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ALGEBRAIC DE RHAM COHOMOLOGY

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We announce the development of a theory of algebraic De Rham cohomology and homology for arbitrary schemes over a field of characteristic zero. Over the complex numbers, this theory is equivalent to singular cohomology. Applications include generalizations of theorems of Lefschetz and Barth on the cohomology of projective varieties.

Introduction

The theory of algebraic De Rham cohomology began when Grothendieck proved his comparison theorem [8] for smooth schemes over $\mathbb{C}$. My interest came from the study of cohomological dimension, with the observation that the classical theorem of Lefschetz about complex cohomology of hyperplane sections was easily proved using De Rham cohomology [12, III §7]. More recently, in attempting to give a purely algebraic proof of the theorems of Barth [4], I was led to develop the theory for singular varieties as well. Independently Lieberman and Herrera [16], and Deligne (unpublished) have proved comparison theorems for De Rham cohomology of singular varieties.

Thus it seems timely to lay purely algebraic foundations for the theory, which is what we do here. We define cohomology and homology groups for schemes of finite type over a field k of characteristic zero. We give purely

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algebraic proofs of finite-dimensionality, functorial properties, exact sequences, and duality theorems. We also develop a theory of local invariants: cohomology and homology of the spectrum of a complete local ring. For both the global and the local invariants, we prove comparison theorems with usual cohomology theories over the complex numbers.

As an application, we give a new proof of the theorems of Barth [4] for smooth schemes in projective space. It depends on an interplay of algebraic and analytic techniques. Meanwhile, these methods have been improved and generalized by Ogus [18], who has purely algebraic proofs of Lefschetz and Barth type theorems for schemes with arbitrary singularities.

In this paper we announce our main results. Full details will appear elsewhere. See also [13] for related results and earlier versions of some of these results.

One of the major outstanding problems in algebraic geometry today is the development of a good cohomology theory for varieties in characteristic p. While our theory is valid only in characteristic zero, perhaps some of its formal aspects (especially the homology) may be useful eventually in developing a good theory in characteristic p.

1. Global Theory

Let k be a field of characteristic zero. Let Y be a scheme of finite type over k. For simplicity, we assume that Y admits a global embedding as a closed subscheme of a scheme X, smooth over k. Let $\Omega^{\cdot}$ denote the complex of sheaves of regular differential forms on X over k with the exterior differentiation d as boundary map. Let $\hat{X}$ be the formal completion of X along Y, and let $\hat{\Omega}^{\cdot}$ be the completion of $\Omega^{\cdot}$. Then we define the <u>algebraic De Rham cohomology</u> of Y, denoted $H^i_{DR}(Y)$, to be the hypercohomology $H^i(\hat{X},\hat{\Omega}^{\cdot})$ of the complex $\hat{\Omega}^{\cdot}$ on the formal scheme $\hat{X}$. We define the <u>algebraic De Rham homology</u> of Y, denoted $H^{DR}_i(Y)$ to be the hypercohomology with supports in Y, $H^{2n-i}_Y(X,\Omega^{\cdot})$, where $n = \dim X$. We will omit the notation "DR" when no confusion can result.

THEOREM 1.1. *The cohomology groups $H^i(Y)$ and homology groups $H_i(Y)$ as defined above are independent of the choice of the ambient scheme X. Cohomology is a contravariant functor in Y; homology is a covariant functor for proper morphisms, and a contravariant functor for open immersions.*

Already the hypothesis of characteristic zero is essential for this theorem. A key step involves an integration of power series similar to the Lemma 17 of Atiyah and Hodge [3]. To demonstrate the functorial aspect of homology, we are led to introduce a canonical injective resolution of $\Omega^{\cdot}$ using the notion of Cousin complex of a sheaf [10, IV §2]. The covariant map is then constructed using the trace map for residual complexes [loc. cit., VI §4].

