum of closed operators. Newburgh's paper does not seem to be well known and the literature contains several papers that reprove some of his results.

Some extensions and refinements of Newburgh's results for closed operators can be found in [5] and [16].

This paper is organized in three sections. In section one the notation is introduced and some preliminary results are presented. In section two several scalar-valued functions on $\mathcal{B}(\mathcal{H})$ are introduced, their semicontinuity properties are discussed, and these results are used to determine points of continuity of the spectral radius and the essential spectral radius. In section three the points of continuity of the set valued maps σ and σ_w are characterized.

§1 Notation and Preliminaries

For a subset X of the complex plane, X^- denotes its closure, int X its interior, and $\partial X = X^- \cap [\mathbb{C} \setminus X]^-$ its boundary. If $\varepsilon > 0$, let $(X)_\varepsilon = \{z \in \mathbb{C} : \text{dist}(z,X) < \varepsilon\}$. For λ in $\mathbb{C}$ and $\varepsilon > 0$, $B(\lambda;\varepsilon)$ denotes the ball of radius ε centered at λ. Finally, $\square$ denotes the empty set.

All Hilbert spaces considered here are separable. In addition to the notation given in the introduction, let $P_n(A) = \{\lambda \in \sigma(A) : \lambda - A \in \mathcal{SF} \text{ and } \text{ind}(\lambda - A) = n\}$ for $n \in \mathbb{Z} \cup \{\pm\infty\}$. Hence, $P_\pm(A) = \cup\{P_n(A) : n \neq 0\}$.

Several facts concerning the left essential spectrum and the Weyl spectrum can be found in [11]. Among them is the following result due to Schechter [19].

1.1 THEOREM. $\sigma_w(A) = \sigma_e(A) \cup P_\pm(A)$.

The following result is undoubtedly known but the authors are unable to find a reference for it. In any case the proof follows easily from results of [11].

1.2 PROPOSITION. *If* $A \in \mathcal{B}(\mathcal{H})$, *then*

$$\sigma_e(A) = [\sigma_{\ell e}(A) \cap \sigma_{re}(A)] \cup P_{+\infty}(A) \cup P_{-\infty}(A) .$$

It follows from the preceding proposition that $\partial P_{\pm}(A) \subseteq \sigma_{\ell e}(A) \cap \sigma_{re}(A)$.

1.3 PROPOSITION. If C is a component of $\sigma_{\ell e}(A) \cap \sigma_{re}(A)$ and $C \cap [P_{\pm}(A)]^{-} = \Box$, then C is a component of $\sigma_{\ell e}(A)$, $\sigma_{re}(A)$, and $\sigma_e(A)$.

Proof. Let D be the component of $\sigma_e(A)$ such that $C \subseteq D$. By Proposition 1.2,

$$D = \{D \cap [\sigma_{\ell e}(A) \cap \sigma_{re}(A)]\} \cup \{D \cap P_{\pm\infty}(A)\} .$$

Let $K = D \cap [\sigma_{\ell e}(A) \cap \sigma_{re}(A)]$; so either $K = C$ or K is not connected. If K fails to be connected, then $K = K_1 \cup K_2$, where K_1 and K_2 are disjoint compact sets that $C \subseteq K_1 \subseteq (C)_\varepsilon$ with $0 < \varepsilon < \frac{1}{2}\text{dist}(C, P_{\pm}(A)^{-})$ ([12], 16.15). $\text{dist}(K_1, P_{\pm\infty}(A)) > 0$, and so

$$\text{dist}(K_1, K_2 \cup [D \cap P_{\pm\infty}(A)]) > 0 .$$

Since $D = K_1 \cup \{K_2 \cup [D \cap P_{\pm\infty}(A)]\}$ is connected and K_1 is nonempty, it must be that $K_2 = D \cap P_{\pm\infty}(A) = \Box$. Thus $D = C$, and C is a component of $\sigma_e(A)$. The other cases follow similarly.

Denote by S the collection of compact subsets of $\mathbb{C}$ and by S_1 the collection of all bounded subsets of $\mathbb{C}$. One can define the Hausdorff metric on S_1 ([9], p. 205), but the distance between two elements of S_1 is positive if and only if their closures are unequal. So this metric is a true metric only if it is restricted to S . Nevertheless it is a pseudometric on S_1 . Also the distance between the empty set and any nonempty bounded set is 1 . If (X, ρ) is a metric space and if $f : X \to S_1$ is a function, then f is said to be upper (lower) semicontinuous at x_0 if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that $\rho(x, x_0) < \delta$ implies $f(x) \subseteq (f(x_0))_\varepsilon$

(respectively, $f(x_0) \subseteq (f(x))_\varepsilon$) . One can show that f is continuous at x_0 if it is both upper and lower semicontinuous at x_0 .

The following result is well known.

1.4 LEMMA. (a) $\sigma : \mathcal{B}(H) \to S$ is upper semicontinuous.

(b) $\sigma_e : \mathcal{B}(H) \to S$ is upper semicontinuous.

(c) If $-\infty \leq n \leq \infty$, $P_n : \mathcal{B}(H) \to S_1$ is lower semicontinuous.

(d) $P_\pm : \mathcal{B}(H) \to S_1$ is lower semicontinuous.

Proof. Parts (a) and (b) follow from [18]. To prove (c), let $A_k \to A$, let $\varepsilon > 0$, and let K be a compact subset of $P_n(A)$ such that $P_n(A) \subseteq (K)_\varepsilon$. Since $\operatorname{ind}(\lambda - A_k) \to \operatorname{ind}(\lambda - A) = n$ for each λ in K , an elementary argument yields the existence of an integer n_0 such that $\operatorname{ind}(\lambda - A_k) = n$ for all λ in K and all $k \geq n_0$. Thus, $P_n(A) \subseteq (K)_\varepsilon \subseteq (P_n(A_k))_\varepsilon$ for $k \geq n_0$. Part (d) follows similarly. (Both parts (c) and (d) can be obtained from Theorem 1 of [14].) ■

If $A \in \mathcal{B}(H)$ and X is a subset of $\sigma(A)$ that is both open and closed in $\sigma(A)$, let $E(X;A)$ denote the corresponding spectral projection ([10], p. 572),

$$E(X;A) = \frac{1}{2\pi i}\int_\Gamma (z - A)^{-1}dz ,$$

for an appropriate choice of contour Γ .

