NOTES ON MULTIVARIATE CONFIDENCE BOUNDS

By MINORU SIOTANI

(Received Feb. 20, 1960)

1. Introduction and summary.

Let x_{il} be an observation from p-variate normal distribution with the mean vector μ_i and the covariance matrix Λ, i.e., $N(\mu_i, \Lambda)$, $l=1, 2, \cdots, n_i$; $i=1, 2, \cdots, k$. Let us use the following usual notations and definitions:

$\bar{x}_i=\sum_{l=1}^{n_i} x_{il}/n_i$, for the mean vector for the i-th sample,

$\bar{x}=\sum_{i=1}^{k} n_i\bar{x}_i \Big/ \sum_{i=1}^{k} n_i$ and $\mu=\sum_{i=1}^{k} n_i\mu_i \Big/ \sum_{i=1}^{k} n_i$, for the weighted grand mean vectors of $\bar{x}_i$'s and μ_i's respectively, and

$$\left(\sum_{i=1}^{k} n_i-k\right)L=\sum_{i=1}^{k}\sum_{l=1}^{n_i}(x_{il}-\bar{x}_i)(x_{il}-\bar{x}_i)' ,$$

for the pooled 'within' covariance matrix of the k samples.

In 1953, S. N. Roy and R. C. Bose [2] obtained the simultaneous confidence bounds on all arbitrary double linear compounds of the difference between k mean vectors μ_i's and their weighted grand mean μ, that is, for all μ_i's, all non-null p-dimensional scalar vector a's and all b_i's subject to $\sum_{i=1}^{k} b_i^2=1$,

$$(1)\qquad \sum_{i=1}^{k} b_i n_i^{1/2} a'(\bar{x}_i-\bar{x})-[(k-1)c_\alpha a'La]^{1/2}$$
$$\leqq \sum_{i=1}^{k} b_i n_i^{1/2} a'(\mu_i-\mu) \leqq \sum_{i=1}^{k} b_i n_i^{1/2} a'(\bar{x}_i-\bar{x})+[(k-1)c_\alpha a'La]^{1/2}$$

where c_α is the 100α per cent point of the largest root of the p-th degree determinantal equation in c: $|L^*-cL|=0$, where L^* is the 'between' covariance matrix, which is defined by

$$(k-1)L^*=\sum_{i=1}^{k} n_i(\bar{x}_i-\bar{x}-\mu_i+\mu)(\bar{x}_i-\bar{x}-\mu_i+\mu)' .$$

In this paper, the following three problems, which are to obtain specific subsets of the set of bounds (1), are considered:

(i) to give simultaneous confidence bounds relating to a specific set of independent comparisons among μ_i's,

(ii) when one of k experiments is a standard experiment or a control, to set the simultaneous confidence bounds relating to the differences between the standard population mean and the other $k-1$ μ_i's and

(iii) to give the simultaneous confidence bounds on the contrasts, $\boldsymbol{a}'(\mu_i-\mu_j)$ for all non-null $\boldsymbol{a}'$ and all $i\neq j=1, 2, \cdots, k$. This problem has been formulated by Roy and Bose in [2].

The method of calculating 100α per cent points of the quantities needed in setting these bounds is also explained.

2. Confidence bounds relating to the independent comparisons Among μ_i's.

Let $\boldsymbol{x}_{il}$ be an observation from i-th p-variate normal distribution $N(\mu_i, \Lambda)$, $(l=1, \cdots, n_i\,; i=1,\cdots, k)$. In practical applications, for example, in analysis of dispersion, we are often interested in the specified set of the independent comparisons among μ_i's rather than the set of all arbitrary comparisons among them. There are $k-1$ independent comparisons among μ_i's. Suppose that we are interested in setting the simultaneous confidence bounds on $\boldsymbol{a}'\eta_i$ for all non-null $\boldsymbol{a}$ and all $i=1, 2, \cdots, k-1$, where

$$\eta_i=\sum_{h=1}^{k} d_{ih}n_h^{1/2}\mu_h, \qquad i=1, 2, \cdots, k-1\ , \tag{2}$$

and d_{ih}'s are given scalars satisfying the conditions

$$\begin{aligned} &\sum_{h=1}^{k} d_{ih}n_h^{1/2}=0, && i=1, 2, \cdots, k-1\ ,\\ &\sum_{h=1}^{k} d_{ih}d_{jh}=0, && i\neq j=1, 2, \cdots, k-1\ . \end{aligned} \tag{3}$$

Without the loss of generality we can assume that $\sum_{h=1}^{k} d_{ih}^2=1$ $(i=1, 2, \cdots, k-1)$. If we define analogous comparisons $\boldsymbol{y}_i$'s among k sample means, $\bar{\boldsymbol{x}}_i$'s, to comparisons η_i's by

$$\boldsymbol{y}_i=\sum_{h=1}^{k} d_{ih}n_h^{1/2}\bar{\boldsymbol{x}}_h, \qquad i=1, 2, \cdots, k-1\ , \tag{4}$$

then it is easily seen that $\boldsymbol{y}_i$ is distributed according to $N(\eta_i, \Lambda)$ independently of $\boldsymbol{y}_j (i\neq j)$.

Now we consider the statement that, for all possible non-null vectors $\boldsymbol{a}$,

$$|\boldsymbol{a}'(\boldsymbol{y}_i-\eta_i)|/[\boldsymbol{a}'\boldsymbol{L}\boldsymbol{a}]^{1/2}\leqq g$$

or

$$\boldsymbol{a}'(\boldsymbol{y}_i-\eta_i)(\boldsymbol{y}_i-\eta_i)'\boldsymbol{a}/\boldsymbol{a}'\boldsymbol{L}\boldsymbol{a}\leqq g^2 \tag{5}$$

where $\boldsymbol{L}$ is the pooled covariance matrix defined in Section 1 and g is a given positive constant. Noting that

$$\hat{T}_i^2=(\boldsymbol{y}_i-\eta_i)'\boldsymbol{L}^{-1}(\boldsymbol{y}_i-\eta_i)=\sup_{a}\boldsymbol{a}'(\boldsymbol{y}_i-\eta_i)(\boldsymbol{y}_i-\eta_i)'\boldsymbol{a}/\boldsymbol{a}'\boldsymbol{L}\boldsymbol{a}\ , \tag{6}$$

it is easy to see that, for given i, the relation: $\hat{T}_i^2\leqq g^2$, is exactly equivalent to the relation (5). Furthermore, considering all i, we can see that the statement: all $T_i^2\leqq g^2$, is precisely equivalent to the statement: $\hat{T}^2_{\text{MAX}}=\max_i\{\hat{T}_i^2\}\leqq g^2$, which is again equivalent to the statement that, for all non-null $\boldsymbol{a}$ and all i,

$$\boldsymbol{a}'\boldsymbol{y}_i-[g^2\boldsymbol{a}'\boldsymbol{L}\boldsymbol{a}]^{1/2}\leqq\boldsymbol{a}'\eta_i\leqq\boldsymbol{a}'\boldsymbol{y}_i+[g^2\boldsymbol{a}'\boldsymbol{L}\boldsymbol{a}]^{1/2}\ . \tag{7}$$

Thus, if we denote the upper 100α per cent point of $\hat{T}^2_{\text{MAX}}$ by $B^2(\alpha;\,p,\,k-1,\,\sum_{i=1}^{k}n_i-k)\equiv B^2(\alpha)$, we have a set of simultaneous confidence bounds, with a confidence coefficient $1-\alpha$,

$$\boldsymbol{a}'\boldsymbol{y}_i-[B^2(\alpha)\boldsymbol{a}'\boldsymbol{L}\boldsymbol{a}]^{1/2}\leqq\boldsymbol{a}'\eta_i\leqq\boldsymbol{a}'\boldsymbol{y}_i+[B^2(\alpha)\boldsymbol{a}'\boldsymbol{L}\boldsymbol{a}]^{1/2} \tag{8}$$

for all non-null $\boldsymbol{a}$ and all i.

