EFFECT OF TIMING-ERROR ON THE POWER SPECTRUM OF SAMPLED-DATA

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(Received Feb. 15, 1960)

0. Introduction and summary

Nowadays it is very common to use time-sampled data to analyse or control an object fluctuating continuously in time. This seems to be motivated by the recent developments of digital methods serving for these purposes. In relation to such procedures there are many papers which treat the noise or error due to quantization ([4], [5]). The effect of time-sampling on the spectral properties is also well-known as folding or aliasing for the case where the fluctuation of object is represented by a stationary stochastic process and the timings are performed without error. As to the cases where timing-errors are present we have seen yet little quantitative description of their effects on the spectral properties of the time-sampled data [3]. In the present paper we treat this problem for the case where the timing is independent of the original process and the intervals between sampling-time points form a stationary process. After the general discussion of this case we treat two special types of time-sampling in more details. The first corresponds to the case where, though it is intended to sample the record at the time points which are the integral multiples of a constant time Δt, the deviations of the sampling-time points from the preassigned ones are present and form a purely random process, i.e., they are random variables which are mutually independent and follow one and the same probability distribution. The second corresponds to the case where the sampling-time points form a renewal process, i.e., the interval lengthes between successive sampling-time points form a purely random process. The results of our analyses show clearly how the timing-error affects the power spectral distribution function of the time-sampled data. The effects are essentially non-linear but the time-sampling of the first type may be described as a low-pass filter with an inner white noise source. We can see further that in practical applications even the time-sampling procedures of the second type may sometimes act approximately as a low-pass filter with an inner white noise source. These results seem to give a mathematical explanation of the fact, mentioned by D. T. Ross in [3], that the time-sampling was sometimes vaguely (and incorrectly) considered to be a filtering. By using the continuous records of the outputs of accelerometers mounted on the frame and on the front axle of an automobile we illustrate these theoretical results by numerical examples. These records were obtained by the members of the research department of the Isuzu Motor Company and were presented to the author by Mr. Itiro Kanesige of the department for the purpose to develop the statistical research of the relations between vehicular oscillations and road surfaces. When we were reading these records using a rule we had to face two main sources of error. The one was the error in the horizontal position of the rule and the other was that due to quantization. The results of our investigation described in this paper enable us to make a quantitative evaluation of the effect of the reading-errors on the forms of the power spectral distribution functions of these time-sampled data.

1. Spectral properties of sampled-data.

We shall here consider a strictly stationary real stochastic process $\{x(t, \omega);\ -\infty < t < \infty\}$ with continuous time parameter t. It is assumed that

a.1. the process has zero-mean and finite second order moments,

a.2. almost all sample functions of the process have finite limit from the right,

$$x(t+, \omega) = \lim_{s \downarrow t} x(s, \omega)$$

for all t, and

a.3. the process is continuous in the sense of mean square,

$$\lim_{s-t \to 0} E\{|x(s, \omega) - x(t, \omega)|^2\} = 0 \ .$$

It is fairly obvious that in almost every physical realizations of stochastic processes these conditions are satisfied. To see further that the condition a.3. really does not seriously restrict the generality of the process, the reader is recommended to consult the book [1] by J. L. Doob. Here we take another strictly stationary stochastic process $\{\Delta\tau_n(\omega);\ -\infty < n < \infty\}$ defined on the same ω space as $x(t, \omega)$ process and with discrete time parameter n. We define $\tau_n(\omega)$ by the followings;

$$\tau_0(\omega)=\varepsilon(\omega)$$
$$\tau_n(\omega)-\tau_{n-1}(\omega)=\Delta\tau_{n-1}(\omega)$$

where $\varepsilon(\omega)$ is a random variable. Now we define a sequence of ω functions $\{x_{\tau,n}(\omega);\ -\infty<n<\infty\}$ as follows;

$$\begin{aligned} x_{\tau,n}(\omega)&=x(\tau_n(\omega),\omega) && \text{if Prob }\{\omega';\ \tau_n(\omega')=\tau_n(\omega)\}>0 \\ &=x(\tau_n(\omega)+,\omega) && \text{otherwise.}\end{aligned}$$

We can see that the function $x_{\tau,n}(\omega)$ is well defined by the assumption a.2. To see that $\{x_{\tau,n}(\omega);\ -\infty<n<\infty\}$ forms a stochastic process we adopt the following discrete approximation procedure described in [2]. Denote by S_τ the set of values which some τ_n takes with positive probability. Obviously S_τ is at most enumerably infinite. For each positive integer q choose finitely many points

$$a_1^{(q)}<a_2^{(q)}<\cdots<a_{n_q}^{(q)}$$

in such a way that first q point of S_τ enumerated in some order are $a_j^{(q)}$'s and that every points in the interval $[-q,q]$ is within distance $1/q$ of some $a_j^{(q)}$ and $\pm\infty$ and 0 are some $a_j^{(q)}$'s. Define the stochastic process $\{\tau_n^{(q)}(\omega);\ -\infty<n<\infty\}$ by

$$\begin{aligned}\tau_n^{(q)}(\omega)&=a_1^{(q)} && \text{if } \tau_n(\omega)\le a_1^{(q)} \\ &=a_j^{(q)} && \text{if } a_{j-1}^{(q)}<\tau_n(\omega)\le a_j^{(q)} \\ &=0 && \text{if } \tau_n(\omega)>a_{n_q}^{(q)}.\end{aligned}$$

If we define

$$x_{\tau^{(q)},n}(\omega)=x(\tau_n^{(q)}(\omega),\omega)\ ,$$

it is obvious that $x_{\tau^{(q)},n}(\omega)$ is a measurable ω function, i.e., a random variable and

$$\lim_{q\to\infty} x_{\tau^{(q)},n}(\omega)=x_{\tau,n}(\omega)$$

holds with probability 1. Hence $x_{\tau,n}(\omega)$ is a random variable. Hereafter we shall sometimes omit the variable ω in the expression of random variables. Now consider a set $\{m_\nu;\ \nu=1,2,\cdots,k\}$ of arbitrary finite number of integers satisfying the relation $m_1<m_2<\cdots<m_k$ and a set $\{\xi_{m_\nu};\ \nu=1,2,\cdots,k\}$ of real numbers. Then we have

$$\begin{aligned}&\text{Prob }\{x_{\tau^{(q)},m_1}\le\xi_{m_1},\ x_{\tau^{(q)},m_2}\le\xi_{m_2},\ \cdots,\ x_{\tau^{(q)},m_k}\le\xi_{m_k}\} \\ &\quad=\sum_{(j_1,j_2,\cdots,j_k)}\text{Prob }\{x(a_{j_1})\le\xi_{m_1},\ x(a_{j_2})\le\xi_{m_2},\ \cdots,\ x(a_{j_k})\le\xi_{m_k} \text{ and} \\ &\qquad\qquad \tau_{m_1}^{(q)}=a_{j_1},\ \tau_{m_2}^{(q)}=a_{j_2},\ \cdots,\ \tau_{m_k}^{(q)}=a_{j_k}\}\end{aligned}$$

where a_{j_ν} denotes $a_{j_\nu}^{(q)}$ for $\nu=1, 2, \cdots, k$ and the summation is taken over all the possible arrangements $(j_1, j_2, \cdots, j_k)$ where j_ν is one of $(1, 2, \cdots, n_q)$. Denote by $\phi^{(q)}(\lambda_1, \lambda_2, \cdots, \lambda_k; m_1, m_2, \cdots, m_k)$ the characteristic function of $(x_{\tau^{(q)},m_1}, x_{\tau^{(q)},m_2}, \cdots, x_{\tau^{(q)},m_k})$ and by $\phi^{(q)}(\lambda_1, \lambda_2, \cdots, \lambda_k \mid a_{j_1}, a_{j_2}, \cdots, a_{j_k}; m_1, m_2, \cdots, m_k)$ the conditional characteristic function of $(x_{\tau^{(q)},m_1}, x_{\tau^{(q)},m_2}, \cdots, x_{\tau^{(q)},m_k})$ conditioned by the random variables $(\tau_{m_1}^{(q)}, \tau_{m_2}^{(q)}, \cdots, \tau_{m_k}^{(q)})$. Then we have

$$\begin{aligned}\phi^{(q)}&(\lambda_1, \lambda_2, \cdots, \lambda_k; m_1, m_2, \cdots, m_k)\\ &= \sum_{(j_1,j_2,\cdots,j_k)} \phi^{(q)}(\lambda_1, \lambda_2, \cdots, \lambda_k \mid a_{j_1}, a_{j_2}, \cdots, a_{j_k}; m_1, m_2, \cdots, m_k)\\ &\qquad \times \operatorname{Prob}(\tau_{m_1}^{(q)}=a_{j_1}, \tau_{m_2}^{(q)}=a_{j_2}, \cdots, \tau_{m_k}^{(q)}=a_{j_k}) .\end{aligned}$$