THEOREM 1.2. *Let X,Y, etc. denote schemes of finite type over k. Then*

a) *If Y is a closed subscheme of X, there is an exact sequence of homology*

$$\cdots \to H_i(Y) \to H_i(X) \to H_i(X-Y) \to H_{i-1}(Y) \to \cdots$$

b) *If* Y_1 *and* Y_2 *are closed subschemes of* X, *there are Mayer-Vietoris exact sequences*

$$\cdots \longrightarrow H^i(Y_1 \cup Y_2) \longrightarrow H^i(Y_1) \oplus H^i(Y_2) \longrightarrow H^i(Y_1 \cap Y_2) \longrightarrow H^{i+1}(Y_1 \cup Y_2) \rightarrow \cdots$$

and

$$\cdots \longrightarrow H_i(Y_1 \cap Y_2) \longrightarrow H_i(Y_1) \oplus H_i(Y_2) \rightarrow H_i(Y_1 \cup Y_2) \longrightarrow H_{i-1}(Y_1 \cap Y_2) \rightarrow \cdots$$

c) *If* Y *is smooth over* k, *of dimension* n, *then*

$$H_i(Y) \cong H^{2n-i}(Y) .$$

d) *If* $f: X' \longrightarrow X$ *is a proper birational map, if* Y *is a closed subscheme of* X, *such that* $f: X'-Y' \longrightarrow X-Y$ *is an isomorphism, where* $Y' = f^{-1}(Y)$, *then there are exact sequences*

$$\cdots \longrightarrow H^i(X) \longrightarrow H^i(X') \oplus H^i(Y) \longrightarrow H^i(Y') \longrightarrow H^{i+1}(X) \rightarrow \cdots$$

and $\cdots \longrightarrow H_i(Y') \longrightarrow H_i(Y) \oplus H_i(X') \longrightarrow H_i(X) \longrightarrow H_{i-1}(Y') \longrightarrow \cdots$

These results are all proved using fairly standard cohomological techniques. Note especially d), which is useful when one wants to apply resolution of singularities to some situation.

THEOREM 1.3. (Duality) *Let* Y *be a scheme proper over* k. *Then* $H^i(Y) \cong \mathrm{Hom}_k(H_i(Y),k)$.

The duality theorem is proved by reducing to a duality theorem for formal completions of coherent sheaves [12, III 3.3] which in turn is a generalization of Serre duality. Here a key point is to show that Serre duality is compatible with exterior differentiation $d: \Omega^i \longrightarrow \Omega^{i+1}$. This boils down to showing that d commutes with the trace map on residual complexes, which is done by explicit computation.

THEOREM 1.4. (Finiteness) *If Y is a scheme of finite type over k, then the groups $H^i(Y)$ and $H_i(Y)$ are finite-dimensional k-vector spaces.*

For this result we use induction on the dimension of Y, and the theorem of resolution of singularities of Hironaka [14]. Then with all the functorial properties listed above, we reduce to the case of a smooth proper scheme over k, in which case the finiteness follows from Serre's finiteness theorems for coherent sheaves.

By now we have enough machinery to do anything one would like as in any other cohomology theory. For example, we can define the cohomology class of a cycle on a smooth scheme. It is sufficient to consider prime cycles. So let Y be a closed integral subscheme of dimension r of a smooth scheme X of dimension n. Let $U \subseteq Y$ be a non-empty open subset, which is smooth. Then one shows $H_{2r}(Y) \longrightarrow H_{2r}(U) \cong H^o(U)$ is an isomorphism. So taking $1 \in H^o(U)$ gives us a fundamental homology class $\eta_Y \in H_{2r}(Y)$. Its image in X gives a homology class $\eta_Y \in H_{2r}(X)$. But since X is smooth, $H_{2r}(X) \cong H^{2n-2r}(X)$, and so we have $\eta(Y) \in H^{2n-2r}(X)$ which is the cohomology class of Y.

THEOREM 1.5. *Let X be a smooth quasi-projective scheme over k, so that one has intersection theory for rational equivalence classes of cycles [7]. Let Y,Z be cycles on X. Then $\eta(Y)$ and $\eta(Z)$ depend only on the rational equivalence class of Y and Z, and we have*

$$\eta(Y.Z) = \eta(Y) \cup \eta(Z)$$

where . denotes intersection of cycle classes, and $\cup$ denotes cup-product.

In case the ground field k is the complex numbers $\mathbb{C}$, we can compare the De Rham cohomology of a scheme Y over $\mathbb{C}$ to the ordinary (topological) invariants of the associated complex-analytic space Y^h.