The distribution of $\hat{T}^2_{\text{MAX}}$ is extremely difficult to obtain, but the approximate evaluation of $B^2(\alpha)$, for ordinary significance level α, can be made by adopting the method obtained by the author in [3], which will be explained later.

Expressing the confidence bounds (8) in terms of μ_i's and $\boldsymbol{x}_i$'s, we have

$$\begin{aligned}\sum_{h=1}^{k}d_{ih}n_h^{1/2}\boldsymbol{a}'\bar{\boldsymbol{x}}_h-[B^2(\alpha)\boldsymbol{a}'\boldsymbol{L}\boldsymbol{a}]^{1/2}&\leqq\sum_{h=1}^{k}d_{ih}n_h^{1/2}\boldsymbol{a}'\mu_h\\&\leqq\sum_{h=1}^{k}d_{ih}n_h^{1/2}\boldsymbol{a}'\bar{\boldsymbol{x}}_h+[B^2(\alpha)\boldsymbol{a}'\boldsymbol{L}\boldsymbol{a}]^{1/2}\ ,\end{aligned} \tag{9}$$

and when $n_1=n_2=\cdots=n_k$, (9) becomes

$$\begin{aligned}\sum_{h=1}^{k}d_{ih}\boldsymbol{a}'\bar{\boldsymbol{x}}_h-[B^2(\alpha)n^{-1}\boldsymbol{a}'\boldsymbol{L}\boldsymbol{a}]^{1/2}&\leqq\sum_{h=1}^{k}d_{ih}\boldsymbol{a}'\mu_h\\&\leqq\sum_{h=1}^{k}d_{ih}\boldsymbol{a}'\bar{\boldsymbol{x}}_h+[B^2(\alpha)n^{-1}\boldsymbol{a}'\boldsymbol{L}\boldsymbol{a}]^{1/2}\ ,\end{aligned} \tag{10}$$

where n denotes the common size of k samples.

3. Confidence bounds when one of k experiments is a standard one.

Suppose that one of k experiments, for example, k-th experiment is made on a standard or a control category. In this case, we are interested in setting the simultaneous confidence bounds relating to the differences between the population mean μ_k in the standard experiment and the other population means μ_i's, that is, the confidence bounds on the comparisons of the type: $\boldsymbol{a}'(\mu_i-\mu_k)$, for all non-null $\boldsymbol{a}$ and all $i=1, 2, \cdots, k-1$.

Considering that, for all permissible non-null $\boldsymbol{a}$,

$$|\boldsymbol{a}'(\bar{\boldsymbol{x}}_i-\bar{\boldsymbol{x}}_k-\mu_i+\mu_k)|/[\boldsymbol{a}'\boldsymbol{L}\boldsymbol{a}]^{1/2}\leqq g$$

or

$$\boldsymbol{a}'(\bar{\boldsymbol{x}}_i-\bar{\boldsymbol{x}}_k-\mu_i+\mu_k)(\bar{\boldsymbol{x}}_i-\bar{\boldsymbol{x}}_k-\mu_i+\mu_k)'\boldsymbol{a}/\boldsymbol{a}'\boldsymbol{L}\boldsymbol{a}\leqq g^2 , \tag{11}$$

we can easily obtain the desired confidence bounds in the same manner as the last section. The result is as follows: for all non-null $\boldsymbol{a}$ and all $i=1, 2, \cdots, k-1$,

$$\boldsymbol{a}'(\bar{\boldsymbol{x}}_i-\bar{\boldsymbol{x}}_k)-[A^2(\alpha)\boldsymbol{a}'\boldsymbol{L}\boldsymbol{a}]^{1/2}\leqq \boldsymbol{a}'(\mu_i-\mu_k)\leqq \boldsymbol{a}'(\bar{\boldsymbol{x}}_i-\bar{\boldsymbol{x}}_k)+[A^2(\alpha)\boldsymbol{a}'\boldsymbol{L}\boldsymbol{a}]^{1/2} , \tag{12}$$

where $A^2(\alpha)\equiv A^2(\alpha\,; p, k-1, \sum_{h=1}^{k} n_i-k)$ is the upper 100α per cent point of the distribution of $\hat{T}_{\text{MAX.C}}$ defined by

$$\hat{T}^2_{\text{MAX.C}}=\max_i \{(\bar{\boldsymbol{x}}_i-\bar{\boldsymbol{x}}_k-\mu_i+\mu_k)'\boldsymbol{L}^{-1}(\bar{\boldsymbol{x}}_i-\bar{\boldsymbol{x}}_k-\mu_i+\mu_k)\} .$$

This statistic $\hat{T}^2_{\text{MAX.C}}$ and $\hat{T}^2_{\text{MAX}}$ in the previous section are similar statistics except that, in $\hat{T}^2_{\text{MAX}}$, $\boldsymbol{y}_i-\eta_i=\sum_{h=1}^{k} d_{ih}n_h^{1/2}(\bar{\boldsymbol{x}}_h-\mu_h)$ is independent of $\boldsymbol{y}_j-\eta_j$ for $i\neq j$, whereas, in $\hat{T}^2_{\text{MAX.C}}$, $(\bar{\boldsymbol{x}}_i-\bar{\boldsymbol{x}}_k-\mu_i+\mu_k)$ is not independent of $(\bar{\boldsymbol{x}}_j-\bar{\boldsymbol{x}}_k-\mu_j+\mu_k)$ for $i\neq j$ and their covariance matrix is $\boldsymbol{\Lambda}/n_k$.