The next lemma will be used very often in this paper. The essence of its proof can be found in the literature (e.g., in [18]), but it is stated and proved here for the convenience of the reader.

1.5 LEMMA. Suppose that $A_n \to A$ in $\mathcal{B}(H)$.

(a) <u>If</u> C <u>is a component of</u> $\sigma(A)$ <u>and</u> U <u>is an open set containing</u> C, <u>then there is an integer</u> n_0 <u>such that for</u> $n \geq n_0$, U <u>contains a component of</u> $\sigma(A_n)$.

(b) <u>If</u> C <u>is a component of</u> $\sigma_e(A)$ <u>and</u> U <u>is an open set containing</u> C, <u>then there is an integer</u> n_0 <u>such that for</u> $n \geq n_0$, U <u>contains a component of</u> $\sigma_e(A_n)$.

<u>Proof</u>. The following proof works for any Banach algebra, so that only part (a) is proved.

By 16.15 of [12], $\sigma(A) = X_1 \cup X_2$ where X_1 and X_2 are disjoint nonempty compact sets with $C \subseteq X_1 \subseteq U$. Choose $\varepsilon > 0$ such that $(X_1)_\varepsilon \cap (X_2)_\varepsilon = \square$ and $(X_1)_\varepsilon \subseteq U$. Let n_1 be an integer such that for $n \geq n_1$, $\sigma(A_n) \subseteq (\sigma(A))_\varepsilon$ (Lemma 1.4 (a)). Thus, $(X_1)_\varepsilon \cap \sigma(A_n)$ is both open and closed in $\sigma(A_n)$ if $n \geq n_1$. It follows that $E((X_1)_\varepsilon \cap \sigma(A_n);A_n) \to E((X_1)_\varepsilon;A)$ and hence there is an integer $n_0 \geq n_1$ such that for $n \geq n_0$, $E((X_1)_\varepsilon \cap \sigma(A_n);A_n) \neq 0$. Hence, $(X_1)_\varepsilon \cap \sigma(A_n) \neq \square$ for $n \geq n_0$; since $(X_1)_\varepsilon \cap \sigma(A_n)$ is both open and closed in $\sigma(A_n)$, $(X_1)_\varepsilon$ must contain a component of $\sigma(A_n)$. ■

For the convenience of the reader, the following result from [4] (Theorem 3.1) is stated. This theorem will be used frequently.

1.6 <u>THEOREM</u> (Apostol and Morrel). <u>If</u> Δ <u>is a nonempty subset of</u> $\mathbb{C}$ <u>and</u> $A \in \mathcal{B}(\mathcal{H})$, <u>then there is a sequence</u> $\{A_n\}$ <u>of operators in</u> $\mathcal{B}(\mathcal{H})$ <u>such that</u> $\sigma(A_n) \subseteq \Delta$ <u>for every</u> n <u>and</u> $||A - A_n|| \to 0$ <u>if and only if</u>:

(a) $P_{\pm}(A) \subseteq \Delta$;

(b) <u>Every component of</u> $\sigma^0(A)$ <u>meets</u> Δ^-.

§2 The Continuity of the Spectral Radius

If $A \in \mathcal{B}(H)$, then the spectral radius of A is defined by $r(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$. It is well known that

$$r(A) = \lim_{n\to\infty} ||A^n||^{1/n} = \inf_n ||A^n||^{1/n} .$$

The first of these equalities asserts that r is the pointwise limit of a sequence of continuous functions from $\mathcal{B}(H)$ into $[0,\infty)$ (viz., $A \to ||A^n||^{1/n}$) . Thus the set of points of continuity of r is a set of the second category. That r has discontinuities follows from the aforementioned example of Kakutani ([13], Solution 87).

The second of the above equalities asserts that r is the infimum of a sequence of continuous functions, and hence that r is upper semicontinuous.

To characterize the points of continuity of r it is necessary to introduce several new functions from $\mathcal{B}(H)$ into $[0,\infty)$.

<u>DEFINITION</u>. If $A \in \mathcal{B}(H)$ and $P_{\pm}(A) = \square$, define $\beta(A) = 0$; if $P_{\pm}(A) \neq \square$, let $\beta(A) = \sup\{|\lambda| : \lambda \in P_{\pm}(A)\}$.

2.1 <u>LEMMA</u>. <u>The function</u> $\beta : \mathcal{B}(H) \to [0,\infty)$ <u>is lower semicontinuous</u>.

This follows from Theorem 1 of [14] and the fact that $\{T \in SF : \text{ind } T \neq 0\}$ is open in $\mathcal{B}(H)$.

<u>DEFINITION</u>. If $A \in \mathcal{B}(H)$, define $\delta(A) \in [0,\infty)$ by

$$\delta(A) = \sup\{\inf\{|\lambda| : \lambda \in C\} : C \text{ is a component of } \sigma_e(A) \cup \sigma_p^0(A)\} .$$

Finally, define $\alpha(A)$ by $\alpha(A) = \max\{\beta(A),\delta(A)\}$.