THEOREM 1.6. (Comparison) Let Y be a scheme of finite type over $\mathbb{C}$. Then there are natural functorial isomorphisms $H^i_{DR}(Y) \cong H^i(Y^h, \mathbb{C})$, the singular cohomology of Y^h with complex coefficients, and $H_i^{DR}(Y) \cong H_i^{BM}(Y^h, \mathbb{C})$, the Borel-Moore homology with locally compact supports and complex coefficients [5].

Like the finiteness theorem, this theorem is proved by induction on the dimension of Y, using resolution of singularities, and the functorial properties of cohomology and homology. Thus one reduces to the case Y smooth proper over k, in which case the result follows from Serre's comparison theorem for coherent sheaves [19].

2. Local Theory

To define local invariants at a point of a scheme, a little reflection shows that the Zariski local ring, or even the Hensel local ring is not local enough to give a reasonable answer. So we use the complete local ring. The definitions are parallel to those in the global situation. Let Y be the spectrum of a complete local ring containing its residue field k of characteristic zero. Embed Y as a closed subscheme of X, the spectrum of a complete regular local ring (hence a ring of formal power series over k). Let $\Omega^\cdot$ be the complex of sheaves of continuous differential forms on X, for the adic topology. Let $\hat{X}$ be the formal completion of X along Y, and let $\hat{\Omega}^{\cdot}$ be the completion of $\Omega^{\cdot}$. Let $P \in Y$ be the closed point. Then we define the *local De Rham cohomology with supports in* P, denoted $H^i_P(Y)$, to be the hypercohomology $H^i_P(\hat{X},\hat{\Omega}^{\cdot})$ of the complex $\hat{\Omega}^{\cdot}$ on $\hat{X}$ with supports in P. We define the *local De Rham homology*, denoted $H_i(Y)$, to be the hypercohomology $H^{2n-i}_Y(X,\Omega^{\cdot})$, where $n = \dim X$.

THEOREM 2.1. The groups $H^i_P(Y)$ and $H_i(Y)$ defined above are independent of the choice of the ambient scheme X.

To prove the finiteness of these local invariants, as in the global case, we must use resolution of singularities. In resolving the singularity of a local ring, we are forced to consider situations which are no longer local. So we consider the category of schemes X, which are proper over the spectrum Y of a complete local ring, and we prove the following more general result.

THEOREM 2.2. (Finiteness) Let $f: X \longrightarrow Y$ be a proper morphism of schemes, where Y is the spectrum of a complete local ring containing its residue field k of characteristic zero. Let $P \in Y$ be the closed point, and let $E = f^{-1}(P)$. Then the De Rham groups $H^i_E(X)$ and $H_i(X)$ (defined analogously) are finite-dimensional k-vector spaces.

For duality, we would like to give a proof using the local duality theorems of Grothendieck [10, V §6] for modules over local rings. We have been unable to do this however, so instead we use resolution of singularities and exact sequences to reduce to the global duality theorem.

THEOREM 2.3. (Duality) *With the notations and hypotheses of the previous theorem, the vector spaces* $H_E^i(X)$ *and* $H_i(X)$ *are naturally dual to each other.*

Since we have defined the local invariants for any complete local ring, we have the possibility of considering the local cohomology at a non-closed point of a scheme. For example, let X be a scheme of finite type over k, let Y be an irreducible closed subscheme of X, and let $\eta \in Y$ be the generic point of Y. We consider the local ring $\mathcal{O}_{\eta,X}$, and choose a field of representatives $k(\eta)$ inside it. Then we can consider the local cohomology $H_\eta^i(\mathrm{Spec}\ \hat{\mathcal{O}}_{\eta,X})$, which is a finite-dimensional vector space over $k(\eta)$. On the other hand, if $y \in Y$ is any closed point, we can consider the local cohomology $H_y^i(\mathrm{Spec}\ \hat{\mathcal{O}}_{y,X})$, which is a vector space over k.

THEOREM 2.4. *In the above situation, there is a non-empty Zariski open set* $U \subseteq Y$ *such that for all closed points* $y \in U$, *we have*

$$H_y^i(\mathrm{Spec}\ \hat{\mathcal{O}}_{y,X}) \otimes_k k(\eta) \cong H_\eta^{i-2r}(\mathrm{Spec}\ \hat{\mathcal{O}}_{\eta,X})$$

where $r = \dim Y$. *In particular the left hand side is independent of the choice of* $y \in U$, *and the right hand side is independent of the choice of the field of representatives* $k(\eta)$.