4. Confidence bounds on the contrast $\boldsymbol{a}'(\mu_i-\mu_j)$.

A set of simultaneous confidence bounds on $\boldsymbol{a}'(\mu_i-\mu_j)$, for all non-null $\boldsymbol{a}$ and all $i\neq j=1, 2, \cdots, k$, has been built up by Roy and Bose [2] in the following form with slight modification:

$$\boldsymbol{a}'(\bar{\boldsymbol{x}}_i-\bar{\boldsymbol{x}}_j)-[R^2_\alpha\boldsymbol{a}'\boldsymbol{L}\boldsymbol{a}]^{1/2}\leqq \boldsymbol{a}'(\mu_i-\mu_j)\leqq \boldsymbol{a}'(\bar{\boldsymbol{x}}_i-\bar{\boldsymbol{x}}_j)+[R^2_\alpha\boldsymbol{a}'\boldsymbol{L}\boldsymbol{a}]^{1/2} \tag{13}$$

for all non-null $\boldsymbol{a}$'s and all pairs (i, j) out of k, where $R^2_\alpha\equiv R^2(\alpha\,; p, k, \sum_{i=1}^{k} n_i-k)$ is the upper 100α per cent point of R^2_{MAX} defined by

$$R^2_{\text{MAX}} = \max_{i,j} \{R^2_{ij}\} = \max_{i,j} \{(\bar{\boldsymbol{x}}_i - \bar{\boldsymbol{x}}_j - \mu_i + \mu_j)' \boldsymbol{L}^{-1} (\bar{\boldsymbol{x}}_i - \bar{\boldsymbol{x}}_j - \mu_i + \mu_j)\} .$$

However, the method for evaluating R^2_α could not be obtained and hence the confidence statement (13) has not been reduced to practical terms. The author has given in [4] (in Japanese) the method for evaluating R^2_α for the ordinary significance levels.

5. Evaluation of the upper 100α per cent point of $\hat{T}^2_{\text{MAX}}$.

The evaluation in this and next sections will be done by using the method given in [3]. This is based on the following well-known formula in probability expressed in terms of $\hat{T}^2_i$'s which are defined in Section 2:

$$\text{(14)} \quad \Pr\{\hat{T}^2_{\text{MAX}} > t^2\} = N \Pr\{\hat{T}^2_1 > t^2\} - \tfrac{1}{2} N(N-1) \Pr\{\hat{T}^2_1 > t^2, \hat{T}^2_2 > t^2\} + \cdots ,$$

where t^2 is a positive constant and $N = k-1$. For the reasonably large value of t^2, a good approximation to $\Pr\{\hat{T}^2_{\text{MAX}} > t^2\}$ is provided by the first one or two terms of the right hand side of (14). Since $\hat{T}^2_{\text{MAX}}$ is defined as $\max_i \{\hat{T}^2_i\}$ or $\max_i \{(\boldsymbol{y}_i - \eta_i)' \boldsymbol{L}^{-1} (\boldsymbol{y}_i - \eta_i)\}$, $\boldsymbol{y}_i - \eta_i$ is distributed according to $N(\boldsymbol{0}, \boldsymbol{\Lambda})$ independently of $\boldsymbol{y}_j - \eta_j (i \neq j)$ and $\boldsymbol{L}$ has the Wishart distribution with parameter matrix $\boldsymbol{\Lambda}$ and with $\nu = \sum_{i=1}^{k} n_i - k$ degrees of freedom, it is easily seen that each $\hat{T}^2_i$ has the Hotelling's T^2-distribution and then the first approximate upper 100α per cent point of $\hat{T}^2_{\text{MAX}}$, which is denoted by $B^2_1(\alpha; p, N, \nu)$, is given by solving

$$\text{(15)} \qquad \alpha/N = \Pr\{\hat{T}^2_1 > B^2_1(\alpha; p, N, \nu)\} ,$$

that is,

$$\text{(16)} \quad B^2_1(\alpha; p, N, \nu) = \nu \left\{ \frac{1}{C(\alpha/N; (\nu-p+1)/2, p/2)} - 1 \right\} \quad \text{(see (3.7) in [3])}$$

where $C(\alpha^*; a, b)$ is the lower $100\alpha^*$ per cent point of the beta-distribution with parameters a and b. By Bonferroni's inequalities, we have, for $t^2 = B^2_1(\alpha; p, N, \nu)$,

$$N\Pr\{\hat{T}^2_1 > B^2_1(\alpha)\} - \tfrac{1}{2} N(N-1) \Pr\{\hat{T}^2_1 > B^2_1(\alpha), \hat{T}^2_2 > B^2_1(\alpha)\}$$
$$< \Pr\{\hat{T}^2_{\text{MAX}} > B^2_1(\alpha)\} < NPr\{\hat{T}^2_1 > B^2_1(\alpha)\}$$

i.e.

$$\text{(17)} \qquad \alpha - \beta(\alpha; p, N, \nu) < \Pr\{\hat{T}^2_{\text{MAX}} > B^2_1(\alpha)\} < \alpha ,$$

where $B^2_1(\alpha) \equiv B^2_1(\alpha; p, N, \nu)$ and $\beta(\alpha; p, N, \nu) = \tfrac{1}{2} N(N-1) \Pr\{\hat{T}^2_1 > B^2_1(\alpha),$

$\hat{T}_2^2 > B_1^2(\alpha)\}$. In order to examine the accuracy of $B_1^2(\alpha)$, we must evaluate the value of $\beta(\alpha; p, N, \nu)$ or $\Pr\{\hat{T}_1^2 > B_1^2(\alpha), \hat{T}_2^2 > B_1^2(\alpha)\}$, which is done (asymptotically) by using the formulas (5.18) —(5.27) in [3]. Since, in the present case, $\boldsymbol{y}_i - \eta_i$ $(i=1, \cdots, N=k-1)$ are mutually independent, the formulas to be calculated are considerably shortened to the following:

$$\Pr\{\hat{T}_1^2 > B_1^2(\alpha),\ \hat{T}_2^2 > B_1^2(\alpha)\} = A_0 + \frac{A_1}{\nu} + \frac{A_2}{\nu^2} + 0(\nu^{-3}) \tag{18}$$

$$A_0 = (\alpha/N)^2\,, \tag{19}$$

$$A_1 = (p/2)\{(\chi^2+p)(\alpha/N)g_{p/2+1} + g_{p/2+1}^2\} - \frac{\chi^2(\chi^2+p)}{2}(\alpha/N)g_{p/2}\,, \tag{20}$$

$$\begin{aligned} A_2 = {} & \frac{\chi^2(\chi^2+p)}{16}\{(g_{p/2}-g_{p/2-1})(\alpha/N)+g_{p/2}^2\} \\ & - \frac{\chi^2\{4\chi^4+(13p-2)\chi^2+7p^2-4\}}{24}(\alpha/N)g_{p/2} \\ & - \frac{p\chi^2(\chi^2+p)}{4}\left[\frac{\chi^2+p}{2}\{(g_{p/2+1}-g_{p/2})(\alpha/N)+g_{p/2+1}g_{p/2}\}\right. \\ & \left. + (g_{p/2+1}-g_{p/2})g_{p/2+1} - (\alpha/N)g_{p/2+1}\right] \\ & - \frac{p(p+2)(p+4)}{3}\{(g_{p/2+3}-2g_{p/2+2}+g_{p/2+1})(\alpha/N) \\ & + 3(g_{p/2+2}-g_{p/2+1})g_{p/2+1}\} \\ & - p(p+2)(p+3)\{(g_{p/2+2}-g_{p/2+1})(\alpha/N-g_{p/2+1})+g_{p/2+1}^2\} \\ & - p(p^2+3p+4)(\alpha/N-g_{p/2+1})g_{p/2+1} \\ & + \frac{p(p+2)(p+4)(p+6)}{16}\{(g_{p/2+4}-3g_{p/2+3}+3g_{p/2+2}-g_{p/2+1})(\alpha/N) \\ & + 4(g_{p/2+3}-2g_{p/2+2}+g_{p/2+1})g_{p/2+1}+3(g_{p/2+2}-g_{p/2+1})^2\} \\ & + \frac{p(p+2)(p+4)(p+5)}{4}\{(g_{p/2+3}-2g_{p/2+2}+g_{p/2+1})(\alpha/N) \\ & - (g_{p/2+3}-5g_{p/2+2}+4g_{p/2+1})g_{p/2+1} - (g_{p/2+2}-g_{p/2+1})^2\} \\ & + \frac{p(p+2)(p^2+12p+23)}{4}\{(g_{p/2+2}-g_{p/2+1})(\alpha/N-2g_{p/2+1})+g_{p/2+1}^2\} \\ & + \frac{p(p+2)(p^2+4p+7)}{8}(g_{p/2+2}-g_{p/2+1})^2 \\ & + \frac{p(2p^2+5p+5)}{2}\{2(\alpha/N-g_{p/2+1})g_{p/2+1}-g_{p/2+1}^2\}\,, \end{aligned} \tag{21}$$