Denote by $\phi(\lambda_1, \lambda_2, \cdots, \lambda_k; m_1, m_2, \cdots, m_k)$ the characteristic function of $(x_{\tau,m_1}, x_{\tau,m_2}, \cdots, x_{\tau,m_k})$. Then we have

$$\phi(\lambda_1, \lambda_2, \cdots, \lambda_k; m_1, m_2, \cdots, m_k)=\lim_{q\to\infty} \phi^{(q)}(\lambda_1, \lambda_2, \cdots, \lambda_k; m_1, m_2, \cdots, m_k) .$$

Hereafter we shall restrict our attention to the case where $\{\tau_n(\omega)\}$ and $\{x(t, \omega)\}$ are mutually independent. For this case we can put

$$\begin{aligned}\phi^{(q)}(\lambda_1, \lambda_2, \cdots, \lambda_k \mid a_{j_1}, a_{j_2}, \cdots, a_{j_k}; m_1, m_2, \cdots, m_k)&\\ =\psi(\lambda_1, \lambda_2, \cdots, \lambda_k; a_{j_1}, a_{j_2}, \cdots, a_{j_k})&\end{aligned}$$

where $\psi(\lambda_1, \lambda_2, \cdots, \lambda_k; a_{j_1}, a_{j_2}, \cdots, a_{j_k})$ denotes the characteristic function of $(x(a_{j_1}, \omega), x(a_{j_2}, \omega), \cdots, x(a_{j_k}, \omega))$. From the continuity assumption a.3. for the $x(t)$ process it follows that $\psi(\lambda_1, \lambda_2, \cdots, \lambda_k; a_{j_1}, a_{j_2}, \cdots, a_{j_k})$ is continuous in $(a_{j_1}, a_{j_2}, \cdots, a_{j_k})$. Obviously $\lim_{q\to\infty} \tau_n^{(q)}(\omega)=\tau_n(\omega)$ with probability 1 and thus the finite dimensional distribution of $(\tau_{m_1}^{(q)}, \tau_{m_2}^{(q)}, \cdots, \tau_{m_k}^{(q)})$ converges to that of $(\tau_{m_1}, \tau_{m_2}, \cdots, \tau_{m_k})$ as $q\to\infty$. Thus taking into account the boundedness of $\psi(\lambda_1, \lambda_2, \cdots, \lambda_k; a_{j_1}, a_{j_2}, \cdots, a_{j_k})$ we have

$$\begin{aligned}\phi&(\lambda_1, \lambda_2, \cdots, \lambda_k; m_1, m_2, \cdots, m_k)\\ &=\int \psi(\lambda_1, \lambda_2, \cdots, \lambda_k; a_{j_1}, a_{j_2}, \cdots, a_{j_k}) dP_{m_1,m_2,\cdots,m_k}(a_{j_1}, a_{j_2}, \cdots, a_{j_k})\end{aligned}$$

where $P_{m_1,m_2,\cdots,m_k}(a_{j_1}, a_{j_2}, \cdots, a_{j_k})=\operatorname{Prob}(\tau_{m_1}\leq a_{j_1}, \tau_{m_2}\leq a_{j_2}, \cdots, \tau_{m_k}\leq a_{j_k})$. From the strict stationarity of the $x(t)$ process

$$\psi(\lambda_1, \lambda_2, \cdots, \lambda_k; a_{j_1}, a_{j_2}, \cdots, a_{j_k})$$

can be represented in the form

$$\hat{\psi}(\lambda_1, \lambda_2, \cdots, \lambda_k; a_{j_2}-a_{j_1}, a_{j_3}-a_{j_2}, \cdots, a_{j_k}-a_{j_{k-1}})$$

and we have

$$\phi(\lambda_1, \lambda_2, \cdots, \lambda_k; m_1, m_2, \cdots, m_k)$$
$$=\int \hat{\psi}(\lambda_1, \lambda_2, \cdots, \lambda_k; b_2, b_3, \cdots, b_k) dP_{(m_2-m_1, m_3-m_2, \cdots, m_k-m_{k-1})}(b_2, b_3, \cdots, b_k)$$

where

$$P_{(m_2-m_1, m_3-m_2, \cdots, m_k-m_{k-1})}(b_2, b_3, \cdots, b_k)$$
$$=\text{Prob}\,\{\tau_{m_2}-\tau_{m_1}\leq b_2, \tau_{m_3}-\tau_{m_2}\leq b_3, \cdots, \tau_{m_k}-\tau_{m_{k-1}}\leq b_k\} .$$

Taking into account of the strict stationarity of the $\Delta\tau_n$ process we can see from this equation that $P_{(m_2-m_1, m_3-m_2, \cdots, m_k-m_{k-1})}$, and so the characteristic function $\phi(\lambda_1, \lambda_2, \cdots, \lambda_k; m_1, m_2, \cdots, m_k)$, is completely determined by the differences m_j-m_{j-1} $(j=2, 3, \cdots, k)$. Thus the process $\{x_{\tau,n}(\omega)\}$ is seen to be strictly stationary. Taking into account the inequalities

$$\left|\frac{\partial}{\partial\lambda_\nu}\psi(\lambda_1, \lambda_2, \cdots, \lambda_k; a_{j_1}, a_{j_2}, \cdots, a_{j_k})\right| \leq E\,|\,x(a_{j_\nu})\,| \leq \sigma$$

$$\left|\frac{\partial^2}{\partial\lambda_\mu\partial\lambda_\nu}\psi(\lambda_1, \lambda_2, \cdots, \lambda_k; a_{j_1}, a_{j_2}, \cdots, a_{j_k})\right| \leq E\,|\,x(a_{j_\mu})x(a_{j_\nu})\,| \leq \sigma^2$$

where $\sigma^2=E\,|\,x(t)\,|^2$, we get

$$\frac{\partial}{\partial\lambda_\nu}\phi(\lambda_1, \lambda_2, \cdots, \lambda_k; a_{j_1}, a_{j_2}, \cdots, a_{j_k})$$
$$=\int \frac{\partial}{\partial\lambda_\nu}\psi(\lambda_1, \lambda_2, \cdots, \lambda_k; a_{j_1}, a_{j_2}, \cdots, a_{j_k}) dP_{m_1, m_2, \cdots, m_k}(a_{j_1}, a_{j_2}, \cdots, a_{j_k})$$

$$\frac{\partial^2}{\partial\lambda_\mu\partial\lambda_\nu}\phi(\lambda_1, \lambda_2, \cdots, \lambda_k; a_{j_1}, a_{j_2}, \cdots, a_{j_k})$$
$$=\int \frac{\partial^2}{\partial\lambda_\mu\partial\lambda_\nu}\psi(\lambda_1, \lambda_2, \cdots, \lambda_k; a_{j_1}, a_{j_2}, \cdots, a_{j_k}) dP_{m_1, m_2, \cdots m_k}(a_{j_1}, a_{j_2}, \cdots, a_{j_k})$$

and

$$E\{x_{\tau,n}(\omega)\}=0 ,$$

$$\rho(k)\equiv E\{x_{\tau,n+k}(\omega)x_{\tau,n}(\omega)\}=\int R(a_{n+k}-a_n)dP_{n,n+k}(a_n, a_{n+k})$$
$$=\int R(\tau)dP_{(k)}(\tau)$$

where $R(\tau)=E\{x(t+\tau, \omega)x(t, \omega)\}$ and $P_{(k)}(\tau)=\text{Prob}\,(\tau_k-\tau_0\leq\tau)$. Clearly this last relation holds for any k positive or negative and we have

$$\rho(0)=\sigma^2 .$$

Thus $x_{\tau,n}$ process is stationary also in the wide sense, i.e., has finite second order moments.