If we think of the local cohomology groups as a measure of the singularity at a point, then this result says that X is equisingular along Y at points of the open set U. This notion is in general weaker than the equisingularity of Zariski [20].

THEOREM 2.5. (Comparison) Let Y be a complex analytic space, let P be a point of Y and let $Y' = \operatorname{Spec} \hat{\mathcal{O}}_{P,Y}$. Then $H^i_P(Y')$ is naturally isomorphic to the local cohomology $H^i_P(Y,\mathbb{C})$, and $H_i(Y')$ is naturally isomorphic to the stalk at P of the Borel-Moore homology sheaf $\mathcal{H}_i(Y,\mathbb{C})$.

To express this result in terms of the local invariants used by Milnor [17] and Lê [15], note that if $\epsilon > 0$ is so small that $B_\epsilon \cap Y$ is contractible, where B_ϵ is a ball of radius ϵ around P, then we have an exact sequence

$$0 \longrightarrow H^0_P(Y,\mathbb{C}) \longrightarrow \mathbb{C} \longrightarrow H^0(B_\epsilon \cap Y-P,\mathbb{C}) \longrightarrow H^1_P(Y,\mathbb{C}) \longrightarrow 0$$

and isomorphisms

$$H^i(B_\epsilon \cap Y-P, \mathbb{C}) \xrightarrow{\cong} H^{i+1}_P(Y,\mathbb{C}) \qquad \text{for } i \geq 1.$$

Remarks: 1. Although we have chosen to work with the local cohomology groups $H^i_P(Y)$, one could also phrase the theory in terms of the punctured space cohomology $H^i(Y-P)$, defined as $H^i(\hat{X}-P, \hat{\Omega}^\cdot)$. The two are related by the long exact sequence of local cohomology

$$\cdots \longrightarrow H^i_P(Y) \longrightarrow H^i(Y) \longrightarrow H^i(Y-P) \longrightarrow H^{i+1}_P(Y) \longrightarrow \cdots ,$$

and the "formal Poincaré lemma", which says that $H^i(Y)$, defined as $H^i(\hat{X},\hat{\Omega}^\cdot)$, is equal to k for $i = 0$, and 0 for $i > 0$. The proof of the formal Poincaré lemma is exactly analogous to the proof of the classical holomorphic Poincaré lemma.

2. One can also define the local De Rham cohomology groups $H^i_Y(X)$ for any closed subset Y of any scheme X over k, which admits an embedding into a smooth scheme. If Y is a point P whose residue field is finite over k, we have a "strong excision theorem" which says that $H^i_P(X) = H^i_P(\text{Spec } \hat{\mathcal{O}}_{P,X})$. One should be careful however, because if X is the spectrum of a non-complete local ring, with closed point P, then $H^i(X)$ and $H^i(X-P)$ may be infinite-dimensional.

3. An Example

To illustrate the relationship between the local and global theories, we calculate the local cohomology and homology of the vertex of the cone over a projective variety. Let Y be a closed subscheme of $\mathbb{P}^n$, let $C(Y) \subseteq \mathbb{P}^{n+1}$ be the projective cone over Y, with vertex P, and let $Y' = \text{Spec } \hat{\mathcal{O}}_{P,C(Y)}$.

By noting that C(Y)-P is a line bundle over Y, and using an algebraic Thom isomorphism, we find that

$$H^i(C(Y)) \cong H^{i-2}(Y) \quad \text{for} \quad i \geq 1$$

$$H^0(C(Y)) = k$$

and

$$H_i(C(Y)) \cong H_{i-2}(Y) \quad \text{for} \quad i \geq 1$$

$$H_0(C(Y)) = k,$$

Now to compute the local invariants, we use the local cohomology sequence for $P \in C(Y)$, and a strong excision theorem which says that $H^i_P(C(Y)) \cong H^i_P(Y')$. We find that

$$H^0_P(Y') = H^1_P(Y') = 0 \ ; \quad H^2_P(Y') \cong H^1(Y) \ ,$$

and there is a long exact sequence

$$0 \to H^0(Y) \xrightarrow{\xi} H^2(Y) \to H^3_P(Y') \to H^1(Y) \xrightarrow{\xi} H^3(Y) \to H^4_P(Y') \to \cdots$$

where ξ is cup-product with the class of the hyperplane section on Y. For homology we find using duality that

$$H_0(Y') = H_1(Y') = 0 \ ; \quad H_2(Y') \cong H_1(Y) \ ,$$

and an exact sequence

$$\cdots \to H_4(Y') \to H_3(Y) \xrightarrow{\xi} H_1(Y) \to H_3(Y') \to H_2(Y) \xrightarrow{\xi} H_0(Y) \to 0,$$

where this time ξ is cap-product with the class of the hyperplane section.