2.2 LEMMA. The function $\delta : \mathcal{B}(H) \to [0,\infty)$ is lower semicontinuous.

Proof. It must be shown that $\{A \in \mathcal{B}(H) : \delta(A) > \rho\}$ is open for each $\rho \geq 0$. If this is false, then there is an A in $\mathcal{B}(H)$ and a sequence $\{A_n\}$ from $\mathcal{B}(H)$ such that $\delta(A_n) \leq \rho < \delta(A)$ and $||A_n - A|| \to 0$ as $n \to \infty$. Let $\rho < \rho_1 < \delta(A)$ and put $U = \{z : |z| > \rho_1\}$. From the definition of $\delta(A)$, there is a component C of $\sigma_e(A) \cup \sigma_p^0(A)$ that is contained in U. If $C \subseteq \sigma_p^0(A)$, then C is a component of $\sigma(A)$ and, by Lemma 1.5, there is an n_0 such that for $n \geq n_0$, U contains a component C_n of $\sigma(A_n)$. Since $C = \{\lambda\}$ for some λ in $\sigma_p^0(A)$, it is easy to see that for sufficiently large n, $C_n = \{\lambda_n\}$ for some λ_n in $\sigma_p^0(A_n)$. Hence, $\delta(A_n) \geq |\lambda_n| > \rho_1$, a contradiction. If C is a component of $\sigma_e(A)$, then Lemma 1.5 (b) implies that U contains a component of $\sigma_e(A_n)$ for sufficiently large n, and hence $\delta(A_n) \geq \rho_1$, a contradiction. ∎

2.3 COROLLARY. The function $\alpha : \mathcal{B}(H) \to [0,\infty)$ is lower semicontinuous.

DEFINITION. For A in $\mathcal{B}(H)$ define $\delta_0(A)$ and $\delta_*(A)$ by

$$\delta_0(A) = \sup\{\inf\{|\lambda| : \lambda \in C\} : C = \text{a component of } \sigma^0(A)\}.$$

$$\delta_*(A) = \sup\{\inf\{|\lambda| : \lambda \in D\} : D = \text{a component of } \sigma(A)\}.$$

2.4 LEMMA. $\delta_*(A) \leq \delta(A) \leq \delta_0(A)$.

Proof. We only prove that $\delta_*(A) \leq \delta(A)$, since the proof of the other half is similar. Suppose $\delta_*(A) > \rho > 0$; it must be shown that $\delta(A) \geq \rho$. Let D be a component of $\sigma(A)$ such that $\inf\{|\lambda| : \lambda \in D\} > \rho$ and let $\lambda_0 \in D$ such that $|\lambda_0|$

equals this infimum. So $|\lambda_0| > \rho$ and $\lambda_0 \in \partial D \subseteq \partial\sigma(A) \subseteq \sigma_p^0(A) \cup [\sigma_{\ell e}(A) \cap \sigma_{re}(A)]$. If $D \subseteq \sigma_p^0(A)$ then $D = \{\lambda_0\}$ is a component of $\sigma_e(A) \cup \sigma_p^0(A)$ and so $\delta(A) > \rho$. If $\partial D \subseteq \sigma_{\ell e}(A) \cap \sigma_{re}(A) \subseteq \sigma_e(A)$, let C = the component of $\sigma_e(A)$ such that $\lambda_0 \in C$. It follows that $C \subseteq D$. Hence, $|\lambda_0| = \inf\{|\lambda| : \lambda \in D\} \le \inf\{|\lambda| : \lambda \in C\} \le \delta(A)$ and $\delta(A) > \rho$. ■

Note that the inequalities in Lemma 2.4 may be strict. Let $\mathbb{D} = \{z : |z| < 1\}$ and $U = \{z : 1 < |z| < 2\}$; let S be the unilateral shift of multiplicity 1 and let T be multiplication by z on $A^2(U)$, the space of square integrable analytic functions on U . If $A = S \oplus T \oplus T \oplus \ldots$, then $\delta_*(A) = 0$, $\delta(A) = 1$, and $\delta_0(A) = 2$.

2.5 <u>LEMMA</u>. $\alpha(A) = \max\{\beta(A), \delta_0(A)\}$.

<u>Proof</u>. By Lemma 2.4, $\delta(A) \le \delta_0(A)$ so $\alpha(A) \le \alpha_0(A) \equiv \max\{\beta(A), \delta_0(A)\}$. If $\alpha_0(A) = \beta(A)$, then clearly $\alpha_0(A) \le \alpha(A)$. So suppose that $\alpha_0(A) = \delta_0(A) > \beta(A)$. Let $\delta_0(A) > \rho > \beta(A)$ and let C be a component of $\sigma^0(A)$ such that $\inf\{|\lambda| : \lambda \in C\} > \rho$. Thus $C \cap [P_\pm(A)]^- = \square$. Hence, C is either a component of $\sigma_p^0(A)$ or, by Proposition 1.3, C is a component of $\sigma_e(A)$; that is, C is a component of $\sigma_e(A) \cup \sigma_p^0(A)$. Hence $\delta(A) > \rho$ and so $\delta_0(A) \le \delta(A)$. Therefore, $\alpha_0(A) \le \alpha(A)$ and equality obtains. ■

2.6 <u>THEOREM</u>. <u>The spectral radius is continuous at</u> A <u>if and only if</u> $r(A) = \alpha(A)$.

<u>Proof</u>. Suppose that $r(A) = \alpha(A)$ and that $A_n \to A$. Since r is upper semicontinuous and α is lower semicontinuous and $\alpha \le r$, $r(A) = \alpha(A) \le \liminf \alpha(A_n) \le \limsup r(A_n) \le r(A)$. Therefore, $r(A) = \lim r(A_n)$.

Now suppose that $r(A) > \alpha(A)$ and choose ρ such that $r(A) > \rho > \alpha(A)$. Let $D = \{z \in \mathbb{C} : |z| \le \rho\}$. Since $\alpha(A) < \rho$, Lemma 2.5 implies that $P_{\pm}(A) \subset D$ and that every component of $\sigma^0(A)$ meets D . By the result of Apostol and Morrel (Theorem 1.6), there is a sequence $\{A_n\}$ in $\mathcal{B}(\mathcal{H})$ such that $\sigma(A_n) \subset D$ for each n and $||A_n - A|| \to 0$ as $n \to \infty$. Thus, $r(A_n) \le \rho$ for each n and $r(A) \ne \lim r(A_n)$. ■

<u>REMARK</u>. It is possible to show that α is continuous at A if and only if r is continuous at A (that is, $r(A) = \alpha(A)$) . The proof of this fact necessitates improving the results of Apostol and Morrel [4] and will appear elsewhere.

2.7 <u>COROLLARY</u>. <u>If</u> $\delta_*(A) = r(A)$, <u>then</u> r <u>is continuous at</u> A .

2.8 <u>COROLLARY</u>. <u>If</u> A <u>is a normal operator</u>, <u>then</u> r <u>is continuous at</u> A <u>if and only if</u> $r(A) = \delta_*(A)$.