Obviously this $x_{\tau,n}$ process is a mathematical representation of the sequence of data which are time-sampled by using the timing impulses situated at the time points τ_n. The purpose of the present paper is to study the spectral properties of this $x_{\tau,n}$ process.

Now we have

$$\begin{aligned}\rho(k)&=\int R(\tau)dP_{(k)}(\tau)=\int_{(\tau)}\left[\int_{(f)}e^{2\pi i f\tau}dP(f)\right]dP_{(k)}(\tau)\\&=\int_{(f)}\left[\int_{(\tau)}e^{2\pi i f\tau}dP_{(k)}(\tau)\right]dP(f)\\&=\int\phi_k(f)dP(f)\end{aligned}$$

where $P(f)$ is the power spectral distribution function of the $x(t)$ process, continuous from the right and with $p(-\infty)=0$, $P(\infty)=\sigma^2$, and

$$\phi_k(f)=\int e^{2\pi i f\tau}dP_{(k)}(\tau)\ .$$

As the $x(t)$ process is real we can assume that $P(-f)=\sigma^2-P(f)$ holds at the continuity points of $P(f)$ and we have

$$\rho(k)=\int_{(f)}\phi_k(f)dP(f)=2\int_{0+}^{\infty}Re(\phi_k(f))dP(f)+P(0)-P(0-)\ .^{*)}$$

This shows that $\rho(k)$ is obtainable as an output power of some (imaginary) filter with signed power transfer function $2Re(\phi_k(f))$ under the input $\{x(t)\}$. Now we can evaluate the power spectrum of the $x_{\tau,n}$ process by using some smoothing process or a filter. Take a convergence factor or a sequence of real numbers c_k such as

$$c_0=1\ ,$$

$$c_{-k}=c_k$$

and

$$\sum_{k=-\infty}^{\infty}|c_k|<\infty\ .$$

We define the smoothing function corresponding to the sequence $\{c_k\}$

$$h(f)=\sum_{k=-\infty}^{+\infty}c_k e^{2\pi i k f}\ .$$

Obviously $h(f)$ is a real continuous even periodic function with period 1 and

$$\int_{-1/2}^{1/2}h(f)df=1\ .$$

*) $Re(\phi_k(f))$ denotes the real part of $\phi_k(f)$.

Here we further assume that $h(f)\geq 0$, i.e., the sequence c_k is positive definite. Now we shall represent by $P_\tau(f)$ the power spectral distribution function of the $x_{\tau,n}$ process, which is continuous to the right, monoton non-decreasing and with $\lim_{f\to -1/2} P_\tau(f)=0$, $P_\tau(\frac{1}{2})=\sigma^2$. Then we have

$$\begin{aligned}\sum_{k=-\infty}^{\infty} c_k\rho(k)e^{-2\pi ikf} &= \sum_{k=-\infty}^{+\infty} e^{-2\pi ikf}c_k\int_{-1/2}^{1/2} e^{2\pi ikf'}dP_\tau(f') \\ &= \int_{-1/2}^{1/2}\left[\sum_{k=-\infty}^{\infty} c_k e^{-2\pi ik(f-f')}\right]dP_\tau(f') \\ &= \int_{-1/2}^{1/2} h(f-f')dP_\tau(f') .\end{aligned}$$

Define

$$h*P_\tau(f)=\int_{-1/2}^{1/2} h(f-f')dP_\tau(f') ,$$

then $h*P(f)$ is continuous with respect to f and

$$\int_{-1/2}^{1/2} h*P_\tau(f)df=\int_{-1/2}^{1/2} dP_\tau(f')=\sigma^2 .$$

Now consider a set $[\{c_k^{(n)};\ -\infty<k<\infty\};\ n=1, 2, 3, \cdots]$ of convergence factors $\{c_k^{(n)}\}$ with the corresponding smoothing functions $h^{(n)}(f)$ for which

$$\lim_{n\to\infty}\int_{\alpha}^{\beta} h^{(n)}(f)df=1$$

holds for any α and β satisfying $-\frac{1}{2}\leq\alpha<0<\beta\leq\frac{1}{2}$. Then we have from the eveness of $h^{(n)}(f)$

$$\lim_{n\to\infty}\int_{0}^{\beta} h^{(n)}(f)df=\lim_{n\to\infty}\int_{\alpha}^{0} h^{(n)}(f)df=\tfrac{1}{2}$$

and for α' and β' such that $0\notin[\alpha', \beta']$ and $-\frac{1}{2}\leq\alpha'<\beta'\leq\frac{1}{2}$ we have

$$\lim_{n\to\infty}\int_{\alpha'}^{\beta'} h^{(n)}df=0 .$$

Now for x and y satisfying $-\frac{1}{2}\leq x<y<\frac{1}{2}$ we have

$$\begin{aligned}\lim_{n\to\infty}\int_{x}^{y} h^{(n)}*P_\tau(f)df &= \lim_{n\to\infty}\int_{x}^{y}\left[\int_{-1/2}^{1/2} h^{(n)}(f-f')dP_\tau(f')\right]df \\ &= \int_{-1/2}^{1/2}\left[\lim_{n\to\infty}\int_{x}^{y} h^{(n)}(f-f')df\right]dP_\tau(f') \\ &= \tfrac{1}{2}[(P_\tau(y)-P_\tau(y-))+(P_\tau(x)-P_\tau(x-))]+P_\tau(y-)-P_\tau(x)\end{aligned}$$

where $P_\tau(x-)=P_\tau(x)=0$ by definition for $x=-\frac{1}{2}$, and

$$\lim_{n\to\infty}\int_{-1/2}^{1/2} h^{(n)}*P_\tau(f)df=\sigma^2=P_\tau(\tfrac{1}{2}) .$$

Thus we have

$$\lim_{y\uparrow 1/2}\lim_{n\to\infty}\left(\int_{-1/2}^{1/2} dP_\tau(f)-\int_{-1/2}^{y} h^{(n)}*P_\tau(f)df\right)=\lim_{y\uparrow 1/2}\lim_{n\to\infty}\int_{y}^{1/2} h^{(n)}*P_\tau(f)df$$
$$=P_\tau(\tfrac{1}{2})-P_\tau(\tfrac{1}{2}-) .$$

For the continuity interval $[x, y]$ of $P_\tau(x)$ we have

$$\lim_{n\to\infty}\int_{x}^{y} h^{(n)}*P_\tau(f)df=P_\tau(y)-P_\tau(x) ,$$

and thus for z in $(-\frac{1}{2}, \frac{1}{2})$

$$\lim_{\substack{y\downarrow z\\ x\uparrow z}}\lim_{n\to\infty}\int_{x}^{y} h^{(n)}*P_\tau(f)df=P_\tau(z)-P_\tau(z-) .^{*)}$$

Further if

$$\lim_{n\to\infty} h^{(n)}*P_\tau(f)=\hat{p}(f)$$

almost everywhere and

$$\int_{-1/2}^{1/2}\hat{p}_\tau(f)df=\sigma^2$$

then we have $dP_\tau(f)=\hat{p}_\tau(f)df$.**)

Here we shall derive concrete results for two types of τ_n process.