4. Applications

One can expect that many theorems about the topology of algebraic varieties, hitherto proved using complex cohomology and transcendental methods, can now be rephrased in terms of algebraic De Rham cohomology, and given algebraic proofs. At the same time, one can expect to eliminate hypotheses of non-singularity, since the algebraic theory is well suited to dealing with singularities, whereas transcendental methods like Hodge theory and Morse theory are not.

For example, one can prove algebraic analogues and generalizations of the analytic theorems of Barth [4]. One approach, due to Ogus [18], is via the study of integrable connections on modules over complete local rings. His method is best suited to dealing with arbitrary singularities, and to proving the local analogues of Barth's theorems. We refer to his paper for an account of his results. Another approach is to use Barth's original method of transplanting cohomology classes. This method seems more suited to comparing the algebraic and analytic results.

The transplanting method is based on the following construction. Let Y be a closed subscheme of $\mathbb{P}^n_k$. Let G be the special linear group $SL(n+1)$ acting on $\mathbb{P}$. We consider the space $G \times \mathbb{P}$ with its two projections p_1 to G and p_2 to $\mathbb{P}$. We embed $G \times Y$ into $G \times \mathbb{P}$ via the group action: $g \times y \longmapsto g \times g(y)$. Let F be a locally free sheaf on $\mathbb{P}$. Then we consider the complex

$$\mathcal{F}^{\cdot} = (p_2^* F \otimes p_1^* \Omega_G^{\cdot})^{\wedge}$$

where $\Omega_G^{\cdot}$ is the complex of sheaves of differential forms on G, and $\wedge$ denotes completion along the subscheme $G \times Y$. Then we study the cohomology of this complex, and that of its analytic analogue, by means of the Leray spectral sequences associated to the two projection maps.

To state our conclusions, we consider the condition (for an integer s).

(Fin-s) For every locally free coherent sheaf F on $\mathbb{P}$, and for every $i < s$, the group $H^i(\hat{\mathbb{P}}, \hat{F})$ is a finite-dimensional k-vector space, where $\wedge$ denotes completion along Y.

Note that (Fin-s) is true if Y is non-singular and $s = \dim Y$ [9, 8.1]. More generally, Ogus [18] has shown that (Fin-s) is true for local complete intersections, and in fact he has given necessary and sufficient conditions for (Fin-s) to hold.

THEOREM 4.1. *Let* Y *be a subscheme of* $\mathbb{P}^n_k$ *(char. k = 0) satisfying the condition* (Fin-s) *for an integer* s. *Then for every* $i < s$, *the graded* $S = k[x_0, \cdots, x_n]$*-module*

$$M^i = \sum_{v \in \mathbb{Z}} H^i(\hat{\mathbb{P}}, \hat{\mathcal{O}}(v))$$

is a free, finitely generated S*-module, generated by its component* $H^i(\hat{\mathbb{P}}, \hat{\mathcal{O}})$ *in degree zero.*

At the same time, we obtain a comparison of algebraic with analytic cohomology groups. Let Y be a subscheme of $\mathbb{P}^n_{\mathbb{C}}$, and let Y^h be the corresponding complex-analytic space.

THEOREM 4.2. *Assume that* Y *satisfies* (Fin-s) *for an integer* s. *Then for every locally free sheaf* F *on* $\mathbb{P}$, *and for every* $i < s$, *the natural maps*

$$H^i(\hat{\mathbb{P}}, \hat{F}) \longrightarrow H^i(\hat{\mathbb{P}}^h, \hat{F}^h) \longleftarrow H^i(Y^h, F^h|_{Y^h})$$

are isomorphisms.