where $\chi^2 \equiv \chi^2(\alpha/N; p)$ is the upper 100 α/N per cent point of the chi-square distribution with p degrees of freedom and $g_m \equiv g_m(\chi^2/2) = [\Gamma(m)]^{-1}(\chi^2/2)^{m-1} e^{-\chi^2/2} (m>0)$ with exception that $g_0(\chi^2/2)=0$, and $g_{-1/2}(\chi^2/2) = -(4\pi)^{-1/2} (\chi^2/2)^{-3/2} e^{-\chi^2/2}$.

Table 5.1. *Values of A_0, A_1 and A_2*

$N(=k-1)$	$\alpha=0.05$			$\alpha=0.01$		
	A_0	A_1	A_2	A_0	A_1	A_2
	$p=1$					
2	$0.0^{3}625$	0.01052	0.05259	$0.0^{4}25$	$0.0^{3}9492$	0.01124
3	$0.0^{3}2778$	$0.0^{2}5917$	0.03312	$0.0^{4}1111$	$0.0^{3}4972$	$0.0^{2}7208$
4	$0.0^{3}1563$	$0.0^{2}3878$	0.02660	$0.0^{5}625$	$0.0^{3}3120$	$0.0^{2}5169$
5	$0.0^{3}1$	$0.0^{2}2775$	0.02202	$0.0^{5}4$	$0.0^{3}2165$	$0.0^{2}3957$
6	$0.0^{4}6944$	$0.0^{2}2102$	0.01867	$0.0^{5}2778$	$0.0^{3}1603$	$0.0^{2}3164$
7	$0.0^{4}5102$	$0.0^{2}1658$	0.01612	$0.0^{5}2041$	$0.0^{3}1247$	$0.0^{2}2618$
8	$0.0^{4}3906$	$0.0^{2}1347$	0.01413	$0.0^{5}1563$	$0.0^{4}9939$	$0.0^{2}2203$
10	$0.0^{4}25$	$0.0^{3}9492$	0.01124	$0.0^{5}1$	$0.0^{4}6840$	$0.0^{2}1651$
12	$0.0^{4}1736$	$0.0^{3}7108$	$0.0^{2}9235$	$0.0^{6}6944$	$0.0^{4}5018$	$0.0^{2}1297$
14	$0.0^{4}1276$	$0.0^{3}5555$	$0.0^{2}7790$	$0.0^{6}5102$	$0.0^{4}3878$	$0.0^{2}1059$
16	$0.0^{5}9766$	$0.0^{3}4480$	$0.0^{2}6698$	$0.0^{6}3906$	$0.0^{4}3089$	$0.0^{3}8846$
	$p=2$					
2	$0.0^{3}625$	$0.0^{2}8505$	0.06393	$0.0^{4}25$	$0.0^{3}7018$	0.01176
3	$0.0^{3}2778$	$0.0^{2}4657$	0.04414	$0.0^{4}1111$	$0.0^{3}3615$	$0.0^{2}7102$
4	$0.0^{3}1563$	$0.0^{2}3000$	0.03306	$0.0^{5}625$	$0.0^{3}2244$	$0.0^{2}4920$
5	$0.0^{3}1$	$0.0^{2}2121$	0.02609	$0.0^{5}4$	$0.0^{3}1545$	$0.0^{2}3670$
6	$0.0^{4}6944$	$0.0^{2}1592$	0.02133	$0.0^{5}2778$	$0.0^{3}1137$	$0.0^{2}2875$
7	$0.0^{4}5102$	$0.0^{2}1246$	0.01791	$0.0^{5}2041$	$0.0^{4}8759$	$0.0^{2}2360$
8	$0.0^{4}3906$	$0.0^{2}1006$	0.01534	$0.0^{5}1563$	$0.0^{4}6982$	$0.0^{2}1912$
10	$0.0^{4}25$	$0.0^{3}7018$	0.01176	$0.0^{5}1$	$0.0^{4}4772$	$0.0^{2}1426$
12	$0.0^{4}1736$	$0.0^{3}5215$	$0.0^{2}9410$	$0.0^{6}6944$	$0.0^{4}3491$	$0.0^{2}1104$
14	$0.0^{4}1276$	$0.0^{3}4050$	$0.0^{2}7767$	$0.0^{6}5102$	$0.0^{4}2678$	$0.0^{3}8867$
16	$0.0^{5}9766$	$0.0^{2}3249$	$0.0^{2}6560$	$0.0^{6}3906$	$0.0^{4}2126$	$0.0^{3}7324$
	$p=3$					
2	$0.0^{3}625$	$0.0^{2}7551$	0.07690	$0.0^{4}25$	$0.0^{3}5966$	0.01197
3	$0.0^{3}2778$	$0.0^{2}4082$	0.05040	$0.0^{4}1111$	$0.0^{3}3047$	$0.0^{2}7032$
4	$0.0^{3}1563$	$0.0^{2}2608$	0.03656	$0.0^{5}625$	$0.0^{3}1881$	$0.0^{2}4768$
5	$0.0^{3}1$	$0.0^{2}1833$	0.02820	$0.0^{5}4$	$0.0^{3}1290$	$0.0^{2}3509$
6	$0.0^{4}6944$	$0.0^{2}1369$	0.02266	$0.0^{5}2778$	$0.0^{4}9460$	$0.0^{2}2612$
7	$0.0^{4}5102$	$0.0^{2}1068$	0.01876	$0.0^{5}2041$	$0.0^{4}7211$	$0.0^{2}2172$
8	$0.0^{4}3906$	$0.0^{3}8596$	0.01589	$0.0^{5}1563$	$0.0^{4}5783$	$0.0^{2}1808$
10	$0.0^{4}25$	$0.0^{3}5966$	0.01197	$0.0^{5}1$	$0.0^{4}3938$	$0.0^{2}1311$
12	$0.0^{4}1736$	$0.0^{3}4408$	$0.0^{2}9430$	$0.0^{6}6944$	$0.0^{4}2873$	$0.0^{2}1006$
14	$0.0^{4}1276$	$0.0^{3}3419$	$0.0^{2}7708$	$0.0^{6}5102$	$0.0^{4}2208$	$0.0^{3}8003$
16	$0.0^{5}9766$	$0.0^{3}2736$	$0.0^{2}6450$	$0.0^{6}3906$	$0.0^{4}1791$	$0.0^{3}6743$