<u>Proof</u>. If A is normal, then $P_{\pm}(A) = \square$ and, since $\mathcal{H}$ is separable, $\sigma(A) = \sigma^0(A)$; so $\alpha(\bar{A}) = \delta_*(A)$. ■

2.9 <u>COROLLARY</u>. <u>Every isometry is a point of continuity of</u> r .

<u>Proof</u>. If A is an isometry then, by the Wold-von Neumann decomposition, $A = S \oplus W$, S , or W, where S is a unilateral shift and W is a unitary operator. In any case $\delta_0(A) = 1$ so that $\alpha(A) = r(A) = 1$. ■

2.10 <u>COROLLARY</u>. <u>If</u> $\sigma(A)$ <u>is totally disconnected</u>, <u>then</u> r <u>is continuous at</u> A . <u>In particular</u>, r <u>is continuous at each compact operator</u>.

<u>Proof</u>. It is clear that $\delta_*(A) = r(A)$. ■

Corollary 2.10 follows from results in [18] where it is shown that $\sigma : \mathcal{B}(\mathcal{H}) \to \mathcal{S}$ is continuous at each operator with totally disconnected spectrum.

We now characterize the points of continuity of the essential spectral radius defined by $r_e(A) = \max\{|\lambda| : \lambda \in \sigma_e(A)\}$. Many of the proofs here are similar to the proofs of the analogous facts used to characterize the points of continuity of r. When this is the case, the appropriate proof will be referenced and the details of the proof will be left to the reader.

<u>DEFINITION</u>. For an operator A in $\mathcal{B}(\mathcal{H})$, define

$$\delta_{0e}(A) = \sup\{\inf\{|\lambda| : \lambda \in C\} : C = \text{a component of } \sigma_{\ell e}(A) \cap \sigma_{re}(A)\},$$

$$\delta_e(A) = \sup\{\inf\{|\lambda| : \lambda \in D\} : D = \text{a component of } \sigma_e(A)\},$$

and $\alpha_e(A) = \max\{\beta(A), \delta_e(A)\}$.

2.11 <u>LEMMA</u>. δ_e <u>is lower semicontinuous</u>.

2.12 <u>COROLLARY</u>. α_e <u>is lower semicontinuous</u>.

2.13 <u>LEMMA</u>. $\delta_e(A) \le \delta_{0e}(A)$.

The proof of Lemma 2.11 is similar to that of Lemma 2.2, while that of Lemma 2.13 follows that of Lemma 2.4.

2.14 <u>LEMMA</u>. $\alpha_e(A) = \max\{\beta(A), \delta_{0e}(A)\}$.

<u>Proof</u>. Let $\alpha_{0e}(A) = \max\{\beta(A), \delta_{0e}(A)\}$. By Lemma 2.13, $\alpha_e(A) \le \alpha_{0e}(A)$. The other half of this inequality is proved as the corresponding fact in Lemma 2.5. ∎

2.15 <u>THEOREM</u>. <u>The essential spectral radius is continuous at</u> A <u>if and only if</u> $\alpha_e(A) = r_e(A)$.

<u>Proof</u>. As in the proof of Theorem 2.6, if $\alpha_e(A) = r_e(A)$, then r_e is continuous at A. So suppose that $\alpha_e(A) < \rho < r_e(A)$ and put $D = \{z \in \mathbb{C} : |z| \le \rho\}$. According to Theorem 4 of [20] (also see [17]), there is a compact operator K on $\mathcal{H}$ such

that $\sigma(A + K) = \sigma_w(A)$. So $\sigma(A + K) = \sigma_e(A) \cup P_\pm(A)$ by Theorem 1.1 and the fact that σ_e and $P_\pm$ are invariant under compact perturbations. Now $\sigma^0(A + K) = \sigma_{\ell e}(A) \cap \sigma_{re}(A)$ and so every component of $\sigma^0(A + K)$ meets D by Lemma 2.14. Since $P_\pm(A + K) = P_\pm(A) \subseteq D$, Theorem 1.6 implies there is a sequence $\{A_n\}$ of bounded operators on $\mathcal{H}$ such that $\sigma(A_n) \subseteq D$ for all n and $A_n \to A + K$. So $A_n - K \to A$ and, since $\sigma_e(A_n - K) = \sigma_e(A_n) \subseteq D$, $r_e(A_n - K) = r_e(A_n) \le \rho$ for all n . Hence, r_e is not continuous at A . ■

REMARKS. As was the case for the points of continuity of α , refinements of the results of Apostol and Morrel [4] may be used to show that α_e is continuous at A if and only if r_e is continuous at A . It is also possible to show that β is continuous at A if and only if $\beta(A) = r_e(A)$. Notice, however, that $\beta(I) = 0 < 1 = r_e(I) = \alpha_e(I)$, so that r_e is continuous at I , but β is not.

2.16 COROLLARY. *If A is a normal operator, then r_e is continuous at A if and only if $\delta_e(A) = r_e(A)$.*

2.17 COROLLARY. *r_e is continuous at every isometry.*

2.18 COROLLARY. *If $\sigma_e(A)$ is totally disconnected, then r_e is continuous at A .*

This section concludes with a discussion of the relationship between the points of continuity of r and r_e .

2.19 PROPOSITION. *If r_e is continuous at A , then r is continuous at A .*

Proof. If $r_e(A) = r(A)$, then $r(A) = r_e(A) = \alpha_e(A) \le \alpha(A) \le r(A)$, so r is continuous at A . Suppose $r_e(A) < r(A)$, and let $\lambda \in \sigma(A)$ such that $|\lambda| = r(A)$. Then $\lambda \in \partial\sigma(A) \subseteq \sigma_e(A) \cup \sigma_p^0(A)$. But $r_e(A) < r(A)$ implies that

$\lambda \notin \sigma_e(A)$, and so, $\lambda \in \sigma_p^0(A)$. That is, $\{\lambda\}$ is a component of $\sigma_e(A) \cup \sigma_p^0(A)$. Therefore, $\delta(A) = |\lambda| = r(A)$, and so, r is continuous at A by Theorem 2.6. ■

The converse of Proposition 2.19 is not true. In fact, let $A = T \oplus D$, where T is multiplication by the independent variable on $L^2[0,1]$ and D is the diagonal operator with entries $\{1 + \frac{1}{n}\}_{n=1}^{\infty}$, each with multiplicity one. Then $\delta(A) = r(A) = 2$, $r_e(A) = 1$, but $\alpha_e(A) = \delta_e(A) = 0$.