1. First we shall consider the case where absolute clock pulses are available and timing-errors (deviations of sampling-time points from the corresponding true clock pulses) form a purely random process. In this case we have

$$\tau_n=n\Delta t+\varepsilon_n \quad \text{and} \quad \Delta\tau_n=\varepsilon_{n+1}-\varepsilon_n+\Delta t$$

where Δt is a fixed non-negative constant and ε_n's are mutually independent random variables following one and the same distribution. We shall call this procedure time-sampling of purely random type.

As

$$\tau_{k+n}-\tau_n=\varepsilon_{k+n}-\varepsilon_n+k\Delta t$$

holds, we have for $k\neq 0$

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*) The symbol $\lim_{\substack{y\downarrow z\\ x\uparrow z}}$ denotes $\lim_{y-x\to 0}$ where x and y are taken to satisfy $x<z<y$.

**) Hereafter we shall use the notation $dP(f)$ to denote the measure function determined by $P(f)$. When $P(f)$ is absolutely continuous and with density function $p(f)$ then we write $dP(f)=p(f)df$.

$$\phi_k(f)=\int e^{2\pi i f\tau}dP_{(k)}(\tau)=e^{2\pi i k\Delta t f}|\phi(f)|^2$$

where

$$\phi(f)=E\{\exp(2\pi i f\varepsilon_n)\}\ .$$

Thus

$$\rho(0)=\sigma^2=\int_{-\infty}^{\infty}dP(f)$$

and for $k\neq 0$

$$\rho(k)=\int_{-\infty}^{\infty}e^{2\pi i k\Delta t f}|\phi(f)|^2 dP(f)\ .$$

Here we shall disregards the trivial case where $\Delta t=0$ and $\varepsilon_n=0$ with probability 1. If we represent by $d|\phi|^2P_A(f)$ the aliased form of $d|\phi(f)|^2P(f)$ for the folding frequency $1/(2\Delta t)$, i.e.,

$$d|\phi|^2P_A(f)=\sum_{\nu=-\infty}^{\infty}d\left|\phi\left(\frac{\nu}{\Delta t}+f\right)\right|^2P\left(\frac{\nu}{\Delta t}+f\right)$$

then we have for $k\neq 0$

$$\rho(k)=\int_{-1/(2\Delta t)}^{1/(2\Delta t)}e^{2\pi i k\Delta t f}d|\phi|^2P_A(f)\ .$$

From this expression we can at once see that $dP_\tau(f)$ is given by the following

$$dP_\tau(f)=d|\phi|^2P_A\left(\frac{f}{\Delta t}\right)+\left[\int_{-\infty}^{\infty}(1-|\phi(f')|^2)dP(f')\right]df\ .$$

Of course we can easily obtain this results by exactly following the smoothing procedure described in the preceding section. For practical applications expressions such as

$$\rho(k)=\int_{-1/2\Delta t}^{1/2\Delta t}e^{2\pi i k\Delta t f}dP_\tau^{\Delta t}(f)\qquad k=\cdots,-2,-1,0,1,2,\cdots,$$

$$dP_\tau^{\Delta t}(f)=d|\phi|^2P_A(f)+\Delta t\left[\int_{-\infty}^{\infty}(1-|\phi(f')|^2)dP(f')\right]df\ ,$$

for $-1/(2\Delta t)\leq f\leq 1/(2\Delta t)$, will be better suited. This last relation clearly shows the effect of timing error on the power spectrum. If there is no timing error we have $\phi(f)=1$ and $dP_\tau^{\Delta t}(f)=dP_A(f)$. Thus time-sampling causes aliasing.*) When the timing-errors are present and are not lattice

*) As to the use of the word "aliasing" see [1]. From the above expression of aliased form of a spectral function we suppose that it will be more natural to consider that the aliased form is obtained by "piling up" the sliced spectrum rather than by "folding".

valued we have $|\phi(f)|^2<1$ for $f\neq 0$. Further when the distribution function of error ε_n has probability density function we have $\lim_{|f|\to\infty}|\phi(f)|=0$. Thus we can see that the time-sampling of purely random type usually acts as a low-pass filter with an inner white noise source. The power of this white noise is the same as that of the higher frequency component excluded by the filter from the original $x(t)$ process.

Estimation of the term $\int_{-\infty}^{\infty}(1-|\phi(f)|^2)dP(f)$.

It will be desirable to get an estimate of the term

$$\Delta t\int_{-\infty}^{\infty}(1-|\phi(f)|^2)dP(f) .$$

If such an estimate is available, by subtracting it from the estimate of $dP_\tau^{\Delta t}(f)$ we can estimate $d|\phi|^2P_\Delta(f)$ which will be a good estimate of $dP(f)$ for f near zero, for proper Δt and $\phi(f)$. Now consider two mutually independent readings of the same $x(t)$ process. We shall represent them as $x_{\tau_{1,n}}$ and $x_{\tau_{2,n}}$ with

$$\tau_{1,n}=n\Delta t+\varepsilon_{1,n} ,$$
$$\tau_{2,n}=n\Delta t+\varepsilon_{2,n}$$

where $\{\varepsilon_{1,n}\}$ and $\{\varepsilon_{2,n}\}$ represent timing-errors which form mutually independent purely random processes with one and the same finite dimensional distribution. Then we can see by using the result for time-sampling of purely random type with $\Delta t=0$ that

$$E|x_{\tau_{1,n}}-x_{\tau_{2,n}}|^2=2(\sigma^2-ER(\varepsilon_{1,n}-\varepsilon_{2,n}))$$

holds where

$$ER(\varepsilon_{1,n}-\varepsilon_{2,n})=E\int_{-\infty}^{\infty}\exp\{2\pi i(\varepsilon_{1,n}-\varepsilon_{2,n})f\}dP(f)=\int_{-\infty}^{\infty}|\phi(f)|^2dP(f) .$$

Thus $\frac{1}{2}(x_{\tau_{1,n}}-x_{\tau_{2,n}})^2$ is an unbiased estimate of

$$\int_{-\infty}^{\infty}(1-|\phi(f)|^2)dP(f)$$

and by using the sample mean of the variable for sufficiently large number of n's we can practically estimate the desired quantity.

2. Next we shall consider the case where the interval lengthes between successive sampling time points form a purely random process.

This is the case where only relative clock pulses are available. Interval length from the former sampling time point τ_{n-1} is strictly measured and when it reaches preassigned value Δt next observation is made about $x(t)$ but with timing-error ε_n. We shall call this procedure the time-sampling of renewal type. In this case $\tau_n=\tau_{n-1}+\Delta t+\varepsilon_n$ and ε_n's are assumed to form a purely random process. Notice that when we are not considering the operation in real-time $\Delta t+\varepsilon_n$ may take negative values.

Now if we define

$$\phi(f)=E\{\exp[2\pi if(\Delta t+\varepsilon_n)]\}$$

we have

$$\begin{aligned}\phi_k(f)&=\phi^k(f) && \text{for } k\geq 0\\ &=\phi^{-k}(-f) && \text{for } k<0\ ,\end{aligned}$$

and

$$\rho(k)=\int_{-\infty}^{\infty}\phi_k(f)dP(f)\ .$$

To evaluate the power spectral distribution function of $x_{\tau,n}$ process we shall use the convergence factors $\{c_k^{(n)}\}$ defined by

$$c_k^{(n)}=c_n^{|k|}$$

where $0<c_n<1$ and $\lim\limits_{n\to\infty}c_n=1$.