Thus the algebraic condition (Fin-s) implies both the conditions (I^*_s) and $(I^*_{s,an})$ of the main theorem of [13, §2]. These conditions in turn imply the theorems of Barth, as is explained in the paper [loc. cit.]. We refer to that paper for complete statements. Note in particular that we recover the original theorems of Barth [4] for a smooth scheme Y in $\mathbb{P}^n_{\mathbb{C}}$, without using the analytic finiteness theorems of Andreotti and Grauert [2].

Another application of algebraic De Rham cohomology, closely related to the above, but much more elementary, is the proof of the Lefschetz theorems. These are

THEOREM 4.3. (Lefschetz) *Let* X *be a scheme proper over* k, *let* Y *be a closed subscheme of* X, *and assume that* X-Y *is affine and smooth over* k. *Then the natural maps*

$$H^i(X) \longrightarrow H^i(Y)$$

are isomorphisms for $i < \dim Y$, *and injective for* $i = \dim Y$.

THEOREM 4.4. (Local Lefschetz Theorem) *Let* X *be the spectrum of a complete local ring, with closed point* P, *and let* Y *be a closed subscheme of* X. *Assume that* X-Y *is affine and smooth. Then the natural maps*

$$H^i_P(X) \longrightarrow H^i_P(Y)$$

are isomorphisms for $i < \dim Y$, *and injective for* $i = \dim Y$.

In the analytic case, these theorems were proved by Bott [6], Andreotti and Frankel [1], and Lê [15] using Morse theory. Using the results of Ogus [18], the hypothesis of non-singularity in both of these theorems can be replaced by the following weaker condition: for every scheme point $x \in X-Y$, $H^i_x(\operatorname{Spec} \hat{\mathcal{O}}_{x,X}) = 0$ for $i < \dim \mathcal{O}_{x,X}$. This holds in particular if X-Y is a local complete intersection.

References

[1] ANDREOTTI, A., and FRANKEL, T.: The Lefschetz theorem on hyperplane sections. Ann. Math. 69, 713-717 (1959).

[2] ANDREOTTI, A., and GRAUERT, H.: Théorèmes de finitude pour la cohomologie des espaces complexes. Bull. Soc. Math. France 90, 193-259 (1962).

[3] ATIYAH, M., and HODGE, W.V.D.: Integrals of the second kind on an algebraic variety. Ann. Math. 62, 56-91 (1955).

[4] BARTH, W.: Transplanting cohomology classes in complex-projective space. Amer. J. Math. 92, 951-967 (1970).

[5] BOREL, A., and MOORE, J.: Homology theory for locally compact spaces. Mich. Math. J. 7, 137-159 (1960).

[6] BOTT, R.: On a theorem of Lefschetz. Mich. Math. J. 6, 211-216 (1959).

[7] CHEVALLEY, C.: Anneaux de Chow et applications. Séminaire Chevalley: Paris 1958.

[8] GROTHENDIECK, A.: On the De Rham cohomology of algebraic varieties. Publ. Math. I.H.E.S. 29, 95-103 (1966).

[9] HARTSHORNE, R.: Ample vector bundles. Publ. Math. I.H.E.S. 29, 63-94 (1966).

[10] ______________: Residues and duality. Springer Lecture Notes 20 (1966).

[11] ______________: Cohomological dimension of algebraic varieties. Ann. Math. 88, 403-450 (1968).

[12] ______________: Ample subvarieties of algebraic varieties. Springer Lecture Notes 156 (1970).

[13] ______________: Cohomology of non-complete algebraic varieties. Compos. Math. 23, 257-264 (1971).

[14] HIRONAKA, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero. Ann. Math. 79, 109-326 (1964).

[15] LÊ DŨNG TRANG: Singularités isolées des intersections complètes. Séminaire Weishu Shih, I.H.E.S. (1969-70).

[16] LIEBERMAN, D., and HERRERA, M.: Duality and the De Rham cohomology of infinitesimal neighborhoods, Invent. Math. 13, 97-124 (1971).

[17] MILNOR, J.: Singular points of complex hypersurfaces. Ann. Math. Studies 61: Princeton 1968.

[18] OGUS, A.: Local cohomological dimensions of algebraic varieties, to appear.

[19] SERRE, J.-P.: Géométrie algébrique et géométrie analytique. Ann. Inst. Fourier 6, 1-42 (1955-56).

[20] ZARISKI, O.: Some open questions in the theory of singularities. Bull. A.M.S. 77, 481-491 (1971).

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