Table 5.1 shows the values of A_0, A_1 and A_2 calculated by the above formulas for $p=1, 2, 3$; $\alpha=0.05, 0.01$ and $N=k-1=2(1)8, 10(2)16$. When $p=1$, it is easily proved that

$$\text{(22)} \qquad \Pr\{\hat{T}_1^2>t^2,\ \hat{T}_2^2>t^2\}=4\int_{-\infty}^{t}\int_{-\infty}^{t} g_\nu(u, v; 0)\,du\,dv\ ,$$

where

$$\text{(23)} \qquad g_\nu(u, v; \rho)=\frac{1}{2\sqrt{1-\rho^2}}\left\{1+\frac{u-2\rho uv+v}{\nu}\right\}^{-(\nu+2)/2}$$

which is the bivariate generalization of Student's t-distribution. Dunnett and Sobel [1] have obtained exact formulas for evaluating the integral of (22) (formulas (10) and (11) in [1]). In order to see the degree of accuracy of the first three terms of our asymptotic formula (18), we shall compare the values obtained by (18) for $p=1$ with the exact values calculated by Dunnett and Sobel's formulas. Table 5.2 shows this comparison for $N=5, 10, 16$; $\nu=20, 30$ and $\alpha=0.05, 0.01$. From this table,

Table 5.2. *Comparison of values by* (18) *with exact values*

N $(k-1)$	ν	values by (18)	exact values	values by (18)	exact values
		$\alpha=0.05$		$\alpha=0.01$	
5	20	$0.0^3 2938$	$0.0^3 2931$	$0.0^4 2472$	$0.0^4 2668$
	30	$0.0^3 2170$	$0.0^3 2176$	$0.0^4 1562$	$0.0^4 1627$
10	20	$0.0^3 1005$	$0.0^3 1022$	$0.0^5 855$	$0.0^5 999$
	30	$0.0^4 6913$	$0.0^4 6973$	$0.0^5 512$	$0.0^5 558$
16	20	$0.0^4 4891$	$0.0^4 5165$	$0.0^4 415$	$0.0^5 521$
	30	$0.0^4 3214$	$0.0^4 3286$	$0.0^5 240$	$0.0^5 274$

it is seen that the discrepancy between the values by our formula (18) and the exact values is not so large at least, for $\nu\geq 20$, $16\geq N\geq 2$, $\alpha=0.05, 0.01$ and $p=1$, and does not matter for the evaluation of the second approximation to the upper 100α per cent point of $\hat{T}^2_{\text{MAX}}$, which will be explained below. Since, for fixed α, N, ν, $\beta(\alpha; p, N, \nu)$ with $p=1, 2, 3$ are found to have the magnitude of the same order, we shall roughly assume that, when $p\geq 2$, at least when $p=2, 3$, our asymptotic formula (18) gives also a satisfactory estimate of $\Pr\{\hat{T}_1^2>B_1^2(\alpha),\ \hat{T}_2^2>B_1^2(\alpha)\}$. From the numerical examination it is seen that the effect of $\beta(\alpha; p, N, \nu)$ is significant, especially when $\alpha=0.05$ and so the second approximation to $B^2(\alpha; p, N, \nu)$ is to be considered. However, since the great deal of labour is necessary to do this and, moreover, the second approximation underestimates the true value, it will be more appropriate for our present evaluation to use the following modified procedure than to use the second approximation itself. The modified second approximate upper 100α per cent point of $\hat{T}^2_{\text{MAX}}$, which is denoted by $B_2^{*2}(\alpha; p, N, \nu)$, is given by solving

$$\alpha+\beta(\alpha; p, N, \nu)=N\Pr\{\hat{T}_1^2=\boldsymbol{y}_1'\boldsymbol{L}^{-1}\boldsymbol{y}_1>B_2^{*2}(\alpha; p, N, \nu)\}\ , \tag{24}$$

where $\beta(\alpha; p, N, \nu)$ is the value calculated for $B_1^2(\alpha)$. This estimate will have the good accuracy for practical applications and is given in Table I at the end of this paper.

6. Evaluation of the upper 100α per cent point of $\hat{T}^2_{\text{MAX}\cdot\text{C}}$ when $n_1=n_2=\cdots=n_k$.

In this paragraph, we shall consider the evaluation of the upper 100α per cent point, $A^2(\alpha; p, N, \nu)$, of $\hat{T}^2_{\text{MAX}\cdot\text{C}}$, which was defined in Section 3. Since the evaluation is extremely complicated in the general case when the sample sizes, $n_1,\cdots,n_k$, are not the same, we shall hereafter treat the special case when $n_1=\cdots=n_k\equiv n$ and consider the extreme statistic $n\hat{T}^2_{\text{MAX}\cdot\text{C}}$, that is,

$$\begin{aligned} n\hat{T}^2_{\text{MAX}\cdot\text{C}}&=\max_i\{n\hat{T}^2_{i\cdot\text{C}}\} \\ &=\max_i\{n(\bar{\boldsymbol{x}}_i-\bar{\boldsymbol{x}}_k-\mu_i+\mu_k)'\boldsymbol{L}^{-1}(\bar{\boldsymbol{x}}_i-\bar{\boldsymbol{x}}_k-\mu_i+\mu_k)\}\ . \end{aligned} \tag{25}$$

In this case we can obtain the good estimate of the upper 100α per cent point, $W^2(\alpha; p, N, \nu)$, of $n\hat{T}^2_{\text{MAX}\cdot\text{C}}$ in the same way as in the last section. Obviously $W^2(\alpha; p, N, \nu)=nA^2(\alpha; p, N, \nu)$.

The first approximation to $W^2(\alpha; p, N, \nu)$ is given by $W_1^2(\alpha; p, N, \nu)$ $(\equiv W_1^2(\alpha))$ satisfying the equation

$$\alpha/N=\Pr\{n\hat{T}^2_{1\cdot\text{C}}=n(\bar{\text{x}}_1-\bar{\text{x}}_k-\mu_1+\mu_k)'\boldsymbol{L}^{-1}(\bar{\text{x}}_1-\bar{\text{x}}_k-\mu_1+\mu_k)>W_1^2(\alpha)\}\ , \tag{26}$$

that is,

$$W_1^2(\alpha; p, N, \nu)=2\nu\left\{\frac{1}{C(\alpha/N; (\nu+1-p)/2, p/2)}-1\right\}\ , \tag{27}$$

where $C(\alpha^*; a, b)$ is again the lower $100\alpha^*$ per cent point of the beta-distribution with parameters a and b.