Here we want to mention a theorem which is well known in the theory of functions and stated as follows; suppose $u(f)$ is a function defined for f in $-\frac{1}{2}\leq f\leq\frac{1}{2}$ and is integrable $[-\frac{1}{2},\frac{1}{2}]$. Then if $u(f)$ is continuous at $f=f_0$ the function

$$u(r,f')=\int_{-1/2}^{1/2}u(f)\frac{1-r^2}{|1-re^{-2\pi i(f-f')}|^2}df\ ,$$

defined for r and f' satisfying $0<r<1$, $-\frac{1}{2}\leq f'\leq\frac{1}{2}$, converges to $u(f_0)$ as $re^{2\pi if'}$ tends to $e^{2\pi if_0}$. A direct consequence of this theorem is that by the present definition of our $\{c_k^{(n)}\}$ the functions

$$h^{(n)}(f)=\sum_{k=-\infty}^{\infty}c_k^{(n)}e^{-2\pi ikf}=\frac{1-|c_n|^2}{|1-e^{-2\pi if}c_n|^2}$$

have the properties which we have postulated in § 1 as necessary for $h^{(n)}(f)$'s to serve for our present purpose. Now we define

$$K_n(\phi, f, f') = \sum_{k=-\infty}^{\infty} c_k^{(n)} \phi_k(f') e^{-2\pi i k f}$$
$$= \frac{1 - |c_n \phi(f')|^2}{|1 - e^{-2\pi i f} c_n \phi(f')|^2} .$$

Then if we use the representation

$$\phi(f') = r' e^{2\pi i s'} \qquad (0 \leq r' \leq 1 , \quad -\tfrac{1}{2} < s' \leq \tfrac{1}{2})$$

we have

$$K_n(\phi, f, f') = \sum_{k=-\infty}^{\infty} e^{-2\pi i k (f - s')} (c_n r')^{|k|} = \frac{1 - |c_n r'|^2}{|1 - e^{-2\pi i (f - s')} c_n r'|^2} ,$$

$$K_n(\phi, f, f') \geq 0$$

and

$$\int_{-1/2}^{1/2} K_n(\phi, f, f') df = (c_n r')^0 = 1 .$$

Now we shall define for $\phi(f') = r' e^{2\pi i s'}$ ($0 \leq r' \leq 1$, $-\frac{1}{2} < s' \leq \frac{1}{2}$) and f ($-\frac{1}{2} \leq f \leq \frac{1}{2}$)

$$\begin{aligned} K(\phi, f, f') &= \frac{1 - |r'|^2}{|1 - e^{-2\pi i (f - s')} r'|^2} && \text{when} \quad r' < 1, \\ &= 0 && \text{when} \quad r' = 1 \end{aligned}$$

and for x and y satisfying $-\frac{1}{2} \leq x < y < \frac{1}{2}$

$$\begin{aligned} \tilde{\chi}_{[x,y],\phi}(f') &= 1 && \text{when } r' = 1 \text{ and } x < s' < y \\ &= \tfrac{1}{2} && \text{when } r' = 1 \text{ and } s' = x \text{ or } s' = y \\ &= 0 && \text{otherwise.} \end{aligned}$$

Then by taking into account the results of the above-mentioned theorem we can get for x and y satisfying $-\frac{1}{2} < x < y < \frac{1}{2}$

$$\lim_{n \to \infty} \int_x^y K_n(\phi, f, f') df = \tilde{\chi}_{[x,y],\phi}(f') + \int_x^y K(\phi, f, f') df ,$$

and for x and y satisfying $-\frac{1}{2} \leq x < y < \frac{1}{2}$

$$\begin{aligned} \lim_{n \to \infty} \int_x^y h^{(n)} * P_\tau(f) df &= \lim_{n \to \infty} \int_x^y \left[\int_{-\infty}^{\infty} K_n(\phi, f, f') dP(f') \right] df \\ &= \lim_{n \to \infty} \int_{-\infty}^{\infty} \left[\int_x^y K_n(\phi, f, f') df \right] dP(f') \\ &= \int_{-\infty}^{\infty} \tilde{\chi}_{[x,y],\phi}(f') dP(f') + \int_x^y \left[\int_{-\infty}^{\infty} K(\phi, f, f') dP(f') \right] df . \end{aligned}$$

Thus we have

$$P_\tau(\tfrac{1}{2})-P_\tau(\tfrac{1}{2}-)=\sigma^2-\lim_{y\uparrow 1/2}\lim_{n\to\infty}\int_{-1/2}^{y} h^{(n)}*P_\tau(f)df$$
$$=\sigma^2-\int_{-\infty}^{\infty}\Big[\lim_{y\uparrow 1/2}\tilde{\chi}_{[-1/2,y],\phi}(f')+\int_{-1/2}^{1/2}K(\phi,f,f')df\Big]dP(f')$$
$$=\int_{-\infty}^{\infty}\tilde{\chi}_{1/2,\phi}(f')dP(f')$$

where $\tilde{\chi}_{1/2,\phi}(f')=1$ if and only if $\phi(f')=e^{\pi i}$ and $=0$ otherwise. If we further define

$$\tilde{\chi}_{s,\phi}(f')=\lim_{\substack{y\downarrow s\\ x\uparrow s}}\tilde{\chi}_{[x,y],\phi}(f') \qquad \text{for } s\neq\pm\tfrac{1}{2}$$
$$=\lim_{y\downarrow -1/2}\tilde{\chi}_{[-1/2,y],\phi}(f') \qquad \text{for } s=-\tfrac{1}{2}$$

we have for s in $[-\frac{1}{2},\frac{1}{2}]$

$$P_\tau(s)-P_\tau(s-)=\int_{-\infty}^{\infty}\tilde{\chi}_{s,\phi}(f')dP(f')\,.$$

We shall hereafter analyse these results in more details. There are three classes of $\phi(f')$. The first is composed of those $\phi(f')$ for which $|\phi(f')|=1$ holds for all f'. The second is composed of those $\phi(f')$ for which the minimum of the absolute values of those f' $(\neq 0)$, for which $|\phi(f')|=1$ hold, takes some positive value f_0 which depends on ϕ. The third is composed of those $\phi(f')$ for which $|\phi(f')|=1$ holds only at $f'=0$. When $\phi(f')$ is of the first class it can be represented in the form

$$\phi(f')=e^{2\pi i f'\Delta\tau}$$

by some real constant $\Delta\tau$ and corresponds to the time-sampling with the length $\Delta\tau$ of sampling interval and without timing-error. Hereafter we shall disregard the trivial case where $\Delta\tau=0$ holds. Now we have

$$K(\phi,f,f')=0$$

and for x and y satisfying $-\frac{1}{2}<x<y<\frac{1}{2}$

$$\lim_{n\to\infty}\int_x^y h^{(n)}*P_\tau(f)df=\int_{-\infty}^{\infty}\tilde{\chi}_{[x,y],\phi}(f')dP(f')$$
$$=\sum_{\nu=-\infty}^{\infty}\left[\frac{P\Big(\dfrac{y+\nu}{\Delta\tau}\Big)+P\Big(\dfrac{y+\nu}{\Delta\tau}-\Big)}{2}-\frac{P\Big(\dfrac{x+\nu}{\Delta\tau}\Big)+P\Big(\dfrac{x+\nu}{\Delta\tau}-\Big)}{2}\right].$$

This is the formula showing the folding or alaising. The line spectra of this case are given by

$$P_\tau(s)-P_\tau(s-)=\sum_{\nu=-\infty}^{\infty}\left\{P\left(\frac{s+\nu}{\Delta\tau}\right)-P\left(\frac{s+\nu}{\Delta\tau}-\right)\right\} \qquad \text{for } -\tfrac{1}{2}<s\leq\tfrac{1}{2}.$$

If the original $P(f)$ has a density function $p(f)$ then we have

$$dP_\tau(f)=\left[\sum_{\nu=-\infty}^{\infty}p\left(\frac{s+\nu}{\Delta\tau}\right)\right]df\,.$$