It is precisely the presence of points in $\sigma_p^0(A)$ in the preceding example that makes A a point of continuity of r .

2.20 <u>PROPOSITION</u>. *If $A \in \mathcal{B}(H)$ and $\sigma_p^0(A) = \square$ and r is continuous at A , then r_e is continuous at A .*

Proof. Since r is continuous at A , $\alpha(A) = r(A)$. As in the proof of Proposition 2.19, if $r_e(A) < r(A)$, then $r(A)$ is attained at a point in $\sigma_p^0(A)$. Since it is assumed here that $\sigma_p^0(A) = \square$, it must be that $r_e(A) = r(A)$. Thus, if $\alpha(A) = \beta(A)$, then $\beta(A) = \alpha_e(A) = r_e(A)$ and r_e is continuous at A . Otherwise, by Lemma 2.5, $\delta_0(A) = r(A)$. So if $\varepsilon > 0$, there is a component C of $\sigma^0(A) = \sigma_{\ell e}(A) \cap \sigma_{re}(A)$ such that $|\lambda| > r(A) - \varepsilon = r_e(A) - \varepsilon$ for all λ in C . Thus, $\sigma_{0e}(A) = r_e(A)$, and so, by Lemma 2.14, $\alpha_e(A) = r_e(A)$ and r_e is continuous at A . ■

3 The Points of Continuity of the Spectrum

Recall that $\mathcal{S}$ denotes the collection of compact subsets of $\mathbb{C}$ furnished with the Hausdorff metric.

3.1 <u>THEOREM</u>. *If $A \in \mathcal{B}(H)$, then the following are logically equivalent statements.*

(a) $\sigma : \mathcal{B}(H) \to \mathcal{S}$ *is continuous at* A .

(b) For each λ in $\sigma(A)\setminus[P_{\pm}(A)]^{-}$ and $\varepsilon > 0$, the ball $B(\lambda;\varepsilon)$ contains a component of $\sigma^0(A)$.

(c) If $\{C_i : i \in I\}$ are the components of $\sigma^0(A)$ and if for each i in I a point λ_i is chosen from C_i, then

$$\sigma(A) = [P_{\pm}(A) \cup \{\lambda_i : i \in I\}]^{-} .$$

(d) If $\{z_j : j \in J\}$ is the collection of points in $\sigma^0(A)$ such that $\{\{z_j\} : j \in J\}$ is the collection of trivial components of $\sigma^0(A)$, then

$$[P_{\pm}(A) \cup \{z_j : j \in J\}]^{-} = \sigma(A) .$$

Before proving this theorem, some of its consequences will be examined. Notice that condition (b) says that each point in $\sigma(A)\setminus[P_{\pm}(A)]^{-}$ is approachable by points in $\sigma^0(A)$. This yields the following corollary.

3.2 COROLLARY. If $\sigma : \mathcal{B}(H) \to S$ is continuous at A, then $\operatorname{int} P_0(A) = \square$.

Proof. $P_0(A) = \{\lambda \in \sigma(A) : A - \lambda$ is Fredholm and $\operatorname{ind}(A - \lambda) = 0\}$ consists of $\sigma_p^0(A)$ together with the interior of $P_0(A)$. Observing that $P_0(A) \subset \sigma(A)\setminus[P_{\pm}(A)]^{-}$, that $[\operatorname{int} P_0(A)] \cap \sigma^0(A) = \square$, and applying part (b) of Theorem 3.1, we see that $\operatorname{int} P_0(A) = \square$. ∎

By the work of Douglas and Pearcy [7] and Apostol, Foias, and Voiculescu ([1], [2], and [3]), an operator A is biquasitriangular if and only if $P_{\pm}(A) = \square$. (Also see [8].) In particular, every normal operator is biquasitriangular.

3.3 COROLLARY. If A is a biquasitriangular operator, then the following statements are logically equivalent.

(a) $\sigma : \mathcal{B}(H) \to S$ is continuous at A.

(b) For each λ in $\sigma(A)$ and $\varepsilon > 0$, $B(\lambda;\varepsilon)$ contains a component of $\sigma^0(A)$.

(c) *If* $\{C_i : i \in I\}$ *are the components of* $\sigma^0(A)$ *and if for each* i *in* I *a point* λ_i *is chosen from* C_i *, then* $\{\lambda_i : i \in I\}^- = \sigma(A)$.

(d) *If* $\{z_j : j \in J\}$ *is the collection of points in* $\sigma(A)$ *such that* $\{\{z_j\} : j \in J\}$ *is the collection of trivial components of* $\sigma^0(A)$ *, then* $\{z_j : j \in J\}^- = \sigma(A)$.

3.4 *COROLLARY*. [18]. *If* $A \in \mathcal{B}(H)$ *and* $\sigma(A)$ *is totally disconnected, then* σ *is continuous at* A .

Any operator with totally disconnected spectrum must have $P_\pm(A) = \square$ and hence is biquasitriangular. In fact, any operator A for which $\sigma(A)$ has a dense collection of trivial components must be biquasitriangular and a point of continuity of σ .

It should be mentioned that there are several compact subsets of $\mathbb{C}$ that have a dense collection of trivial components but which are not totally disconnected. For example,

$$K = \{(x,0) : 0 \le x \le 1\} \cup \{(\tfrac{k}{n},\tfrac{1}{n}) : 0 \le k \le n\} .$$

Proof of Theorem 3.1. The fact that (b) implies (d) was shown the authors by their colleagues J. Ewing, P. R. Halmos, B. Halpern, and R. Kulkarni. It follows in a rather straightforward way from the fact that in a compact metric space a component is the intersection of the closed and open sets that contain it. (See 16.15 of [12].) The details are left to the reader. Clearly (d) implies (c) and it is easy to see that (c) implies (b).