The modified second approximation, $W_2^{*2}(\alpha; p, N, \nu)$, is given by solving

$$(28) \qquad \alpha+\beta^*(\alpha; p, N, \nu)=N\Pr\{n\hat{T}^2_{1\cdot c}>W_2^{*2}(\alpha; p, N, \nu)\} ,$$

where

$$\beta^*(\alpha; p, N, \nu)=\tfrac{1}{2}N(N-1)\Pr\{n\hat{T}^2_{1\cdot c}>W_1^2(\alpha),\ n\hat{T}^2_{2\cdot c}>W_1^2(\alpha)\} .$$

The asymptotic expression for $\Pr\{n\hat{T}^2_{1\cdot c}>W_1^2(\alpha),\ n\hat{T}^2_{2\cdot c}>W_1^2(\alpha)\}$ is written in the form

$$(29) \qquad \Pr\{n\hat{T}^2_{1\cdot c}>W_1^2(\alpha),\ n\hat{T}^2_{2\cdot c}>W_1^2(\alpha)\}=B_0+\frac{B_1}{\nu}+\frac{B_2}{\nu^2}+0(\nu^{-3}) .$$

Since $\sqrt{n}(\bar{\mathbf{x}}_i-\bar{\mathbf{x}}_k-\mu_i+\mu_k)$ is distributed according to $N(0, 2\Lambda)$ and the covariance matrix of $\sqrt{n}(\bar{\mathbf{x}}_i-\bar{\mathbf{x}}_k-\mu_i+\mu_k)$ and $\sqrt{n}(\bar{\mathbf{x}}_j-\bar{\mathbf{x}}_k-\mu_j+\mu_k)$, $(i\neq j)$ is Λ, the calculation of B_0, B_1 and B_2 can be done by the formulas (5.18)—(5.27) in [3] with $\gamma=2$ and $\delta=1$. Table 6.1 gives the values of B_0, B_1 and B_2 for $p=2$ and $\alpha=0.05, 0.01$; $N=k-1=2(1)8, 10(2)16$.

Table 6.1. *Values of B_0, B_1 and B_2 when $p=2$*

$N(=k-1)$	$\alpha=0.05$			$\alpha=0.01$		
	B_0	B_1	B_2	B_0	B_1	B_2
2	$0.0^{2}3156$	0.01832	0.02216	$0.0^{3}3236$	$0.0^{2}3974$	0.01140
3	$0.0^{2}1771$	0.01277	0.01932	$0.0^{3}1833$	$0.0^{2}2618$	$0.0^{2}9086$
4	$0.0^{2}1178$	$0.0^{2}9775$	0.01757	$0.0^{3}1226$	$0.0^{2}1937$	$0.0^{2}7630$
5	$0.0^{3}8590$	$0.0^{2}7900$	0.01587	$0.0^{4}8975$	$0.0^{2}1528$	$0.0^{2}6616$
6	$0.0^{3}6641$	$0.0^{2}6393$	0.01465	$0.0^{4}6959$	$0.0^{2}1257$	$0.0^{2}5862$
7	$0.0^{3}5344$	$0.0^{2}5685$	0.01364	$0.0^{4}5614$	$0.0^{2}1065$	$0.0^{2}5277$
8	$0.0^{3}4428$	$0.0^{2}4977$	0.01279	$0.0^{4}4662$	$0.0^{3}9211$	$0.0^{2}4807$
10	$0.0^{3}3236$	$0.0^{2}3974$	0.01140	$0.0^{4}3418$	$0.0^{3}7221$	$0.0^{2}4098$
12	$0.0^{3}2505$	$0.0^{2}3298$	0.01033	$0.0^{4}2653$	$0.0^{3}5909$	$0.0^{2}3583$
14	$0.0^{3}2019$	$0.0^{2}2813$	$0.0^{2}9460$	$0.0^{4}2142$	$0.0^{3}4984$	$0.0^{2}3192$
16	$0.0^{3}1674$	$0.0^{2}2448$	$0.0^{2}8746$	$0.0^{4}1779$	$0.0^{3}4298$	$0.0^{2}2882$

When $p=1$, we have, for any positive w,

$$(30) \qquad \Pr\{nT^2_{1\cdot c}>w^2,\ nT^2_{2\cdot c}>w^2\}$$
$$=\left[\int_{-\infty}^{-\frac{w}{\sqrt{2}}}\int_{-\infty}^{-\frac{w}{\sqrt{2}}}+\int_{-\infty}^{-\frac{w}{\sqrt{2}}}\int_{\frac{w}{\sqrt{2}}}^{\infty}+\int_{\frac{w}{\sqrt{2}}}^{\infty}\int_{-\infty}^{-\frac{w}{\sqrt{2}}}+\int_{\frac{w}{\sqrt{2}}}^{\infty}\int_{\frac{w}{\sqrt{2}}}^{\infty}\right]g_\nu(u, v; 0.5)dudv ,$$

where $g_\nu(u, v; 0.5)$ is the density of the bivariate Student's t-distribution (23) when $\rho=0.5$, from which the exact value of $\Pr\{n\hat{T}^2_{1\cdot C}>W^2_1(\alpha), n\hat{T}^2_{2\cdot C}>W^2_1(\alpha)\}$ can be calculated. Table 6.2 below corresponds to Table 5.2 in the last section and gives the comparison between the exact values obtained from (30) and the values from the first three terms of (29).

Table 6.2. *Comparison of values by* (29) *with exact values*

N $(k-1)$	ν	values by (29)	exact values	values by (29)	exact values
		$\alpha=0.05$		$\alpha=0.01$	
5	20	$0.0^{2}1371$	$0.0^{2}1367$	$0.0^{3}1925$	$0.0^{3}1911$
	30	$0.0^{2}1242$	$0.0^{2}1241$	$0.0^{3}1623$	$0.0^{3}1622$
10	20	$0.0^{3}5835$	$0.0^{3}5810$	$0.0^{4}8476$	$0.0^{4}8253$
	30	$0.0^{3}5135$	$0.0^{3}5129$	$0.0^{4}6909$	$0.0^{4}6876$
16	20	$0.0^{3}3299$	$0.0^{3}3275$	$0.0^{4}4882$	$0.0^{4}4796$
	30	$0.0^{3}2842$	$0.0^{3}2836$	$0.0^{4}3886$	$0.0^{4}3862$

$$\alpha=0.05\begin{cases} N=5: & B_0=0.0^{3}9926,\ B_1=0.0^{3}7316,\ B_2=0.0^{2}5074 \\ N=10: & B_0=0.0^{3}3817,\ B_1=0.0^{2}3790,\ B_2=0.0^{2}4917 \\ N=16: & B_0=0.0^{3}2002,\ B_1=0.0^{2}2377,\ B_2=0.0^{2}4329 \end{cases}$$

$$\alpha=0.01\begin{cases} N=5: & B_0=0.0^{3}1079,\ B_1=0.0^{2}1513,\ B_2=0.0^{2}3589 \\ N=10: & B_0=0.0^{4}4199,\ B_1=0.0^{2}7293,\ B_1=0.0^{2}2520 \\ N=16: & B_0=0.0^{4}2208,\ B_1=0.0^{3}4405,\ B_2=0.0^{2}1885 \end{cases}$$

It is seen from this table that estimated values have considerably sufficient accuracy. The values, $W^{*2}_2(\alpha)$, obtained by the modified second approximation procedure (28) are contained in Table II at the end of this paper.