As for $\phi(f')$ of the second class we can express it in the form

$$\phi(f')=\exp\left\{2\pi i\left(\frac{f'}{f_0}\right)s_0\right\}\left[\sum_{k=-\infty}^{\infty}p_k\exp\left\{2\pi ik\left(\frac{f'}{f_0}\right)\right\}\right]$$

where s_0 is such that $-\frac{1}{2}<s_0\leq\frac{1}{2}$ and $p_k=\text{Prob}\,\{\Delta t+\varepsilon_n=f_0^{-1}(s_0+k)\}$. In this case $|\phi(f')|=1$ holds only at $f'=\nu f_0$ ($\nu=\cdots,-1,0,1,\cdots$) where $\phi(f')=e^{2\pi i\nu s_0}$. When $s_0\neq 0$ the line spectra are obtained by properly re-scaling the ordinates of the spectra obtained by piling up the line spectra at $f=\nu f_0$ ($\nu=\cdots,-1,0,1,\cdots$) of $P(f)$ sliced at the frequencies $(\mu+\frac{1}{2})(f_0/s_0)$ ($\mu=\cdots,-1,0,1,\cdots$), i.e., $P_\tau(s)-P_\tau(s-)=$ sum of line spectra of $P(f)$ at νf_0's where $\nu f_0=(\mu+s)(f_0/s_0)$ holds for some integer μ.[*] When $s_0=0$ the line spectrum is present only at the origin, or the total power of line spectra at $f=\nu f_0$'s of the original $P(f)$ is transformed into continuous of the d.c. (direct current) component of $x_{\tau,n}$. Now the power part of the $P_\tau(f)$ is seen to have a density function

$$\int_{-\infty}^{\infty}K(\phi,f,f')dP(f')\,.$$

Thus when $P(f)$ has a density function $p(f)$ we have

$$dP_\tau(f)=\left[\int_{-\infty}^{\infty}\frac{1-|\phi(f')|^2}{|1-e^{-2\pi if}\phi(f')|^2}p(f')df'\right]df\,.$$

When $\phi(f')$ belongs to the third class there may be a line spectrum in $P_\tau(f)$ only at the origin and it is equal to that of the line spectrum of the original $P(f')$ at $f'=0$. Thus in this case only the power of the d.c. component of the original process is preserved as line spectrum and becomes the power of the d.c. component of the time-sampled process. Thus if only the d.c. component is absent in the original process we always have absolutely continuous spectrum given by

$$dP_\tau(f)=\left[\int_{-\infty}^{\infty}\frac{1-|\phi(f')|^2}{|1-e^{-2\pi if}\phi(f')|^2}dP(f')\right]df\,.$$

*) When there is line spectrum in $P(f)$ at $f=(\mu+\frac{1}{2})(f_0/s_0)$ it must be piled up at $s=\frac{1}{2}$ in $P_\tau(f)$.

From these results we can see that the present sampling procedure distributes the power $dP(f')$ at f', of the original process, over the range $[-\frac{1}{2}\leq f\leq\frac{1}{2}]$ following the distribution function given by $\{(1-|\phi(f')|^2)/|1-e^{-2\pi if}\phi(f')|^2\}df$. The distribution function given by $\{(1-|\phi(f')|^2)/|1-e^{-2\pi if}\phi(f')|^2\}df$ should be interpreted as $\delta(f-s')df$ for $\phi(f')=1\cdot e^{2\pi is'}$ $(-\frac{1}{2}<s'<\frac{1}{2})$ where $\delta(f-s')$ denotes the Dirac's δ-function and for this frequency f' our time-sampling procedure acts as if there were no timing-errors.*) The distribution given by $\{(1-|\phi(f')|^2)/|1-e^{-2\pi if}\phi(f')|^2\}df$ gives the spectral distribution of randomly phase modulated sinusoidal sequence $\left\{\exp\left[2\pi if'\left(n\Delta t+\sum_{j=1}^{n}\varepsilon_j(\omega)\right)\right];\ n=\cdots,\ -1,\ 0,\ 1,\ \cdots\right\}$. An analogous interpretation is also possible for the case of the time-sampling of purely random type and we can see that our present sampling procedures are essentially non-linear. Taking into account the fact that $\{(1-|\phi(f')|^2)/|1-e^{-2\pi if}\phi(f')|^2\}df$ tends to the uniform distribution as $|\phi(f')|\rightarrow 0$ and tends to the Dirac's δ-function as $|\phi(f')|\rightarrow 1$, we can see that the power $dP(f')$ is conserved near the s' when r' of $\phi(f')=r'e^{2\pi is'}$ is nearly equal to 1 and spread all over the range when r' is nearly equal to 0. When the distribution function of the sampling-time interval is absolutely continuous we have

$$|\phi(f')|\rightarrow 0 \qquad (|f'|\rightarrow\infty)$$

and we can see that the power at the higher frequencies is spread nearly uniformly all over the frequency range of $P_\tau(f)$, while the power near the zero frequency is conserved near the zero frequency.

Thus from the results in this and the preceeding paragraphs we can see that in practical applications of time-sampling procedures of these two types, if in the original process there is some power at some separated very high frequency band, the time-sampling may appear as a low-pass filter with an inner white noise source. This fact will show why time-sampling was sometimes considered to be a filtering while it is essentially a folding which is non-linear.

2. Numerical example**)

Here we shall illustrate the results in the preceeding section by some numerical examples. The estimates of the spectral density functions

*) Obvious modification is necessary for $s'=+\frac{1}{2}$.

**) In this section we shall sometimes use the notations of random variables to represent one of their realizations so long as it does not introduce serious ambiguities.

illustrated in this section were obtained by the following numerical procedure;

a) given a time sampled data $\{x_{\tau,n};\ n=1, 2, \cdots, N\}$ we computed $C(k)$'s $(k=0, 1, 2, \cdots, h)$

$$C(k)=\frac{1}{N}\left(\sum_{n=1}^{N-k}\tilde{x}_{\tau,n+k}\tilde{x}_{\tau,n}\right)$$

where $\tilde{x}_{\tau,n}=x_{\tau,n}-\bar{x}_\tau$ and $\bar{x}_\tau=\frac{1}{N}\sum_{n=1}^{N}x_{\tau,n}$,

b) then the transforms $\tilde{p}(f)$ of this $\{C(k)\}$ where computed for $f=(j/h)(1/2)$ $(j=0, 1, 2, \cdots, h)$

$$\tilde{p}\left(\frac{j}{h}\cdot\frac{1}{2}\right)=C(0)+2\sum_{k=1}^{h-1}C(k)\cos\left(\frac{jk}{h}\pi\right)+C(h)\cos(j\pi)\ ,$$

c) these $\tilde{p}(f)$'s were then further smoothed to give our estimate $p(f)$ for $f=(j/h)(1/2)$ $(j=0, 1, 2, \cdots, h)$

$$p\left(\frac{j}{h}\cdot\frac{1}{2}\right)=0.23\tilde{p}\left(\frac{j-1}{h}\cdot\frac{1}{2}\right)+0.54\tilde{p}\left(\frac{j}{h}\cdot\frac{1}{2}\right)+0.23\tilde{p}\left(\frac{j+1}{h}\cdot\frac{1}{2}\right)$$

where

$$\tilde{p}\left(-\frac{1}{h}\cdot\frac{1}{2}\right)=\tilde{p}\left(\frac{1}{j}\cdot\frac{1}{2}\right)\ \text{and}\ \tilde{p}\left(\frac{h+1}{h}\cdot\frac{1}{2}\right)=\tilde{p}\left(\frac{h-1}{h}\cdot\frac{1}{2}\right).$$

Taking into account of the symmetricity of the present $p(f)$ we have considered the values of $P(f)$ only for positive f. As to the analytical details of the present numerical procedure the reader is recommended to consult the paper [1] by Blackman and Tukey. In the following we shall denote the value of $p(f)/C(0)$ simply as $p(f)$. In Fig. 1 the $p(f)$ of a time-sampled data $\{x_{\tau,n};\ n=1, 2, \cdots, N\}$ is shown where $N=530$ and $h=60$. The data was read from a continuous record of a typical oscillation of the frame of an automobile running over a gravel road. Here $\tau_n=\tau_0+n\Delta t$ and Δt was taken to be 1/50 sec. In this data there may be some errors in τ_n but we shall disregard it now as our concern here is with the comparison of this $p(f)$ with other $p(f)$'s which were obtained from the present data by some artificial random sampling procedures which will be described in the following.