(a) *implies* (c). Suppose that (c) fails; that is, there exists a set of points $\{\lambda_i : i \in I\}$ with λ_i in C_i such that $K = [P_\pm(A) \cup \{\lambda_i : i \in I\}]^- \neq \sigma(A)$. Since $P_\pm(A) \subseteq K$ and each component of $\sigma^0(A)$ meets K , the result of Apostol and Morrel (Theorem 1.6) implies there is a sequence of operators $\{A_n\}$ with $\sigma(A_n) \subset K$ for each n and $||A_n - A|| \to 0$ as $n \to \infty$. Clearly, $\{\sigma(A_n)\}$ does not converge to $\sigma(A)$.

(b) <u>implies</u> (a). Suppose $A_n \to A$ and $\varepsilon > 0$. By Lemma 1.4, there is an integer n_1 such that for $n \geq n_1$,

$$P_{\pm}(A) \subseteq (P_{\pm}(A_n))_{\varepsilon/2} \quad \text{and}$$
$$\sigma(A_n) \subseteq (\sigma(A))_{\varepsilon}\ .$$

By condition (b) (see the argument for Corollary 3.2), $\sigma(A) = \sigma^0(A) \cup P_{\pm}(A)$. If $\sigma^0(A)\setminus(P_{\pm}(A))_{\varepsilon/2} = \square$, then $\sigma(A) = \sigma^0(A) \cup P_{\pm}(A) \subseteq (P_{\pm}(A))_{\varepsilon/2} \subseteq (P_{\pm}(A_n))_{\varepsilon} \subseteq (\sigma(A_n))_{\varepsilon}$; if $\sigma^0(A)\setminus(P_{\pm}(A))_{\varepsilon/2} \neq \square$, then let $\lambda_1, \ldots, \lambda_m \in \sigma^0(A)\setminus(P_{\pm}(A))_{\varepsilon/2}$ be such that

$$\sigma(A)\setminus(P_{\pm}(A))_{\varepsilon/2} \subseteq \bigcup_{k=1}^{m} B(\lambda_k;\varepsilon/4)\ .$$

Let C_k be a component of $\sigma^0(A)$ that is contained in $B(\lambda_k;\varepsilon/4)$. So $C_k \cap P_{\pm}(A)^- = \square$. By Proposition 1.3, C_k is a component of $\sigma_e(A)$ or an isolated point in $\sigma_p^0(A)$. By Lemma 1.5, there is an integer $n_0 \geq n_1$ such that for $n \geq n_0$ and $1 \leq k \leq m$, $B(\lambda_k;\varepsilon/4)$ contains a component of $\sigma_e(A_n)$ or of $\sigma(A_n)$. In either case, $\sigma(A)\setminus(P_{\pm}(A))_{\varepsilon/2} \subseteq (\sigma(A_n))_{\varepsilon/2}$ if $n \geq n_0$. Therefore,

$$\begin{aligned}\sigma(A) &= \sigma^0(A) \cup P_{\pm}(A)\\ &\subseteq [\sigma(A)\setminus(P_{\pm}(A))_{\varepsilon/2}] \cup (P_{\pm}(A))_{\varepsilon/2}\\ &\subseteq (\sigma(A_n))_{\varepsilon/2} \cup (P_{\pm}(A_n))_{\varepsilon}\\ &\subseteq (\sigma(A_n))_{\varepsilon}\ .\end{aligned}$$

This completes the proof. ■

As a final application of Theorem 3.1 (more precisely, of Corollary 3.3 or 3.4), the following corollary is presented.

Recall that if $A = \oplus_{n=1}^{\infty} A_n$, then it is not in general true that $\sigma(A) = [\cup_{n=1}^{\infty} \sigma(A_n)]^-$.

3.5 COROLLARY. *If A is a compact operator and $A = \oplus_{n=1}^{\infty} A_n$, then $\sigma(A) = [\bigcup_{n=1}^{\infty} \sigma(A_n)]^-$.*

Proof. Because A is compact, each A_n must be compact and $\|A_n\| \to 0$. Hence, A is the limit of the sequence $\{A_1 \oplus \ldots \oplus A_n\}$. Since σ is continuous at A and $\sigma(A_1 \oplus \ldots \oplus A_n) = \sigma(A_1) \cup \ldots \cup \sigma(A_n)$, the result follows.∎

Next we will characterize the points of continuity of the Weyl spectrum. Only a condition analogous to (b) in Theorem 3.1 above will be stated; the topologically equivalent conditions analogous to parts (c) and (d) of Theorem 3.1 will not be given.

3.6 THEOREM. *The Weyl spectrum σ_w is continuous at A if and only if for every λ in $\sigma_w(A)\backslash[P_\pm(A)]^-(=\sigma_e(A)\backslash[P_\pm(A)]^-)$ and for every $\varepsilon > 0$, there is a component of $\sigma_e(A)$ that is contained in $B(\lambda;\varepsilon)$.*

Proof. Suppose there is a λ in $\sigma_e(A)\backslash P_\pm(A)^-$ and an $\varepsilon > 0$ such that $B(\lambda;\varepsilon)$ does not contain a component of $\sigma_e(A)$. Then $D = \mathbb{C}\backslash B(\lambda;\varepsilon)$ meets each component of $\sigma_e(A)$ and $P_\pm(A) \subseteq D$. By Theorem 4 of [20] there is a compact operator K such that $\sigma(A + K) = \sigma_w(A) = P_\pm(A) \cup \sigma_e(A)$. Thus $P_\pm(A + K) = P_\pm(A) \subseteq D$ and D meets each component of $\sigma^0(A + K)^- = \sigma_{\ell e}(A) \cap \sigma_{re}(A) = \sigma_e(A)\backslash P_\pm(A)$. By the result of Apostol and Morrel (Theorem 1.6), there is a sequence of operators $\{A_n\}$ with $\sigma(A_n) \subseteq D$ for each n and $A_n \to A + K$. Hence $\sigma_w(A_n - K) \subseteq D$ and it is clear that σ_w is not continuous at A .