7. Acknowledgement.

I wish to express my gratitude to Miss K. Yoshida for her help in numerical work, and also to Miss E. Ozaki for her help in the preparation of this paper.

THE INSTITUTE OF STATISTICAL MATHEMATICS

TABLE Ia. Upper 5% points of $\hat{T}^2_{\text{MAX}}$ when $p=2$. $(N=k-1)$

ν \ N	2	3	4	5	6	7	8	10	12	14	16
20	9.415	10.656	11.569	12.422	12.900	13.418	13.875	14.650	15.294	15.848	16.331
22	9.194	10.385	11.259	11.953	12.529	13.025	13.458	14.195	14.807	15.332	15.798
24	9.015	10.167	11.010	11.678	12.232	12.707	13.124	13.830	14.416	14.918	15.357
26	8.861	9.987	10.805	11.452	11.989	12.448	12.851	13.532	14.096	14.580	15.002
28	8.744	9.837	10.634	11.263	11.785	12.230	12.622	13.283	13.830	14.299	14.707
30	8.639	9.709	10.489	11.103	11.612	12.047	12.428	13.072	13.605	14.060	14.457
32	8.549	9.600	10.364	10.966	11.464	11.890	12.263	12.892	13.411	13.856	14.243
34	8.471	9.505	10.256	10.847	11.336	11.753	12.119	12.735	13.244	13.679	14.058
36	8.402	9.421	10.161	10.742	11.224	11.634	11.993	12.598	13.098	13.525	13.896
38	8.341	9.348	10.077	10.651	11.124	11.528	11.882	12.477	12.969	13.388	13.754
40	8.287	9.282	10.003	10.568	11.036	11.434	11.783	12.370	12.854	13.267	13.626
42	8.238	9.223	9.936	10.495	10.957	11.351	11.695	12.274	12.752	13.159	13.513
44	8.195	9.170	9.875	10.429	10.886	11.275	11.615	12.187	12.659	13.061	13.411
46	8.155	9.123	9.821	10.369	10.822	11.207	11.543	12.109	12.576	12.973	13.319
48	8.119	9.079	9.771	10.315	10.763	11.145	11.478	12.035	12.500	12.893	13.235
50	8.086	9.039	9.726	10.265	10.710	11.088	11.418	11.973	12.430	12.820	13.159
55	8.015	8.953	9.629	10.158	10.595	10.966	11.289	11.834	12.281	12.662	12.994
60	7.956	8.882	9.548	10.070	10.500	10.865	11.184	11.719	12.158	12.533	12.858
65	7.907	8.823	9.481	9.997	10.421	10.781	11.095	11.622	12.057	12.425	12.746
70	7.865	8.772	9.424	9.934	10.353	10.710	11.020	11.541	11.969	12.334	12.650
80	7.798	8.692	9.333	9.834	10.246	10.595	10.900	11.410	11.830	12.187	12.496
90	7.747	8.629	9.262	9.756	10.163	10.508	10.807	11.311	11.724	12.074	12.379
100	7.701	8.580	9.206	9.696	10.097	10.438	10.734	11.231	11.639	11.985	12.286
120	7.645	8.507	9.124	9.605	10.000	10.335	10.626	11.114	11.514	11.854	12.148
150	7.585	8.435	9.042	9.516	9.904	10.233	10.519	10.998	11.391	11.723	12.012
200	7.526	8.364	8.961	9.428	9.809	10.133	10.414	10.884	11.270	11.596	11.879

TABLE Ib. Upper 1% points of $\hat{T}^2_{\mathrm{MAX}}$ when $p=2$, $(N=k-1)$

ν \ N	2	3	4	5	6	7	8	10	12	14	16
20	14.901	16.403	17.507	18.386	19.121	19.751	20.307	21.249	22.035	22.711	23.305
22	14.411	15.828	16.867	17.691	18.379	18.968	19.487	20.366	21.097	21.726	22.277
24	14.021	15.371	16.358	17.136	17.791	18.349	18.839	19.668	20.357	20.949	21.467
26	13.702	14.999	15.945	16.692	17.315	17.847	18.314	19.103	19.759	20.321	20.812
28	13.438	14.690	15.602	16.322	16.920	17.432	17.880	18.637	19.265	19.803	20.273
30	13.214	14.430	15.313	16.010	16.589	17.083	17.515	18.246	18.851	19.369	19.821
32	13.024	14.208	15.067	15.744	16.306	16.786	17.205	17.913	18.499	19.000	19.437
34	12.859	14.016	14.855	15.516	16.063	16.529	16.938	17.626	18.196	18.682	19.107
36	12.714	13.849	14.670	15.316	15.850	16.306	16.705	17.377	17.932	18.406	18.820
38	12.587	13.701	14.507	15.140	15.664	16.110	16.500	17.158	17.701	18.164	18.568
40	12.475	13.571	14.362	14.985	15.499	15.937	16.319	16.964	17.496	17.950	18.346
42	12.374	13.454	14.234	14.846	15.351	15.782	16.158	16.791	17.314	17.758	18.147
44	12.284	13.349	14.118	14.721	15.219	15.643	16.013	16.636	17.150	17.586	17.970
46	12.202	13.255	14.014	14.609	15.100	15.518	15.883	16.497	17.003	17.434	17.810
48	12.128	13.169	13.919	14.507	14.992	15.405	15.765	16.370	16.869	17.294	17.665
50	12.060	13.090	13.832	14.414	14.893	15.301	15.657	16.255	16.748	17.167	17.533
55	11.914	12.922	13.647	14.214	14.682	15.079	15.425	16.008	16.487	16.895	17.250
60	11.795	12.784	13.495	14.051	14.508	14.898	15.236	15.806	16.274	16.672	17.019
65	11.695	12.669	13.368	13.915	14.364	14.746	15.079	15.638	16.097	16.487	16.827
70	11.611	12.572	13.261	13.799	14.242	14.618	14.946	15.496	15.947	16.331	16.665
80	11.476	12.416	13.089	13.615	14.047	14.414	14.733	15.269	15.708	16.082	16.406
90	11.393	12.297	12.958	13.474	13.898	14.258	14.571	15.095	15.526	15.891	16.209
100	11.290	12.203	12.855	13.363	13.781	14.135	14.443	14.959	15.381	15.742	16.054
120	11.169	12.063	12.702	13.199	13.607	13.953	14.254	14.757	15.170	15.520	15.824
150	11.050	11.926	12.551	13.038	13.437	13.775	14.068	14.560	14.962	15.304	15.600
200	10.932	11.792	12.404	12.879	13.270	13.600	13.886	14.366	14.759	15.091	15.380

TABLE IIa. Upper 5% points of $n\hat{T}^2_{\mathrm{MAX.C}}$ when $p=2$. $(N=k-1)$