Fig. 2 shows the effect of timing-error of purely random type. The crosses show the $p(f)$ of the data $\{x^I_{\tau,\nu}\}$ which was time-sampled from the primary data $\{x_{\tau,\nu}\}$ and

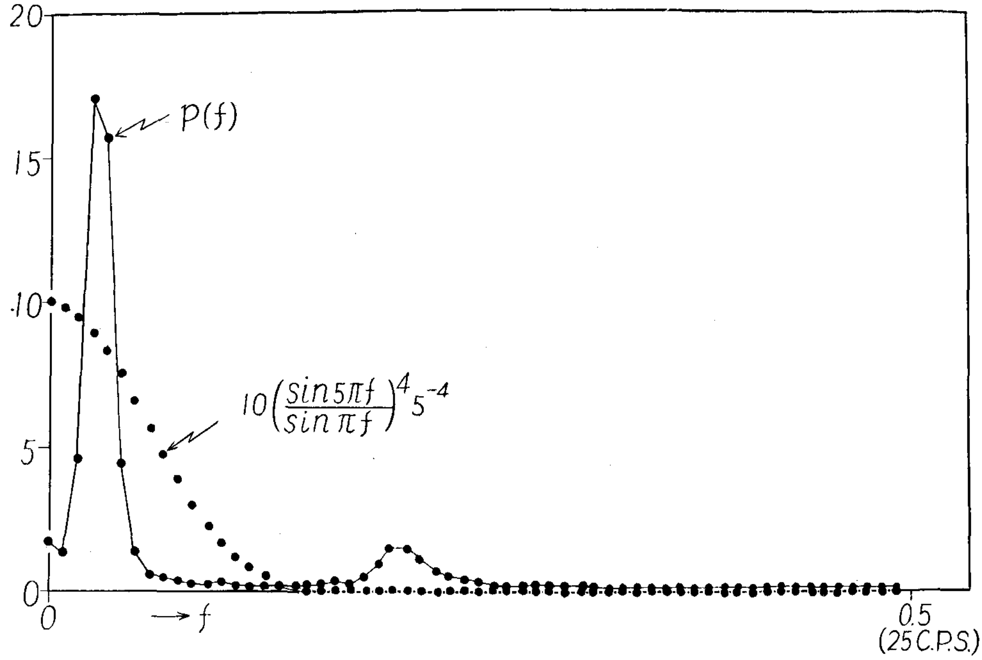

Fig. 1.

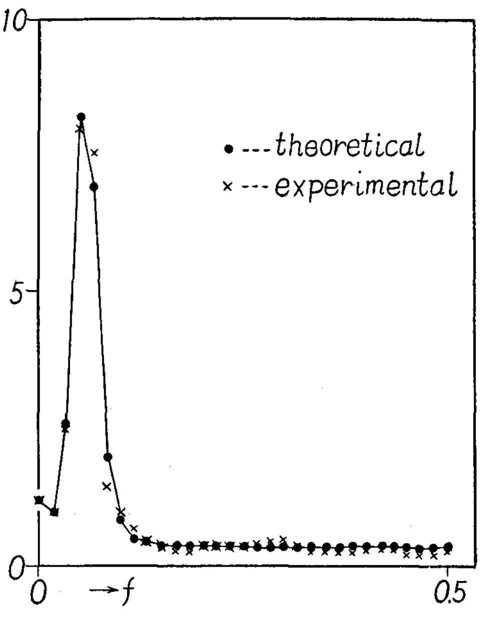

Fig. 2. Effect of time-sampling of purely random type.

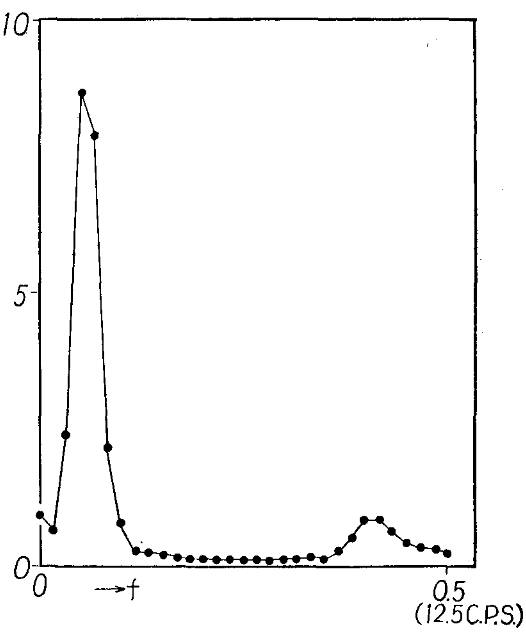

Fig. 3. Effect of pure folding.

$$x^I_{\tau,\nu}=x_{\tau,2\nu+\varepsilon_\nu}$$

where $\{\varepsilon_\nu\}$ is a purely random process such that

$$\Pr\{\varepsilon_\nu=\mu\}=\frac{5-|\mu|}{25}$$

for integer μ in the range $-4\leq\mu\leq4$. Here $N=263$ and $h=30$. In Fig. 2 the dots show the theoretically expected values of $p(f)$ which were obtained by using the results of the preceding section and the $p(f)$ of Fig. 1 in place of the true value of $p(f)$ of $\{x_{\tau,n}\}$. We can see a fairly good agreement. In the present example we have

$$|\phi(f)|^2=\left(\frac{\sin 5\pi f}{\sin \pi f}\right)^4 5^{-4}$$

and its values are plotted, being multiplied by a constant factor 10, for $f=j/2h=j/120$ ($j=0, 1, 2, \cdots, 60$) in Fig. 1.

Fig. 3 shows the $p(f)$ which corresponds to the case where $\varepsilon_\nu\equiv0$ and $x^I_{\tau,\nu}=x_{\tau,2\nu}$ and illustrates the pure folding. By comparing Fig. 2 with Fig. 3 we can clearly see the effect of timing-error. We have further made an experiment of the estimation of the term

$$\int_{-\infty}^{\infty}(1-|\phi(f)|^2)dP(f)\ .$$

By another independent reading we obtained $\{x^{I'}_{\tau,\nu}\}$ and got

$$\frac{1}{2}\left[\frac{1}{263}\sum_{\nu=1}^{263}(x^I_{\tau,\nu}-x^{I'}_{\tau,\nu})^2\right]=0.308\times C(0)\ \text{of}\ x_{\tau,n}\ .$$

Now we can see

$$\frac{1}{2}E(x^I_{\tau,\nu}-x^{I'}_{\tau,\nu})^2=R(0)-\sum_k R(k)\sum_\mu \Pr\{\varepsilon_\nu=\mu+k\}\Pr\{\varepsilon_\nu=\mu\}$$

holds where $R(k)=E\{x_{\tau,n}x_{\tau,n+k}\}$. We computed another estimate of $\frac{1}{2}E(x^I_{\tau,\nu}-x^{I'}_{\tau,\nu})^2$ by putting $C(k)$ of $\{x_{\tau,n}\}$ in place of $R(k)$ in the above formula and it was found to be $0.327\times C(0)$. This last value was used to draw the doted curve of Fig. 2. Thus the present result suggests that for the time-sampling of purely random type, if there is some power at some separated very high frequency band in the original process and the timing-errors are continuously distributed and their range is sufficiently small compared with the wave length of the lower frequency band but sufficiently big compared with the wave length of that high frequency band, then by using the estimate of

$$\int_{-\infty}^{\infty}(1-|\phi(f)|^2)dP(f)$$

described in the former section we may obtain a better estimate of $p(f)$ of the original process. Fig. 4 and 5 show the effect of this correction procedure. There are also presented the order $(1/12)(C(0)^{-1})$ of quantization noise which is assumed approximately to be a white noise. $p(f)$'s in Figs. 4 and 5 were obtained from the data which were read by using a rule, at each timing mark which were 1/100 sec. apart each other, from the continuous records of the outputs of an accelerometer of strain gauge type mounted on the front axle of an automobile running at the speed of 30 km/h and 60 km/h respectively. Here $N=500$, $h=50$ for Fig. 4 and $N=250$, $h=50$ for Fig. 5. We have felt some

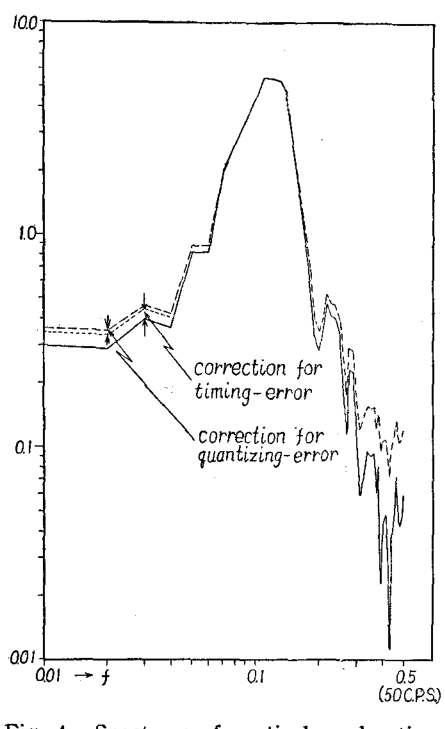

Fig. 4. Spectrum of vertical acceleration of a front axle (at 30km/h).