For the converse, suppose $\sigma_w(A)\backslash P_\pm(A)^-$ satisfies the stated condition. Let $A_n \to A$ and let $\varepsilon > 0$. By Lemma 1.4, there is an integer n_1 such that for $n \geq n_1$,

$$\sigma_e(A_n) \subseteq (\sigma_e(A))_\varepsilon$$

and $$P_\pm(A) \subseteq (P_\pm(A_n))_{\varepsilon/2} .$$

As in the proof of Theorem 3.1, if $\sigma_e(A)\setminus(P_\pm(A))_{\varepsilon/2} = \square$, then $\sigma_w(A) \subseteq (P_\pm(A))_{\varepsilon/2} \subseteq (P_\pm(A_n))_\varepsilon \subseteq (\sigma_w(A_n))_\varepsilon$ for $n \geq n_1$. Otherwise, choose $\lambda_1, \ldots, \lambda_m$ in $\sigma_e(A)\setminus(P_\pm(A))_{\varepsilon/2}$ such that

$$\sigma_e(A)\setminus(P_\pm(A))_{\varepsilon/2} \subseteq \bigcup_{k=1}^{m} B(\lambda_k;\varepsilon/2) .$$

By (b), there is a component C_k of $\sigma_e(A)$ that is contained in $B(\lambda_k;\varepsilon/2)$. By Lemma 1.5, there is an integer $n_2 \geq n_1$ such that for $n \geq n_2$, $B(\lambda_k;\varepsilon/2)$ contains a component of $\sigma_e(A_n)$; thus $B(\lambda_k;\varepsilon/2) \subseteq (\sigma_e(A_n))_\varepsilon$. Therefore, if $n \geq n_2$,

$$\begin{aligned} \sigma_w(A) &\subseteq \sigma_e(A)\setminus(P_\pm(A))_{\varepsilon/2} \cup (P_\pm(A))_{\varepsilon/2} \\ &\subseteq (\sigma_e(A_n))_\varepsilon \cup (P_\pm(A_n))_\varepsilon \\ &= (\sigma_w(A_n))_\varepsilon . \end{aligned}$$

This shows that σ_w is lower semicontinuous at A .

To complete the proof again use the Stampfli result ([17], [20]) to obtain a compact operator K such that $\sigma(A + K) = \sigma_w(A)$. Since $A_n + K \to A + K$ there is an integer $n_0 \geq n_2$ such that if $n \geq n_0$, $\sigma(A_n + K) \subseteq (\sigma(A + K))_\varepsilon$. Therefore, if $n \geq n_0$, then

$$\sigma_w(A_n) = \sigma_w(A_n + K) \subseteq \sigma(A_n + K) \subseteq (\sigma(A + K))_\varepsilon = (\sigma_w(A))_\varepsilon .$$

This completes the proof. ■

In general, there is no inclusion relation between the points of continuity of σ and those of σ_w . For example, let $\alpha_{n,k} = (1 + \frac{1}{n}) \exp(2\pi i\, k/n)$ for $1 \leq k \leq n$ and $n \geq 1$. Let N be the diagonal operator whose eigenvalues are $\Delta = \{\alpha_{n,k} : 1 \leq k \leq n , n \geq 1\}$, each eigenvalue having infinite multiplicity. Hence,

$\sigma(N) = \sigma_e(N) = \Delta \cup \partial\mathbb{D}$. Let S be the unilateral shift of multiplicity one and put $A = S \oplus S^* \oplus N$. Then $\sigma_w(A) = \sigma_e(A) = \Delta \cup \partial\mathbb{D}$, $P_{\pm}(A) = \square$, $P_0(A) = \mathbb{D}$, and $\sigma(A) = \Delta \cup \mathbb{D}^-$. By Theorem 3.6, σ_w is continuous at A but, by Corollary 3.2, σ is not continuous at A .

For another example, let M be the diagonal operator with eigenvalues $\Delta = \{\alpha_{n,k} : 1 \leq k \leq n \ , \ n \geq 1\}$, where each eigenvalue has multiplicity one. Then $\sigma(M) = \partial\mathbb{D} \cup \Delta$, $\sigma_w(M) = \partial\mathbb{D}$, and $P_{\pm}(M) = \square$. Here, σ is continuous at M but σ_w is not.

3.7 <u>PROPOSITION</u>. <u>If</u> σ_w <u>is continuous at</u> A <u>and</u> $\operatorname{int} P_0(A) = \square$, <u>then</u> σ <u>is continuous at</u> A .

<u>Proof</u>. Suppose $A_n \to A$ and $\varepsilon > 0$. Let n_1 be an integer such that for $n \geq n_1$,

$$\sigma(A_n) \subseteq (\sigma(A))_\varepsilon \ ,$$

$$\sigma_w(A_n) \subseteq (\sigma_w(A))_\varepsilon \ ,$$

and

$$\sigma_w(A) \subseteq (\sigma_w(A_n))_\varepsilon \ .$$

By the hypothesis, $\sigma(A) = \sigma_w(A) \cup \sigma_p^0(A)$. Since $[\sigma_p^0(A)]^- \setminus \sigma_p^0(A) \subseteq \sigma_e(A)$, $\sigma_p^0(A) \setminus (\sigma_w(A))_\varepsilon$ consists of a finite number of points $\lambda_1, \ldots, \lambda_m$. An application of Lemma 1.5 yields the existence of an integer $n_0 \geq n_1$ such that $\{\lambda_1, \ldots, \lambda_n\} \subseteq (\sigma(A_n))_\varepsilon$ for $n \geq n_0$. ∎

3.8 <u>PROPOSITION</u>. <u>If</u> $A \in \mathcal{B}(H)$ <u>such that</u> $\sigma_p^0(A) = \square$ <u>and</u> σ <u>is continuous at</u> A , <u>then</u> σ_w <u>is continuous at</u> A .