ν \ N	2	3	4	5	6	7	8	10	12	14	16
16	19.747	22.164	23.878	25.199	26.289	27.153	27.911	29.150	30.130	30.933	31.608
18	19.033	21.312	22.929	24.174	25.200	26.019	26.736	27.910	28.823	29.605	30.251
20	18.489	20.666	22.208	23.397	24.374	25.159	25.844	26.968	27.862	28.597	29.219
22	18.061	20.159	21.644	22.787	23.727	24.484	25.146	26.230	27.094	27.806	28.409
24	17.716	19.750	21.189	22.297	23.206	23.942	24.583	25.636	26.476	27.169	27.756
26	17.432	19.414	20.815	21.894	22.778	23.496	24.121	25.147	25.979	26.644	27.219
28	17.195	19.132	20.502	21.556	22.420	23.123	23.734	24.739	25.542	26.205	26.770
30	16.993	18.893	20.236	21.270	22.116	22.806	23.406	24.392	25.181	25.833	26.388
32	16.819	18.688	20.008	21.024	21.856	22.534	23.124	24.094	24.870	25.513	26.060
34	16.668	18.509	19.810	20.811	21.629	22.298	22.879	23.835	24.601	25.235	25.774
36	16.536	18.353	19.636	20.624	21.430	22.091	22.664	23.608	24.364	24.991	25.525
38	16.418	18.215	19.483	20.458	21.255	21.909	22.475	23.408	24.155	24.775	25.303
40	16.313	18.092	19.346	20.311	21.099	21.746	22.306	23.230	23.956	24.584	25.106
42	16.220	17.981	19.223	20.179	20.959	21.600	22.155	23.070	23.803	24.411	24.930
44	16.135	17.882	19.113	20.060	20.833	21.469	22.019	22.926	23.653	24.257	24.772
46	16.059	17.792	19.014	19.953	20.719	21.350	21.896	22.796	23.517	24.117	24.628
48	15.989	17.710	18.922	19.855	20.615	21.242	21.784	22.677	23.394	23.989	24.497
50	15.926	17.635	18.839	19.766	20.521	21.143	21.681	22.568	23.281	23.872	24.377
55	15.789	17.473	18.660	19.573	20.316	20.929	21.460	22.335	23.037	23.621	24.118
60	15.675	17.340	18.512	19.414	20.147	20.754	21.278	22.142	22.836	23.413	23.906
65	15.581	17.229	18.389	19.281	20.008	20.607	21.126	21.982	22.669	23.240	23.727
70	15.500	17.134	18.284	19.168	19.887	20.483	20.997	21.845	22.526	23.093	23.576
80	15.371	16.982	18.116	18.987	19.695	20.282	20.789	21.625	22.297	22.856	23.333
90	15.271	16.865	17.986	18.848	19.548	20.129	20.630	21.456	22.121	22.674	23.147
100	15.193	16.773	17.884	18.737	19.431	20.007	20.504	21.323	21.982	22.530	22.999
120	15.075	16.636	17.732	18.574	19.257	19.826	20.316	21.125	21.775	22.316	22.780
150	14.960	16.500	17.581	18.412	19.086	19.648	20.131	20.929	21.571	22.105	22.563
200	14.845	16.366	17.433	18.253	18.917	19.472	19.948	20.736	21.362	21.897	22.349

TABLE IIb. Upper 1% points of $n\hat{T}^2_{\mathrm{MXA.C}}$ when $p=2$. ($N=k-1$)

ν \ N	2	3	4	5	6	7	8	10	12	14	16
16	32.341	35.475	37.710	39.437	40.836	42.005	43.006	44.645	45.946	47.016	47.917
18	30.701	33.588	35.645	37.236	38.526	39.605	40.531	42.050	43.260	44.258	45.101
20	29.474	32.179	34.104	35.595	36.804	37.817	38.686	40.114	41.256	42.199	42.998
22	28.522	31.087	32.913	34.325	35.472	36.433	37.259	38.618	39.706	40.606	41.167
24	27.763	30.217	31.964	33.315	34·412	35.332	36.123	37.426	38.471	39.336	40.073
26	27.143	29.509	31.191	32.492	33.549	34.436	35.198	36.456	37.465	38.302	39.015
28	26.628	28.920	30.549	31.809	32.833	33.692	34.431	35.650	36.630	37.444	38.137
30	26.193	28.423	30.007	31.233	32.229	33.065	33.784	34.971	35.926	36.719	37.396
32	25.822	27.999	29.545	30.741	31.713	32.528	33.230	34.391	35.324	36.100	36.763
34	25.499	27.631	29.145	30.316	31.267	32.067	32.754	33.890	34.805	35.566	36.216
36	25.218	27.311	28.797	29.945	30.879	31.663	32.337	33.453	34.351	35.100	35.739
38	24.970	27.029	28.489	29.619	30.536	31.307	31.971	33.068	33.952	34.688	35.318
40	24.750	26.778	28.217	29.329	30.233	30.992	31.646	32.727	33.598	34.324	34.945
42	24.553	26.555	27.974	29.071	29.962	30.711	31.356	32.422	33.282	33.999	34.612
44	24.377	26.354	27.755	28.838	29.718	30.458	31.095	32.148	32.998	33.706	34.312
46	24.217	26.172	27.558	28.628	29.499	30.230	30.859	31.901	32.742	33.442	34.043
48	24.072	26.008	27.379	28.438	29.299	30.023	30.646	31.677	32.509	33.203	33.797
50	23.940	25.857	27.215	28.264	29.117	29.834	30.451	31.472	32.294	32.984	33·573
55	23.655	25.534	26.865	27.891	28.726	29.428	30.032	31.033	31.840	32.514	33.092
60	23.422	25.269	26.575	27.585	28.405	29.095	29.689	30.672	31.466	32.129	32.697
65	23.227	25.047	26.335	27.330	28.138	28.817	29.402	30.372	31.154	31.807	32.368
70	23.062	24.858	26.132	27.114	27.912	28.582	29.160	30.117	30.890	31.536	32.089
80	22.792	24.557	25.806	26.768	27.549	28.206	28.772	29.710	30.467	31.100	31.644
90	22.594	24.330	25.557	26.503	27.272	27.919	28.474	29.398	30.144	30.767	31.300
100	22.434	24.148	25.359	26.294	27.052	27.691	28.241	29.152	29.888	30.504	31.032
120	22.197	23.880	25.068	25.985	26.730	27.355	27.894	28.788	29.510	30.115	30.634
150	21.964	23.615	24.782	25.681	26.411	27.025	27.554	28.431	29.139	29.732	30.241
200	21.734	23.357	24.500	25.382	26.098	26.700	27.219	28.080	28.774	29.356	29.852

REFERENCES

[1] C. W. Dunnett and M. Sobel, "A bivariate generalization of Student's t-distribution, with tables for certain special cases," *Biometrika*, Vol. 41 (1954), pp. 153–169.

[2] S. N. Roy and R. C. Bose, "Simultaneous confidence interval estimation," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 513-536.

[3] M. Siotani, "The extreme value of the gereralized distances of the individual points in the multivariate normal sample," *Ann. Inst. Stat. Math.*, Vol. 10 (1959), pp. 183–203.

[4] M. Siotani "On the range in multivariate case," *Proc. Inst. Stat. Math.*, Vol. 6 (1959) pp. 155–165 (in Japanese).