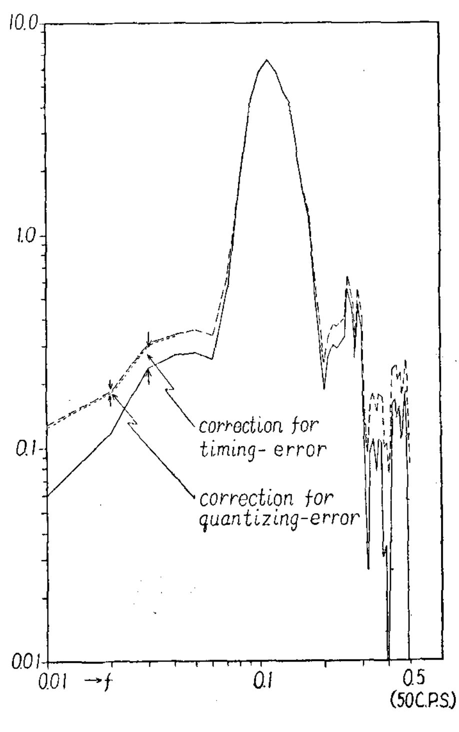

Fig. 5. Sepectrum of vertical aceeleration of a front axle (at 60 km/h).

uncertainty in measuring these data due to the existence of components of very high frequency which might be 100 or 400 cycle per second or higher. We have considered that the uncertainty is mainly due to the fluctuations of the horizontal position of the eye or the rule. As the precise timing marks were available at each 1/100 second such readings will correspond to the time-sampling of purely random type. We can see from the present results that there are more power at higher frequencies in the case of Fig. 5, and this has increased the difficulty in reading the corresponding data. We can see further that timing-error causes little effect on the estimates of $p(f)$ in absolute value. But taking into account of the fact that the present estimate $p(f)$ keeps nearly the same relative accuracy all over the range of f, we have to pay attension, for $p(f)$ at low levels, to the bias of white noise type due to timing-error besides that due to quantization.

In Fig. 6 is illustrated a $p(f)$ of a time-sampled data obtained from the former $\{x_{\tau,\nu}\}$ by a time-sampling procedure of renewal type. The crosses show the values of $p(f)$ of $\{x_{\tau,\nu}^{II}\}$

$$x_{\tau,\nu}^{II}=x_{\tau,\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_\nu} \qquad \nu=1, 2, \cdots, N$$

where $\{\varepsilon_\nu\}$ is a purely random process and

$$\Pr\{\varepsilon_\nu=1\}=\Pr\{\varepsilon_\nu=3\}=\tfrac{1}{4}$$
$$\Pr\{\varepsilon_\nu=2\}=\tfrac{1}{2}\ .$$

Here $N=268$ and $h=30$. The dots represent approximations to the theoretically expected values of $p(f)$ and were obtained by approximately applying the theoretical result of the preceding section to the $p(f)$ of $\{x_{\tau,n}\}$. We can see a fairly good agreement in this case too.

In Fig. 7 are illustrated the values of $C(k)/C(0)$ which were used for the computations of $p(f)$'s of Figs. 1, 2 and 6. The Figs. 1, 2, 3 and 6 show how the present time-sampling procedures act like low-pass filters.

In the present section we have not discussed the sampling variations of our estimates. We did so as our main concern in this section was with the analysis of the biases of our estimates and not of the variances. The discussion of the sampling fluctuations of our estimates is possible at least for the Gaussian case and the reader is recommended to consult the paper [1] for that purpose.

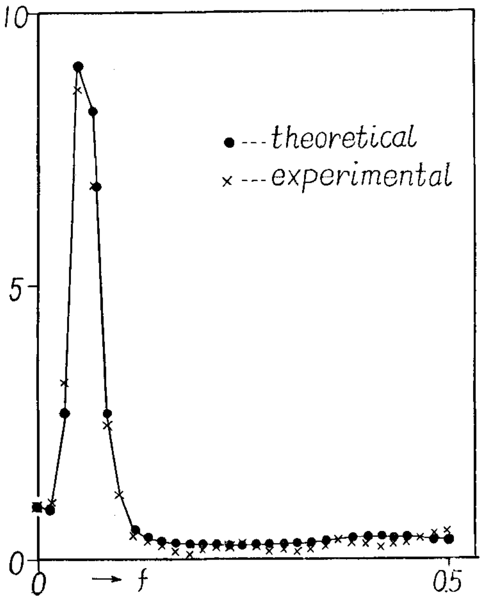

Fig. 6. Effect of time-sampling of renewal type.

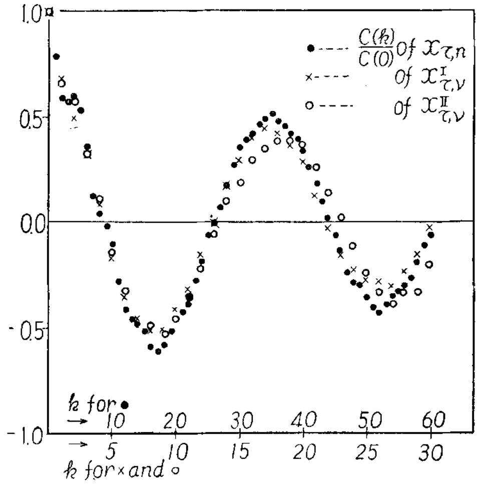

Fig. 7. Correlograms of $\{x_{T,n}\}$, $\{x^{I}_{T,\nu}\}$ and $\{x^{II}_{T,\nu}\}$.

Acknowledgement

I am indebted to the Isuzu Motor Company for providing me with the records of outputs of acceleromers and to Mr. I. Kanesige for stimulating me to the present investigation. I am grateful to Mr. M. Motoo for reading the manuscript of this paper. I am also grateful to Miss Y. Saigusa for programming the computations on FACOM-128 relay computer of our institute and to Mrs. T. Isii for preparing the numerical inputs to the machine and co-operating with Miss Saigusa in computations.

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REFERENCES

[1] R. B. Blackman and J. W. Tukey, "The measurement of power spectra from the point of view of communications engineering", *Bell System Technical Journal* 37, 185–282, 485–569 (1958).

[2] J. L. Doob, "*Stochastic Process*" John Wiley & Sons, Inc. (1953).

[3] D. T. Ross, "Sampling and quantizing", Chaper II in *notes on Analog-Digital Conversion Techniques*, Alfred K. Susskind editor, Technology Press of M.I.T. and John Wiley & Sons, Inc. New York, (1957).

[4] J. W. Tukey, and R. W. Hamming, "Measuring noise color" *Bell Telephon Lab. Memo.*, MM-49-110-119 (1949).

[5] B. Widrow, "A study of rough amplitude quantization by means of Nyquist sampling theory", *Trans. I.R.E.*, PGCT-3, 266–276 (1956)