<u>Proof</u>. The criterion stated in Theorem 3.6 will be used to show that σ_w is continuous at A . If $\lambda \in \sigma_e(A) \setminus [P_{\pm}(A)]^-$ and $\varepsilon > 0$, then Theorem 3.1 implies that for every $\varepsilon > 0$, there is a component C of $\sigma^0(A)$ such that $C \subset B(\lambda;\varepsilon)$. Since $\sigma_p^0(A) = \square$, $\sigma^0(A) = \sigma_{\ell e}(A) \cap \sigma_{re}(A)$, and so, by Proposition 1.3, C is a component of $\sigma_e(A)$. Thus σ_w is continuous at A . ∎

The examples preceding Proposition 3.7 show that these last two results are, in some sense, the best possible for relating the points of continuity of σ and σ_w .

3.9 COROLLARY. *If* $A \in \mathcal{B}(H)$ *and* $\sigma(A) = \sigma_w(A)$, *then* σ *is continuous at* A *if and only if* σ_w *is continuous at* A .

3.10 COROLLARY. *If* $A \in \mathcal{B}(H)$ *and* σ_w *is continuous at* A , *then there is a compact operator* K *such that* σ *is continuous at* $A + K$.

Proof. Let K be a compact operator such that $\sigma(A + K) = \sigma_w(A)$ ([17], [20]). Since $\sigma_w(A) = \sigma_w(A + K)$, $A + K$ is a point of continuity of σ_w . But $\sigma(A + K) = \sigma_w(A)$ so Corollary 3.9 applies.■

The next question that presents itself is "What are the points of continuity of the essential spectrum?" This question will be investigated in a forthcoming paper. It will be shown that if σ_e is continuous at A then so is σ_w .

3.11 PROPOSITION. *If* σ_e *is continuous at* A , *then so is* σ_w .

Proof. Suppose $A_n \to A$ and $\varepsilon > 0$. Let n_1 be an integer such that for $n \geq n_1$,

$$\sigma_e(A) \subseteq (\sigma_e(A_n))_\varepsilon$$

and

$$P_\pm(A) \subseteq (P_\pm(A_n))_\varepsilon .$$

Let K be a compact operator such that $\sigma(A + K) = \sigma_w(A)$ ([17] and [20]). Let $n_0 \geq n_1$ be such that for $n \geq n_0$, $\sigma(A_n + K) \subseteq (\sigma(A + K))_\varepsilon$. Therefore,

$$\sigma_w(A_n) = \sigma_w(A_n + K) \subseteq (\sigma(A + K))_\varepsilon = (\sigma_w(A))_\varepsilon$$

and

$$\sigma_w(A) = P_\pm(A) \cup \sigma_e(A) \subseteq (P_\pm(A))_\varepsilon \cup (\sigma_e(A_n))_\varepsilon = (\sigma_w(A_n))_\varepsilon$$

This completes the proof. ∎

If A is biquasitriangular, then $P_{\pm}(A) = \square$ and so $\sigma_w(A) = \sigma_e(A)$. Hence, σ_e (or σ_w) is continuous at A if and only if every relatively open subset of $\sigma_e(A)$ contains a component of $\sigma_e(A)$.

The converse to Proposition 3.11 is false as the following example illustrates. For a bounded region G in $\mathbb{C}$, let $A^2(G)$ denote the space of analytic functions on G that are square integrable with respect to area measure. Let $G_1 = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1 \text{ and } 0 < \operatorname{Im} z < 1\}$ and $G_2 = \{z \in \mathbb{C} : 1 < \operatorname{Re} z < 2 \text{ and } 0 < \operatorname{Im} z < 1\}$. Let S_k = multiplication by z on $A^2(G_k)$ for $k = 1, 2$. So $\sigma(S_k) = G_k^-$, $\sigma_e(S_k) = \partial G_k$, and $\operatorname{ind}(\lambda - S_k) = -1$ for λ in G_k. If $A = S_1 \oplus S_2$, then σ_w is continuous at A by Theorem 3.6. But, σ_e is not continuous at A. To see this, let $G_3 = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 2,\ 0 < \operatorname{Im} z < 1\}$ and let S_3 = multiplication by z on $A^2(G_3)$. Let $C = \{1 + iy : 0 \leq y \leq 1\}$ and let N be a normal operator with $\sigma(N) = \sigma_e(N) = C$. It follows from the work of Brown, Douglas, and Fillmore [6] that there is a unitary operator U and a compact operator K such that $A = U(S_3 \oplus N)U^{-1} + K$. By the result of Apostol and Morrel (Theorem 1.6), there is a sequence of operators $\{T_n\}$ such that $\sigma(T_n) = \{1\}$ for each n and $||T_n - N|| \to 0$ as $n \to \infty$. Let $A_n = U(S_3 \oplus T_n)U^{-1} + K$. Then $A_n \to A$ as $n \to \infty$. But $\sigma_e(A) = \partial G_1 \cup \partial G_2$, while $\sigma_e(A_n) = \sigma_e(S_3) \cup \sigma_e(T_n) = \partial G_3$ for all n. Hence, $\{\sigma_e(A_n)\}$ does not converge to $\sigma_e(A)$.

An interesting fact is that $A = S_1 \oplus S_2 \oplus S_2$ is a point of continuity of σ_e if S_1 and S_2 are defined as in the preceding paragraph.

One can ask whether the restriction of σ or σ_e to certain subsets of $\mathcal{B}(H)$ is continuous. For example, is the restriction of σ to the set of normal operators continuous? The answer is yes and was obtained by Newburgh [18] as a consequence of a more general fact. The reader can consult [18] for the details.

§4 Concluding Remarks

It is easy to show that r, r_e, σ, and σ_w are continuous on a dense G_δ. In fact, this follows immediately from the results in this paper and the Apostol-Morrel model in [4]. The referee has also pointed out that r_e, σ_w, and σ are discontinuous on a dense F_σ. This follows from the proof of Theorem 4 in [15].

Several of the results of this paper hold in a more general setting. In fact, virtually all the implications that do not use the Apostol-Morrel result (Theorem 1.6) hold for operators on a Banach space. Indeed, most of these implications involve only the basic properties of the Riesz functional calculus and the continuity of the index. These concepts and their properties are valid not only in $\mathcal{B}(H)$, but also in many Banach algebras. The only reason that we are forced to restrict our attention to $\mathcal{B}(H)$ is the existence of Theorem 1.6. It would be interesting to see partial extensions of (1.6) to Banach spaces